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ON FACTORIZATION OF BIINFINITE TOTALLY POSITIVE BLOCK TOEPLITZ MATRICES

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ABSTRACT. We extend the work of Aissen, Schoenberg, Whitney, and Edrei, who characterized the symbol of a totally positive Toeplitz matrix, to characterizing the determinant of the symbol of a block Toeplitz totally positive matrix. As a consequence of our arguments we show that the symbol for the block Toeplitz case may be factored just as in the Toeplitz setting.

1. Introduction. A biinfinite matrix $A = (A_{ij})$, $-\infty < i, j < \infty$, is called totally positive provided that all its minors are nonnegative, i.e., for all $i_1 < \cdots < i_p, j_1 < \cdots < j_p$

$$A\begin{pmatrix} i_1, \ldots, i_p \\ j_1, \ldots, j_p \end{pmatrix} = \begin{vmatrix} A_{i_1, j_1} & \cdots & A_{i_1, j_p} \\ \vdots & & \vdots \\ A_{i_p, j_1} & \cdots & A_{i_p, j_p} \end{vmatrix} \geqq 0. \tag{1.1}$$

In two previous papers, [**5**, **6**], we were concerned with factorization and invertibility of such matrices. Our motivation for these questions arose from certain problems in the theory of spline functions. Consequently, these papers only treated the case where A is banded, i.e., for some integers n and m, with m nonnegative, $A_{ij} = 0$, if $i - j < n$ or $i - j > n + m$. In this case, we say A is m-banded. We will concentrate our attention here on matrices which are block Toeplitz. Thus for some integer N we require

$$A_{ij} = A_{i+N, j+N} \text{ all } i, j \in \mathbf{Z}.$$

Any such matrix has the block structure

$$A = \begin{bmatrix} & & \cdot & \cdot & & \cdot & \\ \cdots & A_{-1} & A_0 & A_1 & A_2 & \cdot & \cdots \\ \cdots & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & \cdots \\ \cdots & \cdot & A_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ & & \cdot & \cdot & \cdot & & \end{bmatrix} \tag{1.2}$$

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where each A_k is the $N \times N$ matrix given by

$$(A_k)_{ij} = A_{i,kN+j},\ 0 \leqq i, j \leqq N - 1.$$

It is customary to associate with A the $N \times N$ matrix-valued formal Laurent series

$$A(z) = \sum_{k=-\infty}^{\infty} A_k z^{-k} \tag{1.3}$$

which is called the symbol of A. The importance of the symbol is based on the easily verified relation

$$(AB)(z) = A(z)\, B(z) \tag{1.4}$$

so that, in particular,

$$(A(z))^{-1} = A^{-1}(z)$$

whenever both sides are defined.

The material we present here arose from an attempt to extend the results obtained in a series of papers [**1**, **2**, **7**, **8**, **9**, **12**] for the scalar Toeplitz case, $N = 1$. In his papers [**7**, **8**, **9**] Edrei obtained the following complete characterization of the (scalar) symbol of a totally positive biinfinite Toeplitz matrix $A = (a_{i-j})$

$$\text{(1.5)}\quad \begin{aligned} &\text{i) } A(z) = \sum_{j=-\infty}^{\infty} a_j z^{-j} = \gamma z^k e^{\alpha z} \prod_{j=1}^{\infty} \left(\frac{1 + \alpha_j z}{1 - \beta_j z} \right), \text{ if } a_j = 0 \text{ for } j > 0, \\ &\text{ii) } A(z) = \sum_{j=-\infty}^{\infty} a_j z^{-j} = \gamma z^k e^{(\alpha z + \beta/z)} \prod_{j=1}^{\infty} \frac{(1 + \alpha_j z)(1 + \delta_j/z)}{(1 - \beta_j z)(1 - \gamma_j/z)} \end{aligned}$$

where in both cases k is an integer, all constants are nonnegative and

$$\sum_{j=-\infty}^{\infty} (\alpha_j + \beta_j + \gamma_j + \delta_j) < \infty.$$

Earlier, for the lower triangular case i) above, Aissen, Schoenberg and Whitney only obtained the representation

$$A(z) = \gamma z^k e^{g(z)} \prod_{j=1}^{\infty} \left(\frac{1 + \alpha_j z}{1 - \beta_j z} \right)$$

where $g(z)$ is some entire function. Later Edrei, [**9**], identified $g(z)$ as αz, $\alpha > 0$, thus proving the conjecture in [**2**].

Our results for the block Toeplitz case are quite similar to those above. To explain this we notice the following matrix theoretic interpretation of Edrei's result. Each factor in Edrei's representation (1.5) is the symbol of a simple totally positive Toeplitz matrix. For instance, the factor $1 + \alpha z$ is the symbol of the one-banded matrix corresponding to the sequence $a_0 = 1$, $a_{-1} = \alpha > 0$, $a_j = 0$, $j \neq 0, -1$. Similarly, $(1 - \beta z)^{-1}$ is the symbol of the lower triangular inverse of the one-banded matrix corresponding to the sequence $b_0 = 1$, $b_{-1} = -\beta < 0$, $b_j = 0$, $j \neq 0, -1$. Therefore we can view (1.5) as a factorization of the (scalar) Toeplitz matrix $A = (a_{i-j})$ into elementary totally positive Toeplitz matrices.

In the block Toeplitz case, we factor the symbol $A(z)$ by directly factoring the associated matrix A. We thereby obtain a factorization of the symbol in terms of a product of symbols of one-banded block Toeplitz factors, inverses of one-banded block Toeplitz matrices and a remaining symbol. For $N > 1$, this symbol as well as its inverse is entire and has determinant identically one. Consequently, we obtain a complete factorization of the determinant of the symbol. This result gives a useful criterion for invertibility of block Toeplitz totally positive matrices over sequence spaces.

The outline of the paper is as follows. First we identify the possible zero patterns of a totally positive block Toeplitz matrix. In case the given matrix has all entries positive we characterize certain "trivial" cases up to which all such matrices can be written as a product of a lower and an upper triangular totally positive block Toeplitz matrix. This factorization is based on some general results in **[3]**. In terms of the symbol this means that the Laurent expansion $A(z)$ is a product of one-sided expansions involving only positive or negative powers of z, respectively. This allows us to restrict our subsequent considerations to the one-sided (lower triangular) case without any loss of generality. This useful reduction was not made in the scalar Toeplitz case, although the LU factorization is apparent from Edrei's factorization (1.5).

§3 is concerned with the factorization of lower triangular matrices. We start by extending the factorization theorem for strictly banded matrices, **[6]**, to arbitrary block Toeplitz banded matrices whose symbol does not identically vanish. We note here that such factorizations are not unique and even for Toeplitz matrices one may find non-Toeplitz factorizations, **[6]**. Therefore, we emphasize that our procedure preserves the block Toeplitz structure. §4 is devoted to characterizing the determinant of the symbol and to the statement of our results for the two-sided case. Finally, in §5 we discuss a class of symbols which arise as solutions of a biinfinite system of ordinary differential equations. These are analogous to the exponential factors appearing in the scalar case. We show that these matrices cannot be factored into a product of one banded matrices or inverses of such. Furthermore, we conjecture that this class comprises all non-factorable symbols of lower triangular totally positive block Toeplitz matrices and that the remaining symbol in our factorization has this form.

2. Preliminary Remarks. We will denote by $\mathscr{A}_N$ the class of all totally positive biinfinite block Toeplitz matrices with block size N. Our first objective is to clarify the circumstances under which the formal expansion $A(z)$ in (1.3) converges in some domain of the complex plane. To this end, let us recall some well-known facts.

LEMMA 2.1. *Let A be totally positive and suppose for some $i, j \in \mathbf{Z}$, $A_{ij} = 0$. Then at least one of the following conditions is satisfied*:

i) $A_{rs} = 0$, for all $r \leqq i$ and $s \geqq j$;
ii) $A_{rs} = 0$, for all $r \geqq i$ and $s \leqq j$;
iii) $A_{is} = 0$, $s \in \mathbf{Z}$;
iv) $A_{rj} = 0$, $r \in \mathbf{Z}$.

The proof follows immediately by checking appropriate 2×2 minors of A having A_{ij} as upper left or lower right corner entry. We shall refer to cases i) and ii) as the (i, j) zero casting an upper right or lower left shadow (of zeroes), respectively.

Since total positivity is preserved whenever a zero row or column is deleted or appended to a totally positive matrix we will assume throughout the sequel that all matrices have a non-zero entry in every row and every column, thereby excluding cases iii) and iv) in Lemma 2.1.

Combining Lemma 2.1 with the N-periodicity of the elements of any $A \in \mathscr{A}_N$ yields.

LEMMA 2.2. *Let A be in $\mathscr{A}_N$. Then either A is full $(A_{ij} \neq 0,\ i, j \in \mathbf{Z})$. banded, or triangular (for lower triangular we have $A_{ij} = 0$ for $i \leqq j + m$ and $A_{ij} \neq 0$ if $i \geqq j + m + N + 1$).*

Later we will refer to the latter case above as a ragged edge.

PROOF. Suppose A is neither full nor banded. Then for each row we have (excluding (iii) and (iv) of Lemma 2.1) $A_{i,i-p} = 0$ and either $A_{i,i-p-1} \neq 0$ or $A_{i,i-p+1} \neq 0$ where the index p may depend on i. The first case as we shall see corresponds to a lower triangular matrix while the second corresponds to an upper triangular matrix. Let us just consider the first case since the reasoning is similar for both cases. By Lemma 2.1 we conclude that $A_{i,i-p}$ throws an upper right shadow. Since $A \in \mathscr{A}_N$ we see that $A_{i+\ell N, i+\ell N-p}$ also throws an upper right shadow for all $\ell \in \mathbf{Z}$ and $A_{i+\ell N, i+\ell N-p-1} \neq 0$. It follows that $A_{r,s} = 0$ for $r - s \leqq p - N$, $r, s \in \mathbf{Z}$. Since A is not banded all zero entries in A must throw upper right shadows. It follows that $A_{rs} \neq 0$ if $r - s \geqq p + N + 1$. Thus we see that $m = p - N$ in Lemma 2.2.

