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HYPERBOLIC OPERATORS IN SPACES OF GENERALIZED DISTRIBUTIONS

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Hyperbolic operators were investigated by L. Ehrenpreis [**4**] in the space of Schwartz distributions and by C. C. Chou [**3**] in the spaces of Roumieu ultradistributions. In this paper we study hyperbolic operators in spaces of Beurling generalized distributions. (See [**1**] and [**2**]).

Let D', E', D'_ω, E'_ω be the spaces of distribution, distributions with compact support, generalized distributions and generalized distributions with compact support in $\mathbf{R}^n$, respectively.

DEFINITION. The convolution operator S, $S \in E'_\omega$, is said to be ω-hyperbolic with respect to $t > 0$ (resp. $t < 0$) if there exists a fundamental solution E^+ (resp. E^-), $E^+; E^- \in D'_\omega$, so that supp $E^+ \subset \{(x, t) \in \mathbf{R}^n \times \mathbf{R}: t \geqq -b_0 + b_1|x|\}$ for some $b_0, b_1 > 0$ (resp. supp $E^- \subset \{(x, t) \in \mathbf{R}^n \times \mathbf{R}: t \leqq b_0 - b_1|x|\}$ for some b_0, $b_1 > 0$).

An operator is said to be ω-hyperbolic if it is ω-hyperbolic with respect to $t > 0$ and $t < 0$. This definition coincides with the definition of hyperbolicity introduced by Ehrenpreis [**4**, Theorem 2] for Schwartz distributions.

For the notation and the properties of generalized distributions we refer to [**2**]. Let $\omega \in \mathscr{M}_c$ (see [**2**, Definition 1.3.23]). Using Proposition 1.2.1 of [**2**] we could extend ω to $\mathbf{C}^n$ without losing any of its original properties; we will assume that ω is the extended function. We use the estimate

$$\omega(\xi) = o(|\xi/\log|\xi|), \text{ as } |\xi| \to \infty, \tag{1}$$

from which it follows that

$$\omega(\xi) \leqq M(1 + |\xi|), \tag{2}$$

for some constant M.

Following Ehrenpreis we prove the following theorem which characterizes ω-hyperbolic operators. The theorem and its proof will be given in the case of ω-hyperbolicity with respect to $t > 0$, the other case could be proved similarly.

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THEOREM. *Let $S \in E'_\omega$ and let $\hat{S}$ denote its Fourier transform. The following conditions are equivalent*:

(i) *S is ω-hyperbolic with respect to $t > 0$.*

(ii) *S is invertible and there exist constants C, A and A_1 so that*

$$\operatorname{Im} \tau \leqq A\left(|\operatorname{Im} Z| + \frac{1}{2(3 + 2MA_1)}\,\omega(\operatorname{Re} Z, \operatorname{Re} \tau)\right),$$

for all $(Z, \tau) \in \mathbf{C}^n \times \mathbf{C}$; $\hat{S}(Z, \tau) = 0$, M is the constant of (2).

(iii) *There exist positive constants C, a and A so that*

$$|\hat{S}(Z, \tau)| \geqq C \exp(-a[|\operatorname{Im} Z| + |\operatorname{Im} \tau| + \omega(\operatorname{Re} z, \operatorname{Re} \tau)],$$

whenever $\operatorname{Im} \tau \geqq A[|\operatorname{Im} Z| + \omega(Z, \tau)$; $(Z, \tau) \in \mathbf{C}^n \times \mathbf{C}$.

We prove the theorem by proving the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

