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THE ORDER OF MAGNITUDE OF THE (C, $\alpha \geqq 0$, $\beta \geqq 0$)-MEANS OF DOUBLE ORTHOGONAL SERIES

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1. Introduction. Let (X, F, μ) be an arbitrary positive measure space and $\{\phi_{ik}(x): i, k = 1, 2, \ldots\}$ an orthonormal system on X. We consider the double orthogonal series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}\phi_{ik}(x), \tag{1.1}$$

where $\{a_{ik}: i, k = 0, 1, \ldots\}$ is a sequence of real numbers for which

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

We denote by $s_{mn}(x)$ the rectangular partial sums of series (1.1):

$$s_{mn}(x) = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{ik}\phi_{ik}(x) \qquad (m, n = 0, 1, \ldots).$$

Let α and β be real numbers, $\alpha > -1$ and $\beta > -1$. We recall that the (C, α, β)-means of series (1.1) are defined

$$\sigma_{mn}^{\alpha\beta}(x) = \frac{1}{A_m^\alpha A_n^\beta}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta-1} s_{ik}(x)$$
$$= \frac{1}{A_m^\alpha A_n^\beta}\sum_{i=0}^{m}\sum_{k=0}^{n} A_{m-i}^{\alpha} A_{n-k}^{\beta} a_{ik}\phi_{ik}(x), \qquad (m, n = 0, 1, \ldots),$$

where

$$A_m^\alpha = \binom{m+\alpha}{m} = \begin{cases} (\alpha+1)(\alpha+2)\cdots(\alpha+m)/m!, & \text{for } m = 1, 2, \ldots, \\ 1, & \text{for } m = 0, \end{cases}$$

(see, e.g., [**9**, p. 77]).

The case $\alpha = \beta = 0$ gives back the rectangular partial sums $s_{mn}(x) = \sigma_{mn}^{00}(x)$. The case $\alpha = \beta = 1$ gives the first arithmetic means with respect to m and n,

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$$\sigma_{mn}^{11}(x) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{k=0}^{n} s_{ik}(x)$$
$$= \sum_{i=0}^{m} \sum_{k=0}^{n} \left(1 - \frac{i}{m+1}\right)\left(1 - \frac{k}{n+1}\right) a_{ik}\phi_{ik}(x).$$

Furthermore, the case $\alpha = 1$ and $\beta = 0$ gives the first arithmetic means with respect to m, i.e.,

$$\sigma_{mn}^{10}(x) = \frac{1}{m+1} \sum_{i=0}^{m} s_{in}(x) = \sum_{i=0}^{m} \sum_{k=0}^{n} \left(1 - \frac{i}{m+1}\right) a_{ik}\phi_{ik}(x),$$

while the case $\alpha = 0$ and $\beta = 1$ gives the first arithmetic means with respect to n.

Before stating the preliminary results, we make the following convention. Given a double sequence $\{f_{mn}(x): m, n = 0, 1, \ldots\}$ of functions in $L^2 = L^2(X, F, \mu)$ and a double sequence $\{\lambda_{mn}\}$ of positive numbers, we write

$$f_{mn}(x) = o_x\{\lambda_{mn}\} \quad \text{a.e.}$$

if

$$f_{mn}(x)/\lambda_{mn} \to 0 \quad \text{a.e. as } \max(m, n) \to \infty$$

and, in addition, there exists a function $F(x) \in L^2$ such that

$$\sup_{m, n \geqq 0} |f_{mn}(x)|/\lambda_{mn} \leqq F(x) \qquad \text{a.e.}$$

2. Preliminary results. The following theorem is well-known (see, e.g., [**3**, Corollary 2]).

THEOREM A. *Under condition* (1.2),

$$s_{mn}(x) = o_x\{\log(m+2) \log(n+2)\} \qquad \text{a.e.} \tag{2.1}$$

In this paper the logarithms are to the base 2.

This theorem is exact in the sense that $\log(t + 2)$ cannot be replaced in it by any sequence $\rho(t)$ tending to ∞ slower as $t \to \infty$ (cf. [**6**]).

The order of magnitude improves when considering $\sigma_{mn}^{11}(x)$ instead of $s_{mn}(x)$.

THEOREM B. *Under condition* (1.2),

$$\sigma_{mn}^{11}(x) = o_x\{\log\log(m+4) \log\log(n+4)\} \qquad \text{a.e.} \tag{2.2}$$

This theorem was proved in [**4**]. It was also pointed [**4**] that Theorem B is the best possible in the same sense as Theorem A is.