This means that any A in $\mathscr{A}_N$ is either full, banded, lower, or upper triangular with respect to some diagonal (with possibly ragged upper or lower edges, respectively). Next note that each entry of the symbol of $A \in \mathscr{A}_N$

with

$$A(z) = (A_{rs}(z))_{r,s=1}^{N}$$

$$A_{rs}(z) = \sum_{k=-\infty}^{\infty} A_{r,kN+s} z^{-k}$$

is the symbol of a Toeplitz matrix in $\mathscr{A}_1$. Consequently, from the scalar case [11], if A is lower triangular then $A(z)$ is convergent in a (deleted) neighborhood of the origin while if A is upper triangular then $A(z)$ is convergent in a (deleted) neighborhood of infinity. If A is full, i.e., $A_{ij} \neq 0$ for all $i, j \in \mathbf{Z}$, we have the following analog to the scalar case (cf. [8]).

THEOREM 2.1. *Let $A \in \mathscr{A}_N$, $A_{ij} \neq 0$ for $i, j \in \mathbf{Z}$. Then either there exists* $0 < R_1 < R_2$ so that

$$A(z) = \sum_{-\infty}^{\infty} A_j z^{-j}$$

converges in the annulus $R_1 < |z| < R_2$ or the blocks A_j have the form

$$A_j = \rho^j A_0 \tag{2.1}$$

for some $\rho > 0$ and all 2×2 minors of A_0 are zero.

REMARK. Note that any block Toeplitz matrix of the form (2.1) where A_0 has nonnegative elements and all its 2×2 minors vanish is totally positive.

PROOF. Since each $A_{rs}(z)$ is the symbol of a Toeplitz matrix in $\mathscr{A}_1$ we know that

$$\begin{vmatrix} A_{r,kN+s} & A_{r,(k+1)N+s} \\ A_{r,(k-1)N+s} & A_{r,kN+s} \end{vmatrix} \geqq 0.$$

Thus we have

$$\frac{A_{r,kN+s}}{A_{r,(k+1)N+s}} \geqq \frac{A_{r,(k-1)N+s}}{A_{r,kN+s}},$$

and so we can define

$$\gamma_{\pm} = \lim_{k \to \pm\infty} \frac{A_{r,kN+s}}{A_{r,(k+1)N+s}}.$$

We know from [6] that γ_+ and γ_- are independent of r and s and hence by the ratio test each of the functions $A_{rs}(z)$ have the same convergence properties. In particular $\sum_{\infty}^{k=0} A_{r,kN+\rho} z^{-k}$ is convergent in $|z| < 1/\gamma_-$ and $\sum_{k=-\infty}^{0} A_{r,kN+s} z^{-k}$ is convergent in $|z| > 1/\gamma_+$. Note that $\gamma_+ \geqq \gamma_-$ so that as long as $\gamma_+ > \gamma_-$ we have a nontrivial annulus of convergence for $A(z)$ with $R_1 = 1/\gamma_+$ and $R_2 = 1/\gamma_-$. If, on the other hand, we have $\gamma_+ = \gamma_-$ then all the ratios must be constant in which case (2. 1) holds with $\rho = 1/\gamma_{\pm}$. The remaining claim follows by considering 2×2 minors of the submatrix of A consisting of the blocks A_0 and ρA_0. This completes the proof of Theorem 2.1.

Let $A(z) = \sum A_j z^{-j}$ be the symbol of A. Then it is easy to verify that for any $\rho > 0$ the symbol of

$$A_\rho = (\rho^{i-j} A_{j-i})$$

is

$$A_\rho(z) = A(\rho z).$$

If $A \in \mathscr{A}_N$ then $A_\rho \in \mathscr{A}_N$ since clearly a positive scaling of rows or columns of A preserves total positivity. According to Theorem 2.1, if $\{A_j\}$ is not the trivial sequence (2.1), we may choose ρ so that $A_\rho(z)$ converges in an annulus containing the unit circle. Moreover, if det $A(z)$ is not identically zero in its annulus of convergence we may choose $\rho > 0$ so that

$$\det A_\rho(z) \neq 0, \text{ for } |z| = 1.$$

Clearly, in this case we have $A_\rho^{-1}(z) = (A_\rho(z))^{-1}$ for $|z| = 1$, where A_ρ^{-1} is the $\ell^\infty(\mathbf{Z})$ inverse of A_ρ. It follows that both A_ρ and A_ρ^{-1} have entries which decay exponentially away from any fixed diagonal. Hence A_ρ and A_ρ^{-1} are bounded maps on $\ell^\infty(\mathbf{Z})$ into itself.

It was shown in [3] that under these circumstances there exist unique lower and upper triangular (relative to some diagonal) totally positive matrices L and U both inducing boundedly invertible maps on $\ell^\infty(\mathbf{Z})$ such that $A_\rho = LU$. The uniqueness of this factorization ensures that both L and U are in $\mathscr{A}_N$ as well. Thus to understand the structure of the elements in $\mathscr{A}_N$ whose symbol converges in some annulus and has determinant not identically zero, it suffices to study the subclass of lower triangular matrices.

3. A factorization procedure. From the preceding discussion it is sufficient to study lower triangular matrices, that is

$$A = (A_{j-i})_{i,j \in \mathbf{Z}'}$$

where for some n

$$A_j = 0, \quad \text{for } j > n. \tag{3.1}$$

In addition, we will assume throughout that

$$\det A(z) \not\equiv 0. \tag{3.2}$$

Let us denote by $\mathscr{A}_N^+$ the class of matrices in $\mathscr{A}_N$ satisfying (3.1) and (3.2). Recall that each entry of $A(z)$ is itself a symbol of some element of $\mathscr{A}_1$. As already remarked, from this it follows immediately that $A(z)$ converges for $A \in \mathscr{A}_N^+$ in some deleted neighborhood of the origin.

Let us describe next how to remove ragged edges of an upper triangular $A \in \mathscr{A}_N$.

LEMMA 3.1. *Let* $A \in \mathscr{A}_N$, det $A(z) \not\equiv 0$, *and assume* $A_{ij} = 0$ *for all* $j < i - m$, *but for some* i_0, $A_{i_0, i_0-m} = 0$. *Then there exists a block diagonal one-banded lower triangular matrix* $M \in \mathscr{A}_N$ *with* $M_{ii} = 1$, $i \in \mathbf{Z}$, *satisfying*

$$A = MB$$

where $B = (B_{ij}) \in \mathscr{A}_N$ *and* $B_{i,i-m} = 0$, *for all* $i \in \mathbf{Z}$.

Thus we see that multiplication by M^{-1} on the left of A removes one band of A. To prove this result we borrow from the ideas of de Boor and Pinkus, [4], which are based on Gaussian elimination. We first make a few simplifying assumptions. We may assume that $m = i_0 = 0$ while $A_{11} \neq 0$ (since if $A_{ii} = 0$ for all i we could choose M to be the identity). Note that $A_{01} \neq 0$ since otherwise we would have $A_{0i} = 0$ for all i contradicting (3.2). It follows that we may "eliminate" the (1, 1) entry using the (0, 1) entry. In matrix notation, setting $t_1 = A_{11}/A_{01}$,

$$(T^1)_{ij} = \begin{cases} 1, & i = j \\ -t_1, & (i, j) = (1, 0), \\ 0, & \text{otherwise}, \end{cases}$$

and

$$A^1 = T^1 A$$

we have

$$(A^1)_{11} = 0$$

where the remaining band structure of A^1 is still the same as that of A. If now $(A^1)_{22} \neq 0$ (otherwise we proceed to the next non-zero element on the respective diagonal) we have again $(A^1)_{1,2} \neq 0$ since Lemma B of [4] ensures that the semiinfinite matrix $(A^1)_{i,j\geq 0}$ is still totally positive so that $(A^1)_{1,2} = 0$ would imply, using Lemma 2.1, that row zero and row one of A were linearly dependent again contradicting (3.2). Thus we may repeat this process to successively eliminate the non-zero diagonal entries or, in matrix form, we have

$$A^{(k)} = T^{(k)} \cdots T^{(1)} A$$

where for $t_\ell = A^{(\ell-1)}_{\ell\ell}/A^{(\ell-1)}_{\ell-1,\ell}$ the $T^{(\ell)}$ are defined recursively as

$$(T^{(\ell)})_{i,j} = \begin{cases} 1, & i = j \\ -t_\ell, & (i, j) = (\ell, \ell - 1) \\ 0, & \text{otherwise}, \end{cases}$$

unless $A_{\ell\ell} = 0$ in which case we set $T^{(\ell)} = I$, the identity matrix. Note that $A_{\ell\ell}^{(\ell-1)} = A_{\ell\ell}$. The construction assures that each $A^{(k)}$ is totally positive on the index set $i, j \geqq 0$. Let us denote the first principal $N \times N$ submatrix of $T^{(N-1)} \cdots T^{(1)}$ with respect to the index set $i, j \geqq 0$ by S. Since $A = A^{(0)}$ is in $\mathscr{A}_N$ we know that $A_{pN,pN} = 0$ for all $p \in \mathbf{Z}$. Thus we conclude that $T^{(pN)} = I$ for $p = 0, 1, \ldots$. Therefore if T is the block diagonal matrix with S on the diagonal we have that the matrix

$$B = TA$$

belongs to $\mathscr{A}_N$. Furthermore, since T^{-1} is a one-banded totally positive block Toeplitz matrix we conclude that the assertion holds with $M = T^{-1}$.