The implication (i) $\Rightarrow$ (ii). For $a > 0$ we denote by $E_{\omega,a}$ the space $E_\omega(\mathbf{R}^n \times (-a, a))$ and by $E^+_{\omega,a}$ the space $E_\omega(\mathbf{R}^n \times (-a, \infty))$ both endowed with the obvious topologies [2]. Let $s > 0$ so that Supp $S \subset \mathbf{R}^n \times (-s, s)$ and let $a > 2s + b_0 + 1$. Denote by $E_{\omega,a}(S)$ the vector space of all $\phi \in E_{\omega,a}$ so that $(S * \phi)(x, t) = 0$ for $|t| < a - s$. We provide $E_{\omega,a}(S)$ with the topology induced by $E_{\omega,a}$. The ω-hyperbolicity of S with respect to $t > 0$ implies that, for all $\psi \in E_{\omega,a}(S)$, there exists a unique $\bar{\psi} \in E^+_{\omega,s}$ so that $(S * \phi)(x, t) = 0$ for all $t \geqq 0$ and $(\bar{\psi} - \psi)(x, t) = 0$ for all (x, t) with $|t| \leqq s$. Moreover, the mapping $\psi \to \bar{\psi}$ from $E_{\omega,a}(S)$ into $E^+_{\omega,s}$ is continuous. The proof, which makes use of the fact that supp $E^+ \subset \{(x, t) \in \mathbf{R}^n \times \mathbf{R}, t \geqq -b_0 + b_1|x|\}$, $b_0, b_1 > 0$, is similar to that of proposition V.1–2 of [3] and will be omitted. Since the embedding $E^+_{\omega,s} \longrightarrow E^+_{\omega_1,s}$, where $\omega_1(\xi) = \log(1 + |\xi|)$, is continuous, it follows that the map $\psi \to \bar{\psi}$ from $E_{\omega,a}(S)$ into $E_{\omega_1,s}$ is continuous. Hence, for any compact subset K of $\mathbf{R}^n \times (-s, \infty)$ and any $m \in N$, there exist a positive constant α and $\phi \in D_\omega(\mathbf{R}^n \times (-a, a))$ so that

$$\text{(3)} \qquad \sup_{\substack{(x,t)\in K\\ |\alpha|\leq m}} |(D^\alpha \psi)(x, t)| \leqq \sup_{\zeta\in\mathbf{C}^n\times\mathbf{C}} \exp(\alpha\,\omega(\xi_1, \xi_2) - H_{K_1}(\eta) - |\eta|)|(\widehat{\psi\phi})(\zeta)|,$$

where $\zeta = (\zeta_1, \zeta_2)$; $\zeta_j = \xi_j + i\,\eta_j$, $\wedge$ denotes the Fourier transform, and H_{K_1} is the support function of K_1, $K_1 = \operatorname{supp} \phi$. Let $(Z, \tau) \in \mathbf{C}^n \times \mathbf{C}$, $Z = Z_1 + iZ_2$, $\tau = \tau_1 + i\,\tau_2$, so that $\hat{S}(Z, \tau) = 0$. By taking the function ψ to be $\psi(x, t) = e^{-i\langle (Z,\tau),\,(x,t)\rangle}$ one gets, from (1),

$$\text{(4)} \quad |\psi(0, 2a)| = e^{2a\tau_2} \leqq \sup_{\zeta\in\mathbf{C}^{n+1}} \exp(\alpha\omega(\xi_1, \xi_2) - H_{K_1}(\eta) - |\eta|))|\hat{\phi}(\zeta_1 + Z, \zeta_2 + \tau|.$$

One also has

$$
(5)\quad \begin{aligned}
-H_{k_1}(\eta) &\leqq -\sup_{\xi\in K_1}\langle \xi, \eta + \operatorname{Im}(Z,\tau)\rangle + \sup_{\xi\in K_1}\langle \xi, \operatorname{Im}(Z,\tau)\rangle\\
&\leqq -\sup_{\xi\in K_1}\langle \xi, \eta + \operatorname{Im}(Z,\tau)\rangle + \sup_{\substack{\xi_1:\\ (\xi_1,\xi_2)\\ \in K_1}}\langle \xi_1, Z_2\rangle + \sup_{\substack{\xi_2:\\ (\xi_1,\xi_2)\\ \in K_1}}\langle \xi_2, \tau_2\rangle\\
&\leqq -\sup_{\xi\in K_1}\langle \xi, \eta + \operatorname{Im}(Z,\tau)\rangle + k_1|Z_2| + a|\tau_2|,
\end{aligned}
$$

for some constant $k_1 > 0$. By using (5) and the subadditivity of ω to estimate the right hand side of (4), one gets

$$
(6)\quad \begin{aligned}
e^{2a\tau_2} &\leqq \exp(\alpha\omega(Z_1,\tau_1) + (k_1+1)|Z_2| + (a+1)|\tau_2|)\\
&\quad\times \sup_{\zeta\in\mathbf{C}^{n+1}} \exp(\alpha\omega(\xi_1 + Z_1, \xi_2 + \tau_1) - H_{K_1}(\eta + \operatorname{Im}(Z,\tau))\\
&\qquad - |\eta + \operatorname{Im}(Z,\tau)|^1 \cdot |\hat{\phi}(\zeta + (Z,\tau))|)\\
&= \exp(\alpha\omega(Z_1,\tau_1) + (k_1+1)|Z_2| + (a+1)|\tau_2|\, \|\phi\|_\alpha^\omega,
\end{aligned}
$$