The orders of magnitude of $\sigma^{10}_{mn}(x)$ and $\sigma^{01}_{mn}(x)$ lie between (2.1) and (2.2). (See, again, [**4**].)

THEOREM C. *Under condition* (1.2),

$$\sigma^{10}_{mn}(x) = o_x\{\log\log(m+4)\log(n+2)\} \quad \text{a.e.} \tag{2.3}$$

and

$$\sigma^{01}_{mn}(x) = o_x\{\log(m+2)\log\log(n+4)\} \quad \text{a.e.} \tag{2.4}$$

3. Main results. Our first goal is to prove that the (C, α, β)-means of series (1.1) are of the same order of magnitude as the right side in (2.2) for all $\alpha > 0$ and $\beta > 0$. Besides, we show that relations (2.3) and (2.4) hold also when $\sigma^{\alpha 0}_{mn}(x)$ is substituted for $\sigma^{10}_{mn}(x)$ and when $\sigma^{0\alpha}_{mn}(x)$ is substituted for $\sigma^{01}_{mn}(x)$ in them, respectively, where $\alpha > 0$ is arbitrary.

THEOREM 1. *If* $\alpha > 0$ *and condition* (1.2) *is satisfied, then*

$$\sigma^{\alpha 0}_{mn}(x) = o_x\{\log\log(m+4)\log(n+2)\} \quad \text{a.e.} \tag{3.1}$$

and

$$\sigma^{0\alpha}_{mn}(x) = o_x\{\log(m+2)\log\log(n+4)\} \quad \text{a.e.}$$

THEOREM 2. *If* $\alpha > 0$, $\beta > 0$ *and condition* (1.2) *is satisfied, then*

$$\sigma^{\alpha\beta}_{mn}(x) = o_x\{\log\log(m+4)\log\log(n+4)\} \quad \text{a.e.} \tag{3.2}$$

These two theorems can be considered as the extension of a result of Tandori [**7**, Theorem 7, p. 101] from single orthogonal series to double ones. Unlike the original proof, we avoid the application of Abel's transformation.

The following two theorems play a crucial role in the proofs of Theorems 1 and 2.

THEOREM 3. *If* $\alpha > 1/2$ *and condition* (1.2) *is satisfied, then*

$$\delta^{\alpha}_{Mn}(x) = \left\{\frac{1}{M+1}\sum_{m=0}^{M}(\sigma^{\alpha-1,0}_{mn}(x) - \sigma^{\alpha 0}_{mn}(x))^2\right\}^{1/2} = o_x\{\log(n+2)\} \quad \text{a.e.}$$

and

$$\left\{\frac{1}{N+1}\sum_{n=0}^{N}(\sigma^{0,\alpha-1}_{mn}(x) - \sigma^{0\alpha}_{mn}(x))^2\right\}^{1/2} = o_x\{\log(m+2)\} \quad \text{a.e.}$$

THEOREM 4. *If* $\alpha > 1/2$, $\beta > 1/2$ *and condition* (1.2) *is satisfied, then*

$$\begin{aligned}\varepsilon^{\alpha\beta}_{MN}(x) &= \left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}(\sigma^{\alpha-1,\beta-1}_{mn}(x) - \sigma^{\alpha\beta}_{mn}(x))^2\right\}^{1/2} \\ &= o_x\{\log\log(\max(M, N)+4)\} \quad \text{a.e.}\end{aligned}$$

On the other hand, taking Theorems 1, 2, 3 and 4 for granted, we can deduce two interesting theorems on the order of magnitude of quadratic averages of the (C, α, β)-means of series (1.1) for $\alpha > -1/2$ and $\beta > -1/2$.

THEOREM 5. *If* $\alpha > 1/2$ *and condition* (1.2) *is satisfied, then*

$$\left\{\frac{1}{M+1}\sum_{m=0}^{M}(\sigma_{mn}^{\alpha-1,0}(x))^2\right\}^{1/2} = o_x\{\log\log(M+4)\log(n+2)\} \quad \text{a.e.} \tag{3.3}$$

and

$$\left\{\frac{1}{N+1}\sum_{n=0}^{N}(\sigma_{mn}^{0,\alpha-1}(x))^2\right\}^{1/2} = o_x\{\log(m+2)\log\log(N+4)\} \quad \text{a.e.} \tag{3.4}$$