3.1 *Factorization of banded matrices.* We now show that a banded matrix in $\mathscr{A}_N^+$ may be represented as a product of one-banded factors in $\mathscr{A}_N^+$. This result is an improvement over the theorems in [**4**, **6**] in the following sense. Both papers deal with factorization of arbitrary totally positive banded matrices. However, while the procedure in [**6**] applied to a block Toeplitz matrix does produce block Toeplitz factors it requires strict bandedness, i.e., whenever a band contains any non-zero entry all its entries are non-zero. On the other hand, the factorization procedure in [**4**] does not require strict bandedness. But it does not guarantee that for block Toeplitz matrices the factors are also block Toeplitz.

THEOREM 3.1. *Suppose $A \in \mathscr{A}_N^+$ is m-banded. Then*

$$A = R_1 \cdots R_m$$

where each R_i is in $\mathscr{A}_N^+$ and is one-banded.

PROOF. If A is strictly m-banded then Theorems 1 and 7 in [**6**] immediately yield the result. Let us assume that $\{A_{i,i-m}\}_{i\in\mathbf{Z}}$ is the lower-most nontrivial band of A. If this band is not strict then applying the procedure in Lemma 3.1 yields

$$A = M^{(1)}A^{(1)}$$

with $M^{(1)} \in \mathscr{A}_N$, one-banded and $A^{(1)} \in \mathscr{A}_N$ is $(m - 1)$-banded. If $A^{(1)}$ has a strict lower band we stop; otherwise continue the procedure until a strict lower band is encountered, which indeed must happen since det $A(z) \not\equiv 0$. This yields

$$A = M^{(1)} \cdots M^{(\ell)}A^{(\ell)}$$

with $M^{(j)} \in \mathscr{A}_N$, one-banded, and det $A(z) =$ det $A^{(\ell)}(z)$, and $A^{(\ell)}$ is $(m - \ell)$-banded. If $\ell = m - 1$ we are done. If $A^{(\ell)}$ is strictly banded then the results in [**6**] complete the proof. If the upper-most non-trivial band of $A^{(\ell)}$ is not strict then we apply Lemma 3.1 to $(A^{(\ell)})^T$ and transposing back we obtain

$$A^{(\ell)} = A^{(\ell+u)}\Gamma^{(u)} \cdots \Gamma^{(1)}$$

where finally $A^{(\ell+u)}$ is strictly $m - (\ell + u)$ banded and the $\Gamma^{(i)} \in \mathscr{A}_N$ are one-banded. Thus another appeal to the results in **[6]** completes the proof.

Note that $\det A(z) = \det A^{(\ell+u)}(z)$. Hence referring to **[14]** we immediately know that the dimension of the null space of A is $m - \ell - u$. As an immediate corollary to this Theorem 3.1 we state

COROLLARY 3.1. *Let $A \in \mathscr{A}_N^+$ be m-banded. Then there are integers ℓ and u as before with $(\ell + u) \leqq m$ and $A(z) = M^{(1)}(z) \cdots M^{(\ell)}(z)\, R_1(z) \cdots R_{m-(\ell+u)}(z)\, \Gamma^{(u)}(z) \cdots \Gamma^{(1)}(z)$ where* $\det M^{(i)}(z) = \det \Gamma^{(j)}(z) = 1$, *$i = 1, \ldots, \ell, j = 1, \ldots, u$, so that*

$$\det A(z) = \prod_{i=1}^{m-(\ell+u)} \det R_i(z),$$

where

$$R_i(z) = \begin{bmatrix} 1 & & & & r_1^i z \\ r_2^i & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ 0 & & & r_N^i & 1 \end{bmatrix} \tag{3.3}$$

and $M^{(i)}$ and $(\Gamma^{(i)})^T$ are as in Lemma 3.1.

We note further that

$$\det A(z) = \prod_{j=1}^{m-(\ell+u)} (1 - (-1)^N \gamma_j z) \tag{3.4}$$

where $\gamma_j = r_1^j \cdots r_N^j$. Finally we remark that we have achieved a formal LU-decomposition of A by setting $L = M^{(1)} \cdots M^{(\ell)}\, R_1 \cdots R_{m-(\ell+u)}$ and $U = \Gamma^{(u)} \cdots \Gamma^{(1)}$. We hasten to add, however, that it is quite possible that L is not boundedly invertible even when A is.

As for non-banded matrices in $\mathscr{A}_N^+$ we first observe that the preceding discussion shows that for $A \in \mathscr{A}_N^+$ we may remove the ragged upper edge of A by applying the one-banded factorization techniques of the previous section yielding $A = LB$ where B is a finite product of one-banded factors and $L \in \mathscr{A}_N^+$ has been normalized so that $L_{ii} = 1$, $i \in \mathbf{Z}$ and $L_{ij} = 0$ if $i < j$, i.e. L is unit lower triangular. Again $\det A(z) \not\equiv 0$ assures that only a finite number of eliminations is needed to remove the ragged upper edge since otherwise $\det A(z)$ would have a zero of infinite order. We record these observations in

THEOREM 3.2. *Let $A \in \mathscr{A}_N^+$ and define $I_r = (\delta_{i,j-r})_{i,j \in \mathbf{Z}} \in \mathscr{A}_N^+$. Then there exists some $r \in \mathbf{Z}$ such that*

(3.5) $$A = I_r LU$$

with $L_{ii} = 1$, $i \in \mathbf{Z}$, $L \in \mathscr{A}_N^+$ *unit lower triangular and*

$$U = DR_1 \cdots R_\ell$$

where each $R_i \in \mathscr{A}_N^+$ *is a one-banded unit upper triangular block diagonal matrix and* $D \in \mathscr{A}_N^+$ *is a diagonal matrix.*

As pointed out in [6], L has a unique lower triangular inverse which we denote by L^{-1}. Furthermore, the matrix $|L^{-1}|$ which is formed by taking the absolute values of the entries of L^{-1}, i.e.,

$$|L^{-1}|_{ij} = |(L^{-1})_{ij}|$$

is in $\mathscr{A}_N^+$.

In order to continue our factorization procedure we require the following lemma whose proof follows from Lemma 2.1 and the reasoning in [6] for a similar case.

LEMMA 3.2. *Suppose* $A \in \mathscr{A}_N^+$ *is not banded. Then the limits*

(3.6) $$\beta_k(A) = \lim_{j \to -\infty} \frac{A_{i,jN+k}}{A_{i,jN+k+1}}$$

exist and do not depend on $i = 1, \ldots, N$. *In fact, we also have for each integer* j

(3.7) $$\beta_k(A) = \lim_{i \to \infty} \frac{A_{i,jN+k}}{A_{i,jN+k+1}},$$

and consequently

(3.8) $$\gamma(A) = \lim_{j \to -\infty} \frac{A_{i,jN+k}}{A_{i,(j+1)N+k}} = \beta_1(A) \cdots \beta_N(A)$$

is independent of i *and* k.

We remark that, when A is unit lower triangular, by Pringsheim's theorem $1/\gamma(A)$ is the first common singularity of each entry of $A(z)$,

$$A_{ik}(z) = \sum_{j=0}^{\infty} A_{i,-jN+k} z^j.$$

Since each $A_{ik}(z)$ is a symbol of an element of $\mathscr{A}_1^+$, (1.5) says that this singularity is a pole.

LEMMA 3.3. *Suppose* $A \in \mathscr{A}_N^+$ *is not banded and is unit lower triangular. Then*

$$A^{(1)} = A(I - B^1) \in \mathscr{A}_N^+$$

where

$$B^1(z) = \begin{bmatrix} 0 & & & & \beta_1(A)z \\ \beta_2(A) & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot\ \cdot & \\ 0 & & & \beta_N(A) & 0 \end{bmatrix}$$

Proof. For arbitrary ℓ, positive m and large positive q consider the section

$$\begin{bmatrix} A_{1,\ell N} & \cdot & \cdot & A_{1,(\ell+m)N} \\ \vdots & & & \vdots \\ A_{N,\ell N} & \cdot & \cdot & A_{N,(\ell+m)N} \\ \vdots & & & \vdots \\ A_{mN,\ell N} & \cdot & \cdot & A_{mN,(\ell+m)N} \\ \vdots & & & \vdots \\ A_{q,\ell N} & \cdot & \cdot & A_{q,(\ell+m)N} \end{bmatrix}$$

Now subtracting $A_{q,\ell N}/A_{q,\ell N+1}$ times the second column from the first one and then subtracting $A_{q,\ell N+1}/A_{q,\ell N+2}$ times the third column from the second and so on yields as the last row

$$(0, 0, \ldots, 0, A_{q,(\ell+m)N}).$$

This operation is known to preserve total positivity for each q, (c.f. **[15]**). Thus sending q to infinity and noting the freedom in ℓ and m one obtains the result.

We are now in position to state our main result of this section.