which implies that

$$
(7)\qquad (a-1)\operatorname{Im}\tau \leqq \alpha\,\omega(Z_1,\tau_1) + k_2|\operatorname{Im} Z|
$$

for some potitive constant k_2. Since S has a fundamental solution E^+ it follows that S is invertible in D'_ω and

$$
\operatorname*{Sup}_{\substack{|x-y|\leqq\\ A_1\omega(x)\\ y\in\mathbf{R}^{n+1}}} |\hat{S}(y)| \geqq C_1 e^{-A_1\omega(x)}, \quad x\in\mathbf{R}^{n+1},
$$

for some positive constants A_1 and C_1. Take

$$
A = \max\left\{\frac{k_2}{a-1}, \frac{2(3+2MA_1)\alpha}{a-1}\right\}.
$$

Then inequality (7) gives

$$
\operatorname{Im}\tau \leqq A\left(|\operatorname{Im} Z| + \frac{1}{2(3+2MA_1}\,\omega(\operatorname{Re} Z, \operatorname{Re}\tau)\right).
$$

The implication (ii) $\Rightarrow$ (iii). Suppose that $(Z,\tau)\in\mathbf{C}^n\times\mathbf{C}$; $Z = Z_1 + iZ_2$, $\tau = \tau_1 + i\tau_2$, so that $\tau_2 \geqq A(|Z_2| + \omega(Z_1,\tau_1))$. The invertibility of S implies that there exist positive constants C_1, A_1 so that

$$
\sup_{\substack{|(Z',\tau')|\leqq A_1\omega(Z_1,\tau_1)\\ (Z_1,\tau_1)\in\mathbf{R}^n\times\mathbf{R}}} |\hat{S}(Z_1 + Z_1', \tau_1 + \tau_1')| \geqq C_1 e^{-A_1\omega(Z_1,\tau_1)}.
$$

Hence there exists a point $(x_1, t_1)\in\mathbf{R}^n\times\mathbf{R}$, $|x_1 - Z_1| + |t_1 - \tau_1| \leqq A_1\omega(Z_1,\tau_1)$ so that

$$
(8)\qquad |\hat{S}(x_1,t_1)| \geqq C_1 e^{-A_1\omega(Z_1,\tau_1)}.
$$

Next, consider the entire function of one complex variable

$$g(\lambda) = \hat{S}(Z + \frac{\lambda}{A_1}(x_1 - Z), \tau + \frac{\lambda}{A_1}(t_1 - \tau)), \qquad \lambda = \lambda_1 + i\lambda_2.$$

We claim that $g(\lambda) \neq 0$ whenever $|\lambda| < 1/4$. For, put $T = \tau + (\lambda/A_1)(t_1 - \tau)$, $\mathscr{Z} = Z + (\lambda/A_1)(x_1 - Z)$. From (ii) it suffices to show that

$$\operatorname{Im} T > A\left(|\operatorname{Im} \mathscr{Z}| + \frac{\omega(\operatorname{Re} \mathscr{Z}, \operatorname{Re} T)}{2(3 + 2\,MA_1)}\right).$$

From the assumption on Im τ and the fact that ω is increasing it follows that

$$\begin{aligned} \operatorname{Im} T &\geqq \left(1 - \frac{\lambda_1}{A_1}\right)\tau_2 - \frac{|\lambda_2|}{A_1}|t_1 - \tau_1| \\ &\geqq A\left(1 - \frac{\lambda_1}{A_1}\right)\Big[|\operatorname{Im} Z| + \omega(Z, \tau)\Big] - \frac{|\lambda_2|}{A_1}|t_1 - \tau_1|. \end{aligned} \tag{9}$$

Also, one has

$$\left(1 - \frac{\lambda_1}{A_1}\right)|\operatorname{Im} Z| \geqq |\operatorname{Im} Z| - \frac{|\lambda_2|}{A_1}|x_1 - Z_1|. \tag{10}$$