THEOREM 6. *If* $\alpha > 1/2$, $\beta > 1/2$ *and condition* (1.2) *is satisfied, then*

$$\left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}(\sigma_{mn}^{\alpha-1,\beta-1}(x))^2\right\}^{1/2} = o_x\{\log\log(M+4)\log\log(N+4)\} \quad \text{a.e.} \tag{3.5}$$

For example, (3.3) is an immediate consequence of Theorems 1 and 3 if we take into account

$$\left\{\sum_{m=0}^{M}(\sigma_{mn}^{\alpha-1,0}(x))^2\right\}^{1/2} \leqq \left\{\sum_{m=0}^{M}(\sigma_{mn}^{\alpha-1,0}(x)-\sigma_{mn}^{\alpha0}(x))^2\right\}^{1/2} + \left\{\sum_{m=0}^{M}(\sigma_{mn}^{\alpha0}(x))^2\right\}^{1/2}.$$

We would like to emphasize that estimates (3.3)–(3.5) in the special cases $\alpha = 1$ and $\beta = 1$ (and even more for $1/2 < \alpha < 1$ and $1/2 < \beta < 1$) are very unexpected in comparison with what would follow from (2.1).

4. Auxiliary results. In this section we consider the numerical series

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} u_{ik}, \tag{4.1}$$

where the u_{ik} are real numbers. The (C, α, β)-means of series (4.1) are defined by

$$\sigma_{mn}^{\alpha\beta} = \frac{1}{A_m^{\alpha}A_n^{\beta}}\sum_{i=0}^{m}\sum_{k=0}^{n}A_{m-i}^{\alpha}A_{n-k}^{\beta}u_{ik} \quad (m, n = 0, 1, \ldots; \alpha > -1, \beta > -1).$$

We recall some identities and inequalities well-known in the literature. For all α and γ,

$$A_m^{\alpha+\gamma+1} = \sum_{i=0}^{m} A_i^{\alpha}A_{m-i}^{\gamma} \tag{4.2}$$

(see, e.g., [9, p. 77, formula (1.10)]). Hence the representations

$$\sigma_{mn}^{\alpha+\gamma,0} = \frac{1}{A_m^{\alpha+\gamma}}\sum_{i=0}^{m}A_{m-i}^{\gamma-1}A_i^{\alpha}\sigma_{in}^{\alpha0} \tag{4.3}$$

and

$$\sigma_{mn}^{\alpha+\gamma,\beta+\delta} = \frac{1}{A_m^{\alpha+\gamma} A_n^{\beta+\delta}} \sum_{i=0}^{m} \sum_{k=0}^{n} A_{m-i}^{\gamma-1} A_{n-k}^{\delta-1} A_i^{\alpha} A_k^{\beta} \sigma_{ik}^{\alpha\beta}$$

easily follow.

We need the following estimate. There exist two positive constants C_1 and C_2 such that

$$C_1 \leqq \frac{A_m^\alpha}{m^\alpha} \leqq C_2 \quad (m = 1, 2, \ldots; \alpha > -1) \tag{4.4}$$

(see, e.g., [**1**, p. 69, formula (25)] or [**9**, p. 77, formula (1.18)]).

In order to formulate the next two Tauberian type results, we consider a double sequence $\{\lambda_{mn}: m, n = 0, 1, \ldots\}$ of positive numbers which is nondecreasing both in m and in n.

LEMMA 1. *If* $\alpha > -1/2$, $\varepsilon > 0$ *and*

$$\frac{1}{\lambda_{Mn}} \left\{ \frac{1}{M+1} \sum_{m=0}^{M} (\sigma_{mn}^{\alpha 0})^2 \right\}^{1/2} \to 0 \quad \textit{as} \max(M, n) \to \infty, \tag{4.5}$$

then

$$\frac{1}{\lambda_{Mn}} \sigma_{Mn}^{\alpha+1/2+\varepsilon, 0} \to 0 \quad \textit{as} \max(M, n) \to \infty. \tag{4.6}$$

Furthermore, if

$$\frac{1}{\lambda_{Mn}} \left\{ \frac{1}{M+1} \sum_{m=0}^{M} (\sigma_{mn}^{\alpha 0})^2 \right\}^{1/2} \leqq B \quad (M, n = 0, 1, \ldots)$$

with a positive number B, then there exists a constant C depending only on α *and* ε *such that*

$$\frac{1}{\lambda_{Mn}} |\sigma_{Mn}^{\alpha+1/2+\varepsilon, 0}| \leqq CB \quad (M, n = 0, 1, \ldots).$$