Theorem 3.3. *The symbol of each $A \in \mathscr{A}_N^+$ has the form*

$$A(z) = I_r(z) \prod_{j=1}^{\infty} (I + A^j(z)) C(z) \Big(\prod_{j=1}^{\infty} (I - B^j(z))^{-1} D \prod_{j=1}^{\ell} (I + U^j(z)) \tag{3.9}$$

with the following properties.

i) *$A^j(z)$, $B^j(z)$ and $U^j(z)$ are $N \times N$ matrices of the form*

$$A^j(z) = \begin{bmatrix} 0 & & & & za_1^j \\ a_j^2 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot\ \cdot & \\ 0 & & & a_N^j & 0 \end{bmatrix} \qquad B^j(z) = \begin{bmatrix} 0 & & & & zb_1^j \\ b_2^j & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot\ \cdot & \\ 0 & & & b_N^j & 0 \end{bmatrix} \tag{3.10}$$

$$U^j(z) = \begin{bmatrix} 0 & u_1^j & & & 0 \\ & 0 & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & 0 & u_{N-1}^j \\ u_N^j/z & & & 0 & \end{bmatrix}$$

where the a_i^j, b_i^j, u_i^j *are all non-negative and*

$$\sum_{i=1}^{N} \sum_{j=1}^{\infty} (a_i^j + b_i^j) < \infty.$$

ii) ℓ *is finite and* $\prod_{i=1}^{N} u_i^j = 0$, $j = 1, \ldots, \ell$, *so that*

$$\det(I + U^j(z)) = 1.$$

iii) D *is a positive constant diagonal matrix.*

iv) $C(z)$ *is the symbol of a matrix* C *in* $\mathscr{A}_N^+$ *such that both* $C(z)$ *and* $C^{-1}(z)$ *are entire,*

$$\det C(z) = 1$$

and

$$\prod_{k=1}^{N} \beta_k(C) = \prod_{k=1}^{N} \beta_k(C^{-1}) = 0.$$

As a corollary we obtain

COROLLARY 3.2. *Let* $A \in \mathscr{A}_N^+$ *be unit lower triangular. Then*

$$\det A(z) = \frac{\prod_1^{\infty} (1 - (-1)^N \gamma_j z)}{\prod_1^{\infty} (1 - \delta_j z)}$$

where γ_j, $\delta_j \geqq 0$, $j = 1, 2, \ldots$ *and* $\sum_{j=1}^{\infty} (\gamma_j^{1/N} + \delta_j^{1/N}) < \infty$.

We wish to prove Theorem 3.3 in several steps. Notice first that if A is banded the result follows from Theorem 3.1 and Corollary 3.1. If A is not banded then, according to Theorem 3.2, $A = I_r LU$ with L unit lower triangular in $\mathscr{A}_N^+$ and the symbol of U has the required structure stated in ii). Thus, without loss of generality, let us assume that $A \in \mathscr{A}_N^+$ is unit lower triangular. The first step in the factorization is to go after the poles.

LEMMA 3.4. *Let* $A \in \mathscr{A}_N^+$ *be unit lower triangular. Then*

$$A(z) = G(z)\Big(\prod_{j=1}^{\infty} (I - B^j(z))^{-1}$$

where $G \in \mathscr{A}_N^+$ is unit lower triangular with an entire symbol and

$$B^j(z) = \begin{bmatrix} 0 & & & & zb_1^j \\ b_2^j & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & b_N^j & 0 \end{bmatrix} \qquad b_i^j \geqq 0, \ \sum_{i=1}^{N} \sum_{j=1}^{\infty} b_i^j < \infty.$$

PROOF. Let $A = A^{(0)}$ and $b_k^1 = \beta_k(A^0)$, $k = 1, \ldots, N$, with the $\beta_k(A)$ as in (3.6). If $b_k^1 = 0$ for some k then the result follows from the remarks after Lemma 3.2 by choosing $G(z) = A(z)$ and $b_j(z) = 0$. If all the $b_k^1 \neq 0$ then set

$$B^1(z) = \begin{bmatrix} 0 & & & & zb_1^1 \\ b_2^1 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & b_N^1 & 0 \end{bmatrix}$$

and conclude from Lemma 3.3 that

$$A^{(1)}(z) = A^{(0)}(z)\,(I - B^1(z))$$

is the symbol of a unit lower triangular matrix in $\mathscr{A}_N^+$. If $A^{(1)}$ is banded we are finished in view of Theorem 3.1. Otherwise, we may repeat the above process thus generating a sequence

$$A^{(n)}(z) = A^{(0)}(z) \prod_{j=1}^{n} (I - B^j(z))$$

where

$$b_k^j = \beta_k(A^{(j-1)}),\ k = 1, \ldots, N,$$

and $A^{(n)} \in \mathscr{A}_N^+$ is unit lower triangular. Again if for some n, $\gamma_n := b_1^n \cdots b_N^n = 0$ we are finished. If this never happens we must be sure that the infinite product

$$\prod_{j=1}^{\infty} (I - B_j(z))$$

converges. To this end, note that

$$(A^{(n)})_{i,i-1} = A_{i,i-1}^{(0)} - (b_i^1 + \cdots + b_i^n) \geqq 0$$

for $i = 1, \ldots, k$, and so

(3.11) $$\sum_{j=1}^{\infty} \sum_{k=1}^{N} b_k^j < \infty.$$

Defining $\|A\|_\infty = \sup_i \sum_j |A_{ij}|$ we have for $|z| \leqq \rho$

$$\| \prod_{j=1}^{n} (I - B^j(z)) \|_\infty \leqq \prod_{j=1}^{n} \| I - B^j(z) \|_\infty \leqq \exp \{ \sum_{j=1}^{n} \|B^j(\rho)\|_\infty \}.$$

Hence by (3.11) we immediately conclude uniform convergence of the infinite products on all compact sets of the complex plane. It follows that $\gamma_n = b_1^n \cdots b_N^n \to 0$ as $n \to \infty$. Since $A^{(n)}(z)$ is analytic for $|z| \leqq 1/\gamma_n$ and $\gamma_n \to 0$, $n \to \infty$, we see that

$$G(z) = A^0(z) \prod_{j=1}^{\infty} (I - B_j(z))$$

is entire. This proves Lemma 3.4.

We remark that it was pointed out in **[6]** that $\gamma_1 \geqq \gamma_2 \geqq \cdots$. This also follows from the fact that γ_i is the first singularity of $A^{(i-1)}(z)$.

Now let G^{-1} be the (unique) unit lower triangular inverse of G. According to an observation made in **[6]** we know that $|G^{-1}| = DG^{-1}D \in \mathscr{A}_N^+$ where $D_{ij} = (-1)^i \delta_{ij}$. Thus we may apply Lemma 3.4 to $|G^{-1}|$ yielding

(3.12) $$|G^{-1}|(z) = L(z) \left(\prod_{j=1}^{\infty} (I - A^j(z)) \right)^{-1}$$

where $L(z)$ is entire, $A^j(z)$ is given by (3.10), $a_i^j \geqq 0$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{N} a_j^i < \infty$. Note that

(3.13) $$\tilde{D}|G^{-1}|((-1)^N z)\tilde{D} = G^{-1}(z) = (G(z))^{-1}$$

where $\tilde{D} = ((-1)^i \delta_{i,j})_{i,j=0}^{N-1}$. Hence

$$(G((-1)^N z))^{-1} = \tilde{D}L(z)\left(\prod_{j=1}^{\infty} (I - A^j(z))\right)^{-1}\tilde{D}$$

which gives

$$G((-1)^N z) = \tilde{D} \prod_{j=1}^{\infty} (I - A^j(z))\tilde{D}\tilde{D}L^{-1}(z)\tilde{D},$$

and finally

$$G(z) = \prod_{j=1}^{\infty} (\tilde{D}(I - A^j((-1)^N z))\tilde{D})|L^{-1}|(z).$$

This simplifies to

(3.14) $$G(z) = \prod_{j=1}^{\infty} (I + A^j(z))|L^{-1}|(z).$$

Thus setting $C(z) = |L^{-1}|(z)$ we see that,

$$A(z) = \prod_{j=1}^{\infty} (I + A^j(z))C(z)(\prod_{j=1}^{\infty} (I - B^j(z)))^{-1} \tag{3.15}$$

and all assertions of Theorem 3.3 have been verified except for the requirements on $C(z)$. However, since $C(z)$ is entire by Lemma 3.4, det $G^{-1}(z)$ has no zeroes and thus by (3.13) det$|G^{-1}|$ (z) also has no zeroes. But (3.12) yields

$$\det G^{-1}(z) = \frac{\det L(z)}{\prod_{j=1}^{\infty} (1 - a_1^j \cdots a_N^j z)},$$

from which we conclude that det $L(z)$ has no zeroes. Since we also know that $L(z)$ is entire it follows that both $C(z)$ and $C^{-1}(z)$ are entire.

In order to complete the proof of Theorem 3.3, it remains to prove that det $C(z) = 1$.

THEOREM 3.4. *Let $A \in \mathscr{A}_N^+$ be unit lower triangular with $A(z)$ entire and* det $A(z)$ *non-vanishing. Then* det $A(z) = 1$.

This theorem is a consequence of the following fact which is of interest in its own right.

THEOREM 3.5. *Let $N > 1$ and $A \in \mathscr{A}_N^+$ be unit lower triangular. Then each entry of $A(z)$ must be of the form* ((1.5) i) *with $\alpha = 0$, i.e.,*

$$\gamma z^k \prod_{j=1}^{\infty} \left(\frac{1 + \alpha_j z}{1 - \beta_j z} \right)$$

with $\sum_{j=1}^{\infty} (\alpha_j + \beta_j) < \infty$ and $\gamma, \alpha_j, \beta_j \geqq 0$, $k \in \mathbf{Z}$.