By using (10) to estimate the right hand side of (9) it follows that

$$\begin{aligned} \operatorname{Im} T &\geqq A|\operatorname{Im}\mathscr{Z}| - A\frac{|\lambda_2|}{A_1}|x_1 - Z_1| + A\left(1 - \frac{\lambda_1}{A_1}\right)\omega(Z, \tau) - \frac{|\lambda_2|}{A_1}|t_1 - \tau_1| \\ &> A|\operatorname{Im}\mathscr{Z}| - \frac{A}{A_1}|\lambda_2|\,(|x_1 - Z_1| + |t_1 - \tau_1|) + A\left(1 - \frac{\lambda_1}{A_1}\right)\omega(Z, \tau) \\ &\geqq A|\operatorname{Im}\mathscr{Z}| + A\left(1 - \frac{\lambda_1}{A_1} - |\lambda_2|\right)\omega(Z, \tau) \geqq A\left(|\operatorname{Im}\mathscr{Z}| + \frac{1}{2}\omega(Z, \tau)\right), \end{aligned} \tag{11}$$

where we assumed without loss of generality that $A > A_1 > 1$. Since$|(\mathscr{Z}, T)| \leqq |(Z, \tau)| + |x_1 - Z_1| + |t_1 - \tau_1| + |(Z_2, \tau_2)$, the monotonicity and the subadditvity of ω together with the fact that $|x_1| - Z_1| + t|_1 - \tau_1| \leqq A_1\omega(Z_1, \tau_1)$ imply that $\omega(Z, T) \geqq 1/(3 + 2MA_1)\omega(Z, T)$, where M is the constant of (2). Hence, (11) becomes

$$\operatorname{Im} T > A\left(|\operatorname{Im}\mathscr{Z}| + \frac{1}{2(3 + 2MA_1)}\omega(\mathscr{Z}, T)\right) \geqq A\left(|\operatorname{Im}\mathscr{Z}| + \frac{\omega(\operatorname{Re}\mathscr{Z}, \operatorname{Re} T)}{2(3 + 2MA_1)}\right), \tag{12}$$

which completes the proof of the claim.

By applying the minimum modulus theorem of Chou [3, Theorem II.2.1] with $R = \lambda_0 = A_1$, $r = 1/6$, $\eta < 1/(96A_1)$, to the function g, one obtains

$$|g(0)| = |\hat{S}(Z, \tau)| \geqq |g(A_1)|^{3(H+1)} \Big/ \sup_{|\lambda| \leqq 3eA_1} |g(\lambda)|^{3H} \sup_{|\lambda| \leqq 1/4} |g(\lambda)|^2. \tag{I}$$

Next, we estimate the denominator of the right hand side of (I). By applying the Paley-Wienner theorem to S it follows that there exist constants C_2, λ_2' so that

$$\begin{aligned} &\sup_{|\lambda|\leqq 3eA_1} |g(\lambda)|^{3H} \sup_{|\lambda|\leqq 1/4} |g(\lambda)|^2 \\ &\leqq \sup_{|\lambda|\leqq 3eA_1} |g(\lambda)|^{3H+2} \\ &= \sup_{|\lambda|\leqq 3eA_1} \left(\left|\hat{S}\left(Z + \frac{\lambda}{A_1}(x_1 - Z), \tau + \frac{\lambda}{A_1}(t_1 - \tau)\right)\right| \right)^{3H+2} \\ &\leqq C_2 \exp\Big(\lambda_2' \omega\Big(Z_1 + \frac{\lambda_1}{A_1} x_1 - \frac{\lambda_1}{A_1} Z_1 + \frac{\lambda_2}{A_1} Z_2, \\ &\qquad \tau_1 + \frac{\lambda_1}{A_1} t_1 - \frac{\lambda_1}{A_1}\tau_1 + \frac{\lambda_2}{A_1}\tau_2\Big) + H_{\operatorname{supp} S}(\operatorname{Im} Z, \operatorname{Im} T)\Big), \end{aligned} \tag{13}$$

where $H_{\operatorname{supp} S}$ is the support function of supp S. Since supp S is compact, one can assume that it is contained in the closed ball $B(0, k)$ for some positive integer k. Hence, one has

$$\begin{aligned} &H_{\operatorname{supp} S}(\operatorname{Im} Z, \operatorname{Im} T) \\ &= \sup_{(x,t)\in \operatorname{supp} S} \langle x, \frac{\lambda_2}{A_1} x_1 - \frac{\lambda_1}{A_1} Z_2 + Z_2 \rangle + \langle t, \frac{\lambda_2}{A_1} t_1 - \frac{\lambda_1}{A_1}\tau_2 + \tau_2 \rangle \\ &\leqq 2k\,|\operatorname{Im} Z| + 2k\,|\operatorname{Im} \tau| + d, \end{aligned} \tag{14}$$