LEMMA 2. *If* $\alpha > -1/2$, $\beta > -1/2$, $\varepsilon > 0$, $\eta > 0$ *and*

$$\frac{1}{\lambda_{MN}} \left\{ \frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} (\sigma_{mn}^{\alpha\beta})^2 \right\}^{1/2} \to 0 \quad \textit{as} \max(M, N) \to \infty,$$

then

$$\frac{1}{\lambda_{MN}} \sigma_{MN}^{\alpha+1/2+\varepsilon, \beta+1/2+\eta} \to 0, \quad \textit{as} \max(M, N) \to \infty.$$

Furthermore, if

$$\frac{1}{\lambda_{MN}} \left\{ \frac{1}{(M+1)(N+1)} \sum_{m=0}^{M} \sum_{n=0}^{N} (\sigma_{mn}^{\alpha\beta})^2 \right\}^{1/2} \leqq B \quad (M, N = 0, 1, \ldots)$$

with a positive number B, then there exists a constant C depending only on α, β, ε *and* η *such that*

$$\frac{1}{\lambda_{MN}} |\sigma_{MN}^{\alpha+1/2+\varepsilon, \beta+1/2+\eta}| \leqq CB \quad (M, N = 0, 1, \ldots).$$

The corresponding result for single series was established by Tandori [7, p. 103]. The basic idea goes back to Zygmund [**8**, p. 360–361]. For the sake of completeness, we show how condition (4.5) implies statement (4.6). By (4.3),

$$\sigma_{MN}^{\alpha+1/2+\varepsilon, 0} = \frac{1}{A_M^{\alpha+1/2+\varepsilon}} \sum_{m=0}^{M} A_{M-m}^{-1/2+\varepsilon} A_m^{\alpha} \sigma_{mn}^{\alpha 0}.$$

Hence, using the Cauchy inequality,

$$|\sigma_{Mn}^{\alpha+1/2+\varepsilon, 0}| \leqq \frac{1}{A_M^{\alpha+1/2+\varepsilon}} \left\{\sum_{m=0}^{M} (\sigma_{mn}^{\alpha 0})^2\right\}^{1/2} \left\{\sum_{m=0}^{M} (A_{M-m}^{-1/2+\varepsilon} A_m^{\alpha})^2\right\}^{1/2}.$$

Taking into account (4.2), (4.4) and (4.5), it is not hard to check that

$$\begin{aligned} |\sigma_{Mn}^{\alpha+1/2+\varepsilon, 0}| &= O\left\{\frac{1}{(M+1)^{\alpha+1/2+\varepsilon}}\right\} o\{(M+1)^{1/2} \lambda_{Mn}\} \, O\{(M+1)^{\alpha+\varepsilon}\} \\ &= o\{\lambda_{Mn}\} \quad \text{as } \max(M, n) \to \infty, \end{aligned}$$

which is (4.6) to be proved.

In the sequel, we often make use of the representation

$$\text{(4.7)} \qquad \sigma_{mn}^{\alpha-1, \beta} - \alpha_{mn}^{\alpha\beta} = \frac{1}{\alpha A_m^{\alpha} A_n^{\beta}} \sum_{i=1}^{m} \sum_{k=0}^{n} A_{m-i}^{\alpha-1} A_{n-k}^{\beta} \, i u_{ik} \quad (\alpha > 0, \beta > -1),$$

which can be deduced from the identities

$$A_m^{\alpha-1} = \frac{\alpha}{\alpha+m} A_m^{\alpha} \quad \text{and} \quad A_{m-i}^{\alpha} = \frac{\alpha+m-i}{\alpha} A_{m-i}^{\alpha-1}.$$

Finally, the inequality

$$\text{(4.8)} \qquad \sum_{m=i}^{\infty} \left(\frac{A_{m-i}^{\alpha-1}}{A_m^{\alpha}}\right)^2 = O\left\{\frac{1}{i}\right\} \quad (i = 1, 2, \ldots; \alpha > 1/2)$$

will be useful in the proofs (see, e.g., [1, p. 110]).

5. Proof of Theorem 1. We will prove the first statement, i.e., relation (3.1). The companion statement can be derived in a similar way. Now, the proof of (3.1) is done on the basis of Theorem 3, which will be proved in §7, and of the following consequence of Lemma 1.