PROOF. Since any 2×2 submatrix of a symbol $A(z)$, $A \in \mathscr{A}_N$ is the symbol of some $\tilde{A} \in \mathscr{A}_2$ it is sufficient to prove the assertion for $N = 2$. Let

$$(A(z) = \begin{bmatrix} A_{11}(z) & A_{12}(z) \\ A_{21}(z) & A_{22}(z) \end{bmatrix}$$

and hence

$$(A(z))^{-1} = A^{-1}(z) = \begin{bmatrix} A_{22}(z) & -A_{12}(z) \\ -A_{21}(z) & A_{11}(z) \end{bmatrix} / \det A(z)$$
$$= \begin{bmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{bmatrix}.$$

Since from (3.13) we have

$$A^{-1}(z) = \tilde{D}|A^{-1}|(z)\tilde{D}$$

we conclude that $B_{11}(z)$ is the symbol of an element of $\mathscr{A}_1$ which gives

(3.16) $$B_{11}(z) = e^{\gamma_1 z} \prod_{j=1}^{\infty} \left(\frac{1 + \alpha_j z}{1 - \beta_j z} \right) = \frac{A_{22}(z)}{\det A(z)}.$$

On the other hand, by (3.15) and the fact that $A_{22}(z)$ is also a symbol of an element of $\mathscr{A}_1$ we get

$$B_{11}(z) = e^{\gamma_2 z} \left(\prod_{j=1}^{\infty} \frac{1 + \gamma_j^{(1)} z}{1 - \gamma_j^{(2)} z} \right) / \det C(z)$$

where $C(z)$ and $C^{-1}(z)$ are both entire. Consequently, we have for some $\gamma \in \mathbf{R}$

(3.17) $$\det C(z) = e^{(\gamma_2 - \gamma_1) z} = e^{\gamma z}.$$

Next, we show that $\gamma \geqq 0$. In view of (3.15) we have

$$\det A(z) = e^{\gamma z} \prod_{j=1}^{\infty} \left(\frac{1 + \delta_j^{(1)} z}{1 - \delta_j^{(2)} z} \right) = A_{11}(z)\, A_{22}(z) - A_{12}(z)\, A_{21}(z).$$

Again since the $A_{ij}(z)$ are symbols of elements of $\mathscr{A}_1$ we obtain

$$\det A(z) = e^{\alpha z} \prod_{j=1}^{\infty} \left(\frac{1 + \beta_j^{(1)} z}{1 - \beta_j^{(2)} z} \right) - e^{\beta z} \prod_{j=1}^{\infty} \left(\frac{1 + \alpha_j^{(1)} z}{1 - \alpha_j^{(2)} z} \right)$$

where α, β, $\alpha_j^{(\ell)}$, $\beta_j^{(\ell)}$, $\delta_j^{(\ell)} \geqq 0$, $\ell = 1, 2$, $j = 1, 2, \ldots$ and $\sum_{j=1}^{\infty} \sum_{\ell=1}^{2} (\alpha_j^{(\ell)} + \beta_j^{(\ell)} + \delta_j^{(\ell)}) < \infty$. Note that the conclusion of the theorem follows as soon as we show $\alpha = \beta = 0$.

On clearing the denominators we obtain

(3.18) $$e^{\gamma z} \prod_{j=1}^{\infty} (1 + g_j z) = e^{\alpha z} \prod_{j=1}^{\infty} (1 + a_j z) - e^{\beta z} \prod_{j=1}^{\infty} (1 + b_j z)$$

where a_j, b_j, g_j are real numbers satisfying

$$\sum_{j=1}^{\infty} (|a_j| + |b_j| + |g_j|) < \infty.$$

Rewriting (3.18) as

$$e^{(\gamma/2) z} \prod_{j=1}^{\infty} (1 + g_j z) = e^{(\alpha - \gamma/2) z} \prod_{j=1}^{\infty} (1 + a_j z) - e^{(\beta - \gamma/2) z} \prod_{j=1}^{\infty} (1 + b_j z)$$

we see that if $\gamma < 0$ the right hand side of this equation goes to zero as $z \to -\infty$, since each product is of exponential type zero. On the other hand, for the same reason, the left hand side is not bounded. Thus we conclude that $\gamma \geqq 0$.

Now since by (3.13) we get for $N = 2$

$$\det C^{-1}(z) = \det |C^{-1}|(z), \; |C^{-1}| \in \mathscr{A}_2^+,$$

we can apply the same argument to $|A^{-1}|$ and conclude $-\gamma \geqq 0$. Therefore $\gamma = 0$ and we obtain

$$\det C(z) = \det C^{-1}(z) = 1.$$

Now equation (3.18) reads

$$\prod_{j=1}^{\infty} (1 + g_j z) = e^{\alpha z} \prod_{j=1}^{\infty} (1 + a_j z) - e^{\beta z} \prod_{j=1}^{\infty} (1 + b_j z).$$

Therefore the same argument used above shows that $\alpha = \beta = 0$. This completes the proof of Theorem 3.5.

We return to the proof of Theorem 3.4. We now know that each entry of $A(z)$ is of the form

$$A_{ij}(z) = \gamma z^k \prod_{\ell=1}^{\infty} (1 + \alpha_\ell z)$$

where γ, k, α_ℓ depend on (i, j). Since $\det A(z)$ is a linear combination of products of this type it is of exponential type zero. But $\det A(z)$ has by assumption no zeroes and no poles and so it is a constant. This completes the proof of Theorem 3.4 because $\det A(z) = \det A(0) = 1$.

As an example satisfying the hypotheses of Theorem 3.4 consider the matrix

$$A = \lim_{n \to \infty} (I + B/n)^n$$

where

$$B_{ij} = \begin{cases} 0, & i - j \neq 1, \\ b_i, & i - j = 1, \\ b_{i+N} = b_i, & i \in \mathbf{Z}. \end{cases}$$

Then

$$A(z) = \exp(B(z)) \tag{3.19}$$

and

$$B(z) = \begin{bmatrix} 0 & & & & zb_1 \\ b_2 & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & b_N & 0 \end{bmatrix}.$$

Since $\det A(z) = \exp(tr\, B(z))$ it follows that $\det A(z) = 1$.

Let us point out that there exist matrices $A \in \mathscr{A}_N$, $N > 1$, such that both $A(z)$ and $A^{-1}(z)$ are entire but A is not of the form (3.19). As an example let $N = 2$ and define for any $a, b, c, d \geqq 0$

(3.20) $$A(z) = \exp\begin{pmatrix} 0 & az \\ b & 0 \end{pmatrix} \exp\begin{pmatrix} 0 & cz \\ d & 0 \end{pmatrix}.$$

Clearly $A \in \mathscr{A}_2^+$ but it is easy to see that it is not of the form (3.19).

In fact, the fifth section is devoted to the discussion of a subclass of $\mathscr{A}_N^+$ which strictly contains the class of matrices of the form (3.19) or (3.20) but still satisfies the hypothesis of Theorem 3.4.

4. The Two-Sided Case. Suppose now that $A \in \mathscr{A}_N$ but neither A nor A^T is in $\mathscr{A}_N^+$. We know from Theorem 2.1 that $A(z)$ is either the trivial function

$$\sum_{-\infty}^{\infty} \rho^j z^j A_0$$

where A_0 is as in Theorem 2.1, or $A(z)$ converges in a non-trivial annulus. In the latter case, as discussed in section 2, we may rescale $A(z)$ to $A(\rho z) = A_\rho(z)$ so that $A_\rho \in \mathscr{A}_N$ and det $A_\rho(z) \neq 0$ for $|z| = 1$. Thus A_ρ is bounded and boundedly invertible on ℓ^∞ (**Z**) and we may factor A_ρ for some r as

$$A_\rho = I_r \, LDU$$

withe $L \in \mathscr{A}_N^+$ unit lower triangular $U^T \in \mathscr{A}_N^+$ is unit lower triangular and $D \in \mathscr{A}_N^+$ is diagonal. We recall that for any block Toeplitz matrix A

$$A^T(z) = [A(z^{-1})]^T.$$

Thus we may state

THEOREM 4.1. *Let $A \in \mathscr{A}_N$ be neither lower nor upper triangular with respect to any diagonal. If* det $A(z)$ *converges in some annulus and is not identically zero then*

$$A(z) = I_r(z) \, D_1 \, L(z) \, D_2 \, U(z)$$

where $L(z)$ and $U^T(z)$ can be factored as in Theorem 3.3 *and D_1, D_2 are diagonal matrices in $\mathscr{A}_N$.*

We obtain two corollaries from this theorem.

COROLLARY 4.1. *Let $A \in \mathscr{A}_N$, $N \geqq 2$, with* det $A(z)$ *not identically zero and $A(z)$ convergent in some annulus. Then*

$$\det A(z) = \gamma z^k \prod_{j=1}^{\infty} \frac{(1 - (-1)^N \gamma_j z)(1 - (-1)^N \alpha_j/z)}{(1 - \delta_j z)(1 - \beta_j/z)}$$

where all constants are non-negative and

$$\sum_{j=1}^{\infty} (\alpha_j^{1/N} + \beta_j^{1/N} + \gamma_j^{1/N} + \delta_j^{1/N}) < \infty.$$

Recall that for any block Toeplitz matrix A, A is a bounded mapping from $\ell^\infty(\mathbf{Z})$ to $\ell^\infty(\mathbf{Z})$ implies $A(z)$ is defined for $|z| = 1$ and A is boundedly invertible if and only if $\det(A(z)) \neq 0$ on $|z| = 1$. Thus we conclude from Corollary 4.1.

COROLLARY 4.2. *Let $A \in \mathscr{A}_N$ be a bounded mapping on $\ell^\infty(\mathbf{Z})$ then A is boundedly invertible if and only if*

$$\det A((-1))^N \neq 0.$$

The proof is immediate since the only possible zeroes on $|z| = 1$ are at $(-1)^N$ as can be seen from Corollary 4.1.

5. Symbols accessible by flows. Consider the biinfinite system of ordinary differential equations,

$$(5.1) \qquad \begin{aligned} Y'(t) &= Y(t)\,\Lambda(t) \\ Y(t_0) &= I \end{aligned}$$

where

$$(5.2) \qquad \Lambda_{ij}(t) = \delta_{i,j+1}\, r_i(t),$$

and $r_i \in L^\infty[0, 1]$. The solution to (5.1) will be denoted by $Y(t) = Y_\Lambda(t) = Y(t_0; t)$. As we will see, for all $t \geqq t_0$, $Y(t_0; t)$ is totally positive. Therefore, if in addition $r_i = r_{i+N}$, $i \in \mathbf{Z}$ and $t \geqq t_0$ it follows $Y(t_0; t) \in \mathscr{A}_N^+$. We now make the following conjecture.