for some constant d. Also, one has

$$\begin{aligned} &\lambda_2' \omega\Big(Z_1 + \frac{\lambda_1}{A_1}(x_1 - Z_1) + \frac{\lambda_2}{A_1} Z_2, \tau_1 + \frac{\lambda_1}{A_1}(t_1 - \tau_1) + \frac{\lambda_2}{A_1}\tau_2\Big) \\ &\leqq \lambda_2'\omega(Z_1, \tau_1) + \lambda_2'\omega(x_1 - Z_1, t_1 - \tau_1) + \lambda_2'\omega(Z_2, \tau_2) \\ &\leqq \lambda_2'\omega(Z_1, \tau_1) + \lambda_2' M(1 + |x_1 - Z_1| + |t_1 - \tau_1|) + \lambda_2' M(1 + |Z_2| + |\tau_2|) \\ &\leqq M\Big(2\lambda_2' + \Big(\frac{1}{M} + A_1\Big)\lambda_2'\omega(Z_1, \tau_1) + \lambda_2'|\operatorname{Im} Z| + \lambda_2'|\operatorname{Im}\tau|\Big), \end{aligned} \tag{15}$$

where M is the constant of (2).

Now, using inequalities (14) and (15) to estimate the right hand side of (13), one gets

$$\begin{aligned} &\sup_{|\lambda|\leqq 3eA_1} |g(\lambda)|^{3H+2} \sup_{|\lambda|\leqq 1/4} |g(\lambda)|^2 \\ &\leqq C_2' \exp(\lambda_2'(1 + MA_1)\omega(Z_1, \tau_1) + (2k + M\lambda_2')(|\operatorname{Im} Z| + \operatorname{Im}\tau)), \end{aligned} \tag{16}$$

where $C_2' = C_2 e^{D+2M\lambda_2'}$. Next, using inequalities (8) and (16) to estimate the right hand side of (I) one gets

$$|\hat{S}(Z, \tau)| \geqq C \exp(-a(|\operatorname{Im} Z| + \operatorname{Im}\tau + \omega(\operatorname{Re} Z, \operatorname{Re}\tau))),$$

where $C = 1/C_2' \cdot C_1^{3(H+1)}$ and $a = \max\{(3H + 3 + M)A_1\lambda_2' + 1, 2k + M\lambda_2'\}$.

The implication (iii) $\Rightarrow$ (i). We want to find a fundamental solution

E for S, $E \in D'_\omega$, so that supp $E \subset \{(x, t) \in \mathbf{R}^n \times \mathbf{R};\ t \geqq -b_0 + b_1|x|\}$ for some positive constants b_0 and b_1. For any $Z \in \mathbf{C}^n$, we define, in $\mathbf{C}$, the curve $\Gamma(Z) = \{\tau = \tau_1 + i\tau_2;\ \tau_1 \in \mathbf{R},\ \tau_2 = A(|\mathrm{Im}\ Z| + \omega(\mathrm{Re}\ Z, \tau_1))\}$ directed in the sense that τ_1 is increasing. Let us assume that $n = 1$. Define the linear form E on D_ω by

$$\text{(II)} \qquad \langle_i, \phi\rangle = \iint_{\substack{\mathrm{Im} Z = -\lambda_1 \Gamma(Z) \\ \lambda_1 > 0}} \frac{\hat{\phi}(Z, \tau)}{\hat{S}(-Z, -\tau)}\, d\tau dZ.$$

From the condition on the growth of $\hat{S}$ one has for $\tau \in \Gamma(Z)$, $Z = x - i\lambda_1$, $\lambda_1 > 0$,

$$\text{(17)} \qquad \begin{aligned} \frac{1}{\hat{S}(-Z, -\tau)} &\leqq \frac{1}{C} \exp(a(\lambda_1 + \mathrm{Im}\ \tau + \omega(x, \tau_1))) \\ &\leqq \frac{1}{C} \exp(1 + A)a\lambda_1 + a(1 + A)\omega(x, \tau_1)), \end{aligned}$$

where a and A are the constants of condition (iii). By applying the standard Paley-Weiner theorem to $\phi \in D$ and using (17), it follows that the integral on the right hand side of (II) converges. Thus the linear form E is weli defined. Next, we show that E is continuous. Since D_ω is a Montel space, it suffices to show that E is sequentially continuous. Suppose that $\phi_j \to 0$ in D_ω as $j \to \infty$; it follows that, for all $\lambda_2 > 0$,

$$\sup_{\substack{(Z,\tau) \in \mathbf{C} \times \mathbf{C} \\ |(-\lambda_1, \tau_2)| \leqq \lambda_2 \omega(x, \tau_1)}} |\hat{\phi}_j(Z, \tau)|\, e^{\lambda_2 \omega(x, \tau_1)} \to 0 \quad \text{as } j \to \infty.$$