CONJECTURE 1. *Let A be in $\mathscr{A}_N$ such that $A(z)$ is entire and* $\det A(z) = 1$. *Then there is a Λ of the form* (5.2) *with $r_i = r_{i+N}$ so that if Y is the solution to* (5.1) *then*

$$A = Y(0; 1).$$

The remainder of this section explores the supporting evidence for this conjecture. We begin with the requisite background material.

Let $\mathscr{M}$ be the Banach space of all biinfinite matrices which map $\ell^\infty(\mathbf{Z})$ into inself. The matrix Γ is in $\mathscr{M}$ if and only if

$$(5.3) \qquad \|\Gamma\|_\infty = \sup_i \sum_{j=-\infty}^{\infty} |\Gamma_{ij}| < \infty$$

Let $C_{\mathscr{M}}[0, 1]$ be the continuous $\mathscr{M}$-valued functions on $[0, 1]$ with the supremum norm. $C_{\mathscr{M}}[0, 1]$ is a Banach algebra with identity $e = (\delta_{ij})$ and norm

$$(5.4) \qquad |\Gamma(\cdot)| = \operatorname*{ess\,sup}_{0 \leq t \leq 1} \|\Gamma(t)\|_\infty$$

and so $C_{\mathscr{M}}[0, 1]$ is isometrically embedded in $L^\infty_{\mathscr{M}}[0, 1]$.

For reasons that will soon be apparent we wish to study the Volterra integral equation on $C_{\mathscr{M}}[0, 1]$

(5.5) $$Y(t) = I + \int_{t_0}^{t} Y(\sigma)\, \Gamma(\sigma)\, d\sigma.$$

Setting

(5.6) $$(TY)(t) = \int_{t_0}^{t} Y(\sigma)\, \Gamma(\sigma)\, d\sigma,$$

then clearly $\|T\| \leqq |\Gamma|$ and we also have $\lim_{n\to\infty}\|T^n\|^{1/n} = 0$. Consequently for $\Gamma \in L^{\infty}_{\mathscr{M}}[0, 1]$ the Neumann series $I + TI + \cdots + T^n I + \cdots$ converges to the unique solution of (5.5). Moreover, $Y(t)$ is Lipschitz continuous and so satisfies (5.1) *a.e.* for $\Gamma = \Lambda$. These standard remarks guarantee us that (5.1) has a unique solution. Moreover, since

(5.7) $$T^n I = \int_{t_0}^{t}\int_{t_0}^{t_1} \cdots \int_{t_0}^{t_{n-1}} \Gamma(t_n) \cdots \Gamma(t_1)\, dt_n \cdots dt_1$$

we see that $Y(t)$ is lower triangular whenever $\Gamma(t)$ is lower triangular. In particular, for the choice (5.2) $Y(t)$ is lower triangular. Thus the equation

(5.8) $$\begin{aligned} Y'(t) &= Y(t)\,\Gamma(t),\ t \geqq t_0, \\ T(t_0) &= I, \end{aligned}$$

for Γ lower triangular generates a flow in the space of lower triangular matrices. When will $Y(t)$ also be totally positive for $t \geqq t_0$ and any t_0? If it were then the off diagonal entires of Γ are nonnegative because for $i \neq j$

(5.9) $$\begin{aligned} Y_{ij}(t) &= \int_{t_0}^{t} (Y(\sigma)\,\Gamma(\sigma))_{ij}\, d\sigma \\ &= \int_{t_0}^{t} \Gamma_{ij}(\sigma) d\sigma + 0(t - t_0)^2 \\ &= (t - t_0)\,\Gamma_{ij}(t_0) + 0(t - t_0)^2. \end{aligned}$$

Next observe that $\Gamma(t)$ must be tridiagonal for all t. We will content ourselves by showing only that $\Gamma_{i,i-2}(t) = 0$ for all i. A similar argument gives $\Gamma_{i,i+k}(t) = 0$ for $|k| > 1$. Since $Y(t)$ is totally positive we have

$$0 \leqq Y(t)\begin{pmatrix} i, & i+1 \\ i-1, & i \end{pmatrix} = -\int_{t_0}^{t} \Gamma_{i+1,i-1}(\sigma) d\sigma + 0(t - t_0)^2$$

which we derive by making use of (5.9). Therefore $\Gamma_{i+1,i-1}(\sigma) = 0$ since we have already shown it is nonnegative. Hence when Γ is lower triangular we may express it as $\Gamma(t) = D(t) + \Lambda(t)$ where $\Lambda(t)$ has the form (5.2). From this equation we can obtain

$$Y_{\Gamma}(t) = Y_{\tilde{\Lambda}}(t)\tilde{D}(t)$$

where $\tilde{D}$ is the diagonal matrix

$$\bar{D}(t) = \exp \int_{t_0}^{t} D(\sigma)d\sigma$$

and $\tilde{\Lambda} = \tilde{D}\Lambda$. Thus, modulo a diagonal scaling, we see that the only $\Gamma(t)$ which may yield a flow in lower triangular totally positive matrices has the form (5.2).

Let us now explicitly compute the entries $Y_{ij}(t)$ of the solution (5.1) when $t_0 = 0$. Since Y is lower triangular and

$$Y'_{ij}(t) = Y_{i,j+1}(t)r_{j+1}(t) \tag{5.10}$$

for all $i \geqq j$ we get $Y_{ii}(t) = 1$. Therefore it follows that

$$Y_{i,i-1}(t) = \int_0^t r_i(\tau)d\tau \tag{5.11}$$

and in general for $i > j$ one has

$$Y_{ij}(t) = \int_0^t r_{j+1}(t_1) \int_0^{t_1} r_{j+2}(t_2) \cdots \int_0^{t_{i-j-1}} r_i(t_{i-j})dt_{i-j} \cdots dt_1. \tag{5.12}$$

We also write this as

$$Y_{ij}(t) = \int_{\Sigma_{i-j} \leqq t} r_{j+1} \cdots r_i$$

where $\sum_{i-j} \leqq t$ represents the simplex $\{(t_1, \ldots, t_{i-j}): 0 \leqq t_{i-j} < \cdots < t_1 \leqq t\}$. Note that

$$|Y_{ij}(t)| \leqq \frac{|\Lambda|^{i-j}}{(i-j)!}, \; 0 \leqq t \leqq 1, \; i > j. \tag{5.13}$$

Similarly, if $Y^{-1}(t)$ denotes the lower triangular inverse of $Y(t)$ then for

$$W(t) = DY^{-1}(t)D, \; D = ((-1)^i \delta_{ij}) \tag{5.14}$$

we have

$$W'(t) = \Lambda(t)W(t),$$
$$W(0) = I.$$

Therefore, we obtain as before

$$W_{ij}(t) = \int_{\Sigma_{i-j} \leqq t} r_i \cdots r_{j+1}. \tag{5.15}$$

We will now show that $Y(t)$ is totally positive for each fixed $t > 0$. This could be directly verified using (5.12) but in our context it is appropriate to use a modified Euler's method to make this conclusion. To this end, note that for $0 = t_0 < t_1 < \cdots < t_n = t$ we have

$$Y(t_{i+1}) - Y(t_i) = \int_{t_i}^{t_{i+1}} Y(\sigma)\Lambda(\sigma)d\sigma = Y(t_i)\int_{t_i}^{t_{i+1}} \Lambda(\sigma)d\sigma + 0(\Delta t_i^2), \tag{5.16}$$

$\Delta t_i = t_{i+1} - t_i$. It follows that

$$(5.17) \qquad Y(t_{i+1}) = Y(t_i)\,(I + \int_{t_i}^{t_{i+1}} \Lambda(\sigma)d\sigma) + 0(\Delta t_i^2)$$

and by a standard argument we get

$$(5.18) \qquad Y(t) = \lim_{\substack{\Delta_n \to 0 \\ n\to\infty}} \prod_{i=0}^{n-1} (I + \int_{t_{n,i}}^{t_{n,i+1}} \Lambda(\sigma)d\sigma),$$

$\Delta_n = \max_i \Delta t_{n,i}$. Each factor on the right of (5.18) is one-banded with nonnegative entries and hence $Y(t)$ is totally positive. Furthermore, if we introduce the matrices

$$A_i^n = \int_{t_{n,i}}^{t_{n,i+1}} \Lambda(\sigma)d\sigma,\ 1 \leqq i \leqq n - 1$$

then

$$(5.19) \qquad \lim_{n\to\infty} \max \|A_i^n\|_\infty = 0$$

as $\Delta_n \to 0$. This suggests introducing the subclass $\mathscr{P}_N$ of biinfinite matrices in $\mathscr{A}_N$ such that $A \in \mathscr{P}_N$ whenever its symbol is a uniform limit on compact subsets of C,

$$(5.20) \qquad A(z) = \lim_{n\to\infty} \prod_{i=1}^{n} (I + A_i^n(z))$$

for some $\{A_i^n\} \subset \mathscr{A}_N$ such that (5.19) holds and the symbol of A_i^n has the form

$$A_i^n(z) = \begin{bmatrix} 0 & & & & za_{i,1}^n \\ a_{i,2}^n & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & a_{i,N}^n & 0 \end{bmatrix}.$$

Equation (5.18) therefore shows every matrix accessible by the flow (5.1), (5.2) is in $\mathscr{P}_N$. We will next show the converse is also true. For this purpose, we let

$\mathscr{D}_N = \{A\colon A \in \mathscr{A}_N,\ A = Y(1)$, where $Y(t)$ solves (5.1) for some Λ as in (5.2)$\}$.