Hence, it follows that

$$\text{(18)} \qquad \begin{aligned} |\langle E, \varphi_j\rangle| &\leqq \frac{1}{C} e^{a(A+1)\lambda_1} \iint_{\substack{\mathrm{Im} Z = -\lambda_1 \Gamma(\hat{Z}) \\ \lambda_1 > 0}} |\hat{\varphi}_j(Z, \tau)|\, e^{(1+A)a\omega(x, \tau_1)} |d\tau|\, dZ \\ &\leqq \frac{1}{C} e^{a(1+A)\lambda_1} e^{2MA\lambda_3} \sup_{\substack{(Z,\tau) \in \mathbf{C}^2 \\ |(-\lambda_1, \tau_2)| \\ \leqq \lambda_3 \omega(x, \tau_1)}} |\hat{\varphi}_j(Z, \tau)|\, e^{(1+MA)\lambda_3 \omega(x, \tau_1)} \\ &\quad \times \int_{\substack{Z \in \mathbf{C} \\ \mathrm{Im} Z = -\lambda_1}} \int_{\Gamma(Z)} \frac{e^{a(A+1)\omega(x, \tau_1)}}{(1 + |Z| + |\tau|)^{b\lambda_3}} |d\tau| dZ, \end{aligned}$$

where M and b are (fixed) positive constants. Since the double integral on the right hand side of (18) converges for λ_3 large enough and the supremum on the same side converges to zero for all $\lambda_3 > 0$ as $j \to \infty$, it follows that $\langle E, \varphi_j\rangle \to 0$ as $j \to \infty$, i.e., E is continuous. Hence $E \in D'_\omega$.

Next, we show that $S * E = \delta$. From the definition of E it follows that

$$\langle S * E, \varphi\rangle = \langle E, \check{S}_* \varphi\rangle = \int\limits_{\substack{Z \in \mathbf{C} \\ \operatorname{Im} Z = -\lambda_1}} \int\limits_{\Gamma(Z)} \hat{\varphi}(Z, \tau)\, d\tau dZ, \quad \varphi \in D_\omega.$$

Since $\hat{\varphi}$ is entire, it follows from Cauchy's theorem and a change of variable that

$$\langle S_* E, \varphi\rangle = \int\limits_{x \in \mathbf{R}} \int\limits_{\tau \in \mathbf{R}} \hat{\varphi}(zx, \tau) d\tau dx = \varphi(0, 0) = \langle \delta, \varphi\rangle, \quad \text{i.e., } S_* E = \delta.$$

Finally, we prove that supp $E \subset \{(x, t) \in \mathbf{R} \times \mathbf{R},\, t \geqq -b_0 + b_1 |x|$, for some $b_0, b_1 > 0\}$. Take $b_0 = 3a(1 + 1/A)$, $b_1 = 1/A$ where a and A are the constants of condition (iii) above. Let $\varphi \in D_\omega$ so that supp $\varphi \subset \{(x, t) \in \mathbf{R}^2; t < -b_0 + b_1 |x|\}$. We prove that $\langle E, \varphi\rangle = 0$ .Let's assume that $x > 0$. There exists an ε_1, $0 < \varepsilon_1 < 0.1$ so that $t \leqq -b_0 - \varepsilon_1 + b_1 x$. Since $D^{(k, \ell)} \varphi(Z, \tau) = (D_1^k D_2^\ell \varphi)(Z, \tau) = Z^k \tau^\ell \hat{\varphi}(Z, \tau)$, for all $k, \ell \in \mathbf{N}$, it follows from the Paley-Wiener theorem applied to $D^{(k, \ell)} \varphi$ that, for all $\lambda > 0$, there exists a constant $C_{\lambda, \varepsilon_1}$ so that

$$\begin{aligned} |\hat{\varphi}(Z, \tau)| &\leqq C_{\lambda, \varepsilon_1} |Z|^{-k} |\tau|^{-\ell} e^{-\lambda \omega(\operatorname{Re} Z,\ \operatorname{Re} \tau)} \\ &\qquad + H_{\operatorname{supp} \varphi}(\operatorname{Im} Z, \operatorname{Im} \tau) + \varepsilon_1 |(\operatorname{Im} Z, \operatorname{Im} \tau)| \end{aligned} \tag{19}$$