THEOREM 5.1.

$$\mathscr{D}_N = \mathscr{P}_N.$$

PROOF. We wish to show that any $A \in \mathscr{P}_N$ is accessible by the flow (5.1), (5.2). To get a feel for $\mathscr{P}_N$ we let $A(z) = \lim_{n\to\infty} A^n(z)$ where

$$A^n(z) = \prod_{j=1}^{n} (I + A_j^n(z))$$

then

$$A_{i,i-1}^n = \sum_{j=1}^{n} a_{ji}^n.$$

Thus, $\lim_{n\to\infty} \sum_{j=1}^{n} a_{ji}^n$ exists for each i. For our later use we define

$$\Sigma(A) = \lim_{n\to\infty} \sum_{i=1}^{N} \sum_{j=1}^{n} a_{ji}^n.$$

Furthermore, we observe that

$$\det A(z) = \lim_{n\to\infty} \det A^n(z) = \lim_{n\to\infty} \prod_{j=1}^{n} (1 - (-1)^N \gamma_j^n z)$$

where

$$\gamma_j^n = \prod_{k=1}^{N} a_{j,k}^n.$$

Since $\max_{j,k}|a_{j,k}^n| \to 0$, as $n \to \infty$, we have for $N \geqq 2$ $\sum_{j=1}^{n} \gamma_j^n \to 0$ and so we get $\det A(z) \equiv 1$. This observation is consistent with Theorem 5.1 because for every $A \in \mathscr{D}_N$ with $A = Y(1)$ we have

$$(Y'(t))(z) = Y(t)\,(\Lambda(t))\,(z) \tag{5.21}$$

and so by the formula

$$\det (Y(1))(z) = \exp \int_0^1 \operatorname{trace}(\Lambda(\sigma))\,(z) d\sigma$$

we get $\det A(z) = 1$ too.

Returning to the proof we first observe that

$$A(z) = \lim_{n\to\infty} \prod_{j=1}^{n} \exp A_j^n(z). \tag{5.22}$$

This is seen by setting

$$\exp A_j^n(z) = I + A_j^n(z) + R_j^n(z).$$

Then

$$\|R_j^n(z)\|_\infty \leqq \|A_j^n(z)\|_\infty^2 \exp \|A_j^n(z)\|_\infty. \tag{5.23}$$

Therefore

$$\prod_{j=1}^{n} \exp (A_j^n(z)) = A^n(z) + G^n(z).$$

The remainder $G^n(z)$ may be estimated as follows. Setting $\varepsilon_{n,j} = \max_{|z|\leq\rho}\|A_j^n(z)\|_\infty$ and $\varepsilon_n = \max_{1\leq j\leq n}\varepsilon_{n,j}$ so that $\sum_{j=1}^n \varepsilon_{n,j} \leqq \max(1, \rho)\sum(A)$ we note that by (5.23)

$$\max_{|z|\leq\rho}\|G^n(z)\|_\infty \leqq \exp(2\textstyle\sum(A)\max(1, \rho))\displaystyle\sum_{\ell=1}^n \sum_{1\leq j_1<\cdots<j_\ell\leq n}(\varepsilon_{n,j_1}\cdots\varepsilon_{n,j_\ell})^2.$$

Recalling that $\varepsilon_n \to 0$, as $n \to \infty$, it is easy to verify that the right hand side of the above inequality tends to zero as $n \to \infty$. Thus we have verified (5.22).

As a next step we will construct for a given $A \in \mathscr{P}_N$ a sequence of matrices $\Lambda_n(t)$ of type (5.2) so that $Y_{\Lambda_n}(1)$ converges pointwise to A.

To this end, we define

$$\Delta_{n,j} = \Big(\sum_{k=1}^N a_{j,k}^n\Big)\Big/\sum_{j=1}^n\sum_{k=1}^N a_{j,k}^n.$$

Excluding the obvious case when A is the identity matrix, we have $\sum(A) > 0$ and so $\max_j \Delta_{n,j} \to 0$ as $n \to \infty$ and

$$\sum_{j=1}^n \Delta_{n,j} = 1.$$

Therefore by defining

$$t_{n,i} = \sum_{j=1}^i \Delta_{n,j}, \ 1 \leqq i \leqq n,$$
$$t_{n,0} = 0,$$

we obtain a partition $\{t_{n,i}\}$ of [0, 1]. Next we introduce the step functions

$$r_k^n(t) = \begin{cases} a_{j,k}^n/\Delta_{n,j}, & \text{if } t \in [t_{n,j-1}, t_{n,j}), \\ 0, & \text{if } t \notin [0, 1]. \end{cases}$$

Since $\|r_k^n\|_\infty \leqq \sum_{j=1}^n\sum_{k=1}^n a_{j,k}^n$ we get

$$\overline{\lim_{n\to\infty}}\|r_k^n\|_\infty \leqq \textstyle\sum(A), \quad 1 \leqq k \leqq N. \tag{5.24}$$

Setting

$$\Lambda^n(t)(z) = \begin{bmatrix} 0 & & & & zr_1^n(t) \\ r_2^n(t) & 0 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ 0 & & & r_N^n(t) & 0 \end{bmatrix} \tag{5.25}$$

it follows that

(5.26) $$\exp(A_j^n(z)) = \exp(\Delta_{j,n} \Lambda^n(t)(z))$$

for $t \in [t_{n,j-1}, t_{n,j})$. Therefore, the matrix whose symbol is

(5.27) $$B_n(z) = \prod_{j=1}^{n} \exp(A_j^n(z))$$

satisfies, on one hand

(5.28) $$B_n(z) = Y_{\Lambda^n}(1)(z)$$

while, on the other hand (5.22) provides

(5.29) $$\lim_{n\to\infty} B_n(z) = A(z).$$

We complete the proof by showing that there is a Λ as in (5.2) with $|\Lambda| \leqq \Sigma(A)$ satisfying $Y_\Lambda(1) = A$.

It is easy to see from (5.24) and (5.25) that the $\{Y_{\Lambda^n}\}_{n=1}^{\infty}$ are equicontinuous and bounded and in addition the sequences $\{\Lambda_{j,j-1}^n\}_{n=1}^{\infty}$ are bounded in $L^\infty[0, 1]$. Thus we may select a subsequence (again calling them $\{\Lambda^n\}$ and $\{Y_{\Lambda^n}\}$) so that $Y_{\Lambda^n} \to Y$ by the Arzela-Ascoli Theorem and $\Lambda_{j,j-1}^n \to \Lambda_{j,j-1}$ weak star $L^\infty[0, 1]$ by Alaoglou's Theorem where $|\Lambda| \leqq \Sigma(A)$. We need to verify that $Y = Y_\Lambda$. To this end, observe that

$$\begin{aligned} Y_{\Lambda^n}(t) &= I + \int_0^t Y_{\Lambda^n}(\sigma)\Lambda^n(\sigma)d\sigma \\ &= I + \int_0^t Y(\sigma)\Lambda^n(\sigma)d(\sigma) + \int_0^t [Y_{\Lambda^n} - Y](\sigma)\Lambda^n(\sigma)\, d\sigma. \end{aligned}$$

Letting $n \to \infty$ and using $Y_{\Lambda^n} \to Y$ and $\Lambda^n \to \Lambda$ we conclude that

$$Y(t) = I + \int_0^t Y(\sigma)\Lambda(\sigma)d\sigma.$$

This completes the proof of Theorem 5.1.

We finally discuss whether or not matrices in $\mathscr{D}_N$ can be factored as $A = RB$ or $A = BR$ where $R, B \in \mathscr{A}_N$ and R is one-banded. So far we have observed for every $A \in \mathscr{D}_N$ $\det A(z) = 1$ and $\gamma(A) = \prod_{j=1}^{N} \beta_j(A) = 0$ because $A(z)$ is entire (see (3.6)). We approach this issue by determining when the $\beta_i(A) = 0$, $i = 1, \ldots, N$. As we shall see this means that $A \in \mathscr{D}_N$ cannot be factored.

Let us now clarify the circumstances which assure us that $\beta_i(A) = 0$, $i = 1, \ldots, N$. This requires introducing the subclass $\mathscr{S}\mathscr{D}_N$ of $\mathscr{D}_N$

(5.30) $$\mathscr{S}\mathscr{D}_N = \Big\{A\colon A = Y(1) \text{ for some } \Lambda \text{ as in (5.2), satisfying for all } i \text{ and all } \varepsilon,\ 0 < \varepsilon < 1,\ \int_{1-\varepsilon}^{1} \Lambda_{i,i-1}(\sigma)d\sigma > 0\Big\}.$$

THEOREM 5.2. *A necessary and sufficient condition for $A \in \mathscr{D}_N$ to satisfy*

$\beta_i(A) = 0$, $i = 1, \ldots, N$ is that $A \in \mathcal{SD}_N$. Moreover, any $A \in \mathcal{SD}_N$ cannot be factored as $A = RB$ or BR when $R, B \in \mathcal{A}_N$, R is one-banded but not diagonal and B has an entire symbol.

We should keep in mind that there are matrices $A \in \mathcal{D}_N$ which are banded with banded inverse. For such matrices the $\beta_i(A)$ are not defined. As a simple example of this possibility we let $N = 2$, $r_1(t) = 1$ and $r_2(t) = 0$, then

$$Y_A(1)(z) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

PROOF. In order to prove the first assertion, we need the following lemma.

LEMMA 5.1. *Let $A \in \mathcal{A}_N^+$ with $A(z)$ entire and $A, B \in \mathcal{A}_N^+$ be unit lower triangular, such that $\beta_i(A)$ and $\beta_i(B)$ exist for all i. Then*

$$\beta_i(AB) = \beta_i(B), \quad i = 1, \ldots, N. \tag{5.31}$$

PROOF. Let $C = AB$ and compute for $i > j$

$$\frac{C_{ij}}{C_{i,j+1}} = \frac{\sum_{\ell=j}^{i} A_{i\ell}B_{\ell j}}{\sum_{\ell=j+1}^{i} A_{i\ell}B_{\ell,j+1}} = \sum_{\ell=j+1}^{i} W_{ij\ell}(B_{\ell j}/B_{\ell,j+1}) + V_{ij} \tag{5.32}$$

where

$$W_{ij\ell} = \frac{A_{i\ell}B_{\ell,j+1}}{\sum_{k=j+1}^{i} A_{ik}B_{k,j+1}},$$

and

$$V_{ij} = A_{ij}/\sum_{\ell=j+1}^{i} A_{i\ell}B_{\ell,j+1}.$$

Now for fixed j, ℓ and $i > k + N > j$

$$W_{ij\ell} \leqq \frac{A_{i,\ell}B_{\ell,j+1}}{A_{i,\ell+N}B_{\ell+N,j+1}}$$

and also for $i \to \infty$

$$A_{ik}/A_{i,k+N} \to \prod_{k=1}^{N} \beta_k(A) = 0.$$

Thus $W_{ij\ell}, V_{ij} \to 0$ as $i \to \infty$ for fixed j, ℓ. But then (5.32) is seen to be a summability method which gives

$$\lim_{i\to\infty} \frac{C_{ij}}{C_{i,j+1}} = \lim_{i\to\infty} \frac{B_{ij}}{B_{i,j+1}},$$

as was to be shown.

Returning to the proof, we let $A = Y_\Lambda(1)$ be in $\mathscr{S}\mathscr{D}_N$. Note that

$$Y_\Lambda(1) = Y_\Lambda(t)Y_\Lambda(t;\ 1)$$

where $Y_\Lambda(t;\ \cdot)$ solves

$$V' = V\Lambda,\ V(t) = I.$$

Since $Y_\Lambda(t)(z)$ is entire for each $t \in (0,\ 1)$, Lemma 5.1 yields

$$\beta_i(A) = \beta_i(Y_\Lambda(t;\ 1)). \tag{5.33}$$

But from (5.12) we have

$$\lim_{i\to\infty} \frac{(Y_\Lambda(t;\ 1))_{i,j}}{(Y_\Lambda(t;\ 1))_{i,j+1}} \leqq (1 - t)\|r_{j+1}\|_\infty.$$

This inequality holds for all $t \in (0,\ 1)$ and hence from (5.33) we conclude that all the $\beta_i(A)$, $i = 1, \ldots, N$, vanish.

Conversely let $\beta_i(A) = 0$, $i = 1, \ldots, N$ and suppose $A \in \mathscr{D}_N$. Then $A = Y_\Lambda(1)$. Let $t^* = \min\{t\colon t \leqq 1 \text{ and } Y_\Lambda(t) = Y_\Lambda(1)\}$ and set $\Lambda^*(t) = t^*\Lambda(tt^*)$. Hence $A = Y_{\Lambda^*}(1)$ and $\int_{1-\varepsilon}^1 \Lambda^*(t)dt \neq 0$ for $0 < \varepsilon < 1$. If A is not in $\mathscr{S}\mathscr{D}_N$ then we must have

$$\int_{1-\varepsilon_0}^1 \Lambda^*_{i,i-1}(t)dt = 0 \tag{5.34}$$

for some i and $\varepsilon_0 > 0$. Again we have

$$A = Y_{\Lambda^*}(1 - \varepsilon_0)\ Y_{\Lambda^*}(1 - \varepsilon_0;\ 1).$$

From (5.12) and (5.34) it follows that $Y_{\Lambda^*}(1 - \varepsilon_0;\ 1)$ is a non-diagonal banded matrix in $\mathscr{A}_N^+$. Hence by Theorem 3.1 it is a product of one-banded nondiagonal matrices. In particular, there exists $B \in \mathscr{A}_N$ with $\det B(z) = 1$ and R a one-banded nondiagonal matrix in $\mathscr{A}_N^+$ such that

$$A = BR.$$

Since $\det A(z) = \det B(z) = 1$, it follows that $\det R(z) = 1$ from which we conclude R is a block diagonal matrix. Consequently, R^{-1} is block diagonal too, and $DR^{-1} D \in \mathscr{A}_N^+$. Hence we can find a column of R^{-1} (say the j^{-th}) so that the only two nonzero entries are

$$R_{jj}^{-1} = 1 \text{ and } R_{j+1,j}^{-1} = c < 0.$$

Thus

$$\frac{B_{ij}}{A_{i,j+1}} = \frac{A_{ij} + cA_{i,j+1}}{A_{i,j+1}} = \frac{A_{ij}}{A_{i,j+1}} + c$$

is negative for large i because $\beta_i(A)$ is zero. This same argument can be used to conclude that $A \in \mathscr{S}\mathscr{D}_N$ has no one-banded factors of the type mentioned in the theorem. This completes the proof of Theorem 5.2.

Theorem 5.2 leads to the following characterization of the matrices in $\mathscr{D}_N$.

THEOREM 5.3. *A is in* $\mathscr{D}_N$ if and only if

$$A = B \prod_{-\infty}^{0} R_i \tag{5.35}$$

where each R_i *is one-banded with* $\det R_i(z) = 1$ *and* $B \in \mathscr{S}\mathscr{D}_N$.

Before proving Theorem 5.3 we observe that if conjecture 1 is valid the central remaining factor $C(z)$ in Theorem 3.3 has the form

$$C(z) = B(z) \prod_{i=-\infty}^{0} R_i(z)$$

where $\beta_i(B) = 0$, $i = 1, \ldots, N$ and $B \in \mathscr{S}\mathscr{D}_N$. Consequently, by relabeling all one-banded factors appearing in (3.9) and (5.35) we would get the representation (3.9) with the remaining central factor in $\mathscr{S}\mathscr{D}_N$. This would give a complete factorization of $A(z)$ analogous to the scalar case.

Proof of Theorem 5.3. Suppose $B \in \mathscr{S}\mathscr{D}_N$ and $\prod_{i=-\infty}^{0} R_i$ exists where $\det R_i(z) = 1$, $i = 0, \ldots, -\infty$. Since $\mathscr{D}_N$ is closed under pointwise limits and under multiplication

$$B \prod_{i=-\infty}^{0} R_i \in \mathscr{D}_N \tag{5.36}$$

follows as soon as we have shown that any one-banded matrix $R \in \mathscr{A}_N^+$ such that $\det R(z) = 1$ is in $\mathscr{D}_N$. For such an R we can assume without loss of generality that $R_{ii} = 1$, $i \in \mathbf{Z}$, and

$$R_{i+1,i} = \begin{cases} 0, & i = kN,\ k \in \mathbf{Z}, \\ r_i \geqq 0, & i = kN + j,\ j = 1, \ldots, N-1. \end{cases}$$

Now let

$$T_{ii}^j = 1,\ i \in \mathbf{Z},$$

$$T_{i+1,i}^j = \begin{cases} r_j, & i = kN + j,\ k \in \mathbf{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

Notice that

$$T^{\ell} = Y_{\Lambda^{\ell}}(1)$$

where

$$\Lambda_{i+1+kN,i+kN}^{\ell}(t) = \delta_{i,\ell} r_{\ell},\ k \in \mathbf{Z},$$

and

$$\prod_{\ell=0}^{N-1} T^{\ell} = R$$

proving (5.36).

Conversely, let $A \in \mathscr{D}_N$ and assume $A = Y_A(1)$ where $A(t)$ does not vanish identically in a left neighborhood of 1. If $\int_{1-\varepsilon}^{1} A_{i,i-1}(t)dt > 0$ for all $i \in \mathbf{Z}$ and all $1 \geqq \varepsilon > 0$ we have by Theorem 5.2 that $\beta_i(A) = 0$, $i = 1, \ldots, N$, whence the assertion follows with $A = B$, $R_i = I$, all i. If for some i_0 and some $\varepsilon > 0$, $\int_{1-\varepsilon}^{1} A_{i_0,i_0-1}(t)dt = 0$ we pick the largest ε with this property, ε_1 say, and write

$$A = Y_A(1) = Y_A(1 - \varepsilon_1)Y_A(1 - \varepsilon_1; 1).$$

As before we easily conclude that $Y_A(1 - \varepsilon; 1)$ is banded and det $Y_A(1 - \varepsilon_1; 1)(z) = 1$. Hence $Y_A(1 - \varepsilon_1; 1)$ can be factored into one-banded matrices R^i, with det $R^i(z) = 1$. We now repeat the same argument on the interval $(0, 1 - \varepsilon_1)$ if any of the $\beta_i(Y_A(1 - \varepsilon_1)) \neq 0$, thereby obtaining the desired factorization.

REMARK. The following example shows that there may indeed be an infinite number of factors in (5.35). Let $N = 2$ and $A_{i+2k,i-1+2k}(t) = r_i(t)$, $i = 0, 1$, $k \in \mathbf{Z}$ when

$$r_0(t) = \begin{cases} 1, & t \in [2^{-2k-1}, 2^{-2k}),\ k = 0, 1, \ldots, \\ 0, & t \in [2^{-2k}, 2^{-2k+1}),\ k = 1, 2, \ldots, \end{cases}$$

$$r_1(t) = 1 - r_0(t).$$

We end this paper with another conjecture which would follow from Conjecture 1. This additional conjecture may be of some use in resolving Conjecture 1.

CONJECTURE 2. *Let* $A \in \mathscr{A}_N$ *with an entire symbol. Then there are banded matrices* $R_n \in \mathscr{A}_N$ *such that*

$$A(z) = \lim_{n\to\infty} R_n(z)$$

uniformly on compact subsets of C.

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