For $Z = x - i\lambda_1$, $\lambda_1 > 0$, $\operatorname{Im} \tau = \lambda_1 A + A\omega(x, \tau_1)$ and $t \leqq -b_0 - \varepsilon_1 + b_1 x$, it follows that

$$\begin{aligned} &H_{\operatorname{supp} \varphi}(\operatorname{Im} Z, \operatorname{Im} \tau) \\ &= \sup_{(x, t) \in \operatorname{supp} \varphi} \langle -\lambda_1, x\rangle + \langle t, \lambda_1 A + A\omega(x, \tau_1)\rangle \\ &\leqq \operatorname*{Sup}_{(x, t) \in \operatorname{supp} \varphi} \langle -\lambda_1, x\rangle + \left\langle \frac{x}{A} - b_0 - \varepsilon_1, \lambda_1 A + A\omega(x, \tau_1)\right\rangle \\ &\leqq -b_0 A \lambda_1 - \varepsilon_1(\lambda_1 A + A\omega(x, \tau_1)) + (k_1 - b_2 - \varepsilon_1) A\omega(x, \tau_1), \end{aligned} \tag{20}$$

for some constant k_1. Using (20) to estimate the right hand side of (19) one gets

$$|\hat{\varphi}(Z, \tau)| \leqq C_{\lambda, \varepsilon_1} e^{\varepsilon_1 \lambda_1 + \varepsilon_1 \lambda_1 A - b_0 \lambda_1 A} |Z|^{-k} |\tau|^{-\ell} e^{-(\lambda + b_0 A - k_1 A - \varepsilon_1 A)\omega(x, \tau_1)}. \tag{21}$$

Using condition (iii) and the estimate (21) it follows that

$$\begin{aligned} |\langle E, \varphi\rangle| &\leqq C C_{\lambda, \varepsilon_1} e^{-(b_0 A - \varepsilon_1 - \varepsilon_1 A - aA - a)\lambda_1} \\ &\qquad \times \int\limits_{\operatorname{Im} Z = -\lambda_1} \int\limits_{\Gamma(Z)} |Z|^{-k} |\tau|^{-\ell} e^{-(\lambda + b_0 A - k_1 A - \varepsilon_1 A - a)\omega(Z, \tau)} |d\tau| dZ. \end{aligned} \tag{22}$$

By choosing λ, k and ℓ large enough, it follows that the double integral on the right hand side of (22) is bounded and (22) gives

$$|\langle E, \varphi\rangle| \leqq C'_{\lambda, \varepsilon_1} \exp(-\lambda_1(b_0 A - \varepsilon_1 - A\varepsilon_1 - aA - a)). \tag{23}$$

By the choice of b_0 and ε_1 one has $b_0 A - \varepsilon_1 - \varepsilon_1 A - aA - a > 0$, and by sending λ_1 to infinity it follows that $\langle E, \varphi \rangle = 0$. In case $x < 0$, $t \geqq -b_0 - b_1 x$ we consider the curve $\Gamma(Z)$; $Z = x + i\lambda_1$, $\lambda_1 > 0$ and we send λ_1 to infinity. This proves the implication in case $n = 1$. For $n > 1$ we consider the corresponding hyperplanes and the argument is similar to the case $n = 1$. This completes the proof of the theorem.

References

1. A.Beurling, *Ouasi-analyticity and general distributions, lectures* 4 *and* 5, A. M. S. Summer Institute, Stanford, 1961 (mimeographed).

2. G. Bjork, *Linear partial differential operators and generalized distributions*, Akriv for Matematik **6** (1966), 351–407.

3. C. C. Chou, *La Transformation de Fourier complexe et l'equation de convolution*. Lecture Notes in Mathematics, **325**, Springer, 1973.

4. L. Ehrenpreis, *Solutions of some problems of divisions* V. Amer, J. of Math. **84** (1962), 324–348.

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