COROLLARY 1. *If* $\alpha > -1/2$, $\varepsilon > 0$ *and*

$$(5.1)\qquad \left\{\frac{1}{M+1}\sum_{m=0}^{M}(\sigma_{mn}^{\alpha 0}(x))^2\right\}^{1/2} = o_x\{\log\log(M+4)\log(n+2)\}\quad \text{a.e.},$$

then

$$(5.2)\qquad \sigma_{Mn}^{\alpha+1/2+\varepsilon,0}(x) = o_x\{\log\log(M+4)\log(n+2)\}\quad \text{a.e.}$$

Proof of (3.1). By Theorem C, (3.1) holds for $\alpha = 1$. Hence, by Theorem 3, we obtain (5.1) for $\alpha = 0$. Thus, by Corollary 1, we get (5.2) for $\alpha = 0$. Applying again Theorem 3, we find (5.1) for $\alpha = -1/2 + \varepsilon$. Hence, again by Corollary 1, we obtain (5.2) for $\alpha = -1/2 + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this is equivalent to (3.1).

6. Proof of Theorem 2. It relies on Theorem 4, which will be proved in §8, and on the following consequence of Lemma 2.

Corollary 2. *If* $\alpha > -1/2$, $\beta > -1/2$, $\varepsilon > 0$, $\eta > 0$ *and*

$$(6.1)\qquad \left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}(\sigma_{mn}^{\alpha\beta}(x))^2\right\}^{1/2} = o_x\{\log\log(M+4)\log\log(N+4)\}\quad \text{a.e.},$$

then

$$(6.2)\qquad \sigma_{MN}^{\alpha+1/2+\varepsilon,\beta+1/2+\eta}(x) = o_x\{\log\log(M+4)\log\log(N+4)\}\quad \text{a.e.}$$

Proof of (3.2). By Theorem B, (3.2) holds for $\alpha = \beta = 1$. Hence, by Theorem 4, we get (6.1) for $\alpha = \beta = 0$. Thus, by Corollary 2, we obtain (6.2) for $\alpha = \beta = 0$. Using again Theorem 4, we find (6.1) for $\alpha = -1/2 + \varepsilon$ and $\beta = -1/2 + \eta$. Hence, by Corollary 2, we get (3.2) for $\alpha = 2\varepsilon$ and $\beta = 2\eta$. Taking into account that ε and η are arbitrary positive numbers, the proof of (3.2) is complete.

7. Proof of Theorem 3. It is enough to prove the first statement, since the second one can be proved in a similar way.

For simplicity in notation we suppose $M \geqq 1$, i.e., neglect the case $M = 0$. Then there exists an integer $p \geqq 0$ such that $2^{p-1} < M \leqq 2^p$. Clearly,

$$\delta_{Mn}^{\alpha}(x) \leqq \sqrt{2}\,\delta_{2^p,n}^{\alpha}(x).$$

Thus, it suffices to derive

$$(7.1)\qquad \delta_{2^p,n}^{\alpha}(x) = o_x\{\log(n+2)\}\quad \text{a.e.}$$

Next we take into consideration the decomposition

$$(7.2)\qquad (\delta_{2^p,n}^{\alpha}(x))^2 = \sum_{r=-2}^{p-1} 2^{r-p+2} \times \frac{1}{2^{r+2}}\sum_{m=2^r+1}^{2^{r+1}}(\sigma_{mn}^{\alpha-1,0}(x) - \sigma_{mn}^{\alpha 0}(x))^2,$$

where we make the convention that for $r = -2$ and -1, by 2^r we mean -1 and 0, respectively. So, by proving

$$(7.3)\quad {}^1\delta^{\alpha}_{pn}(x) = \left\{\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}} (\sigma^{\alpha-1,0}_{mn}(x) - \sigma^{\alpha 0}_{mn}(x))^2\right\}^{1/2} = o_x\{\log(n+2)\} \quad \text{a.e.}$$

we also prove (7.1). For convenience, we assume $p \geqq 0$ from now on.

PROOF OF (7.3). *Step* 1. First we prove (7.3) for the special case $n = 2^q$ ($q \geqq -1$; in the case $q = -1$ we take 2^q to equal 0):

$$(7.4)\quad {}^1\delta^{\alpha}_{p,2^q}(x) = o_x\{q+2\} \quad \text{a.e.}$$

Actually we prove (7.4) with $(q+2)^{1/2+\eta}$ on the right side instead of $q+2$, where $\eta > 0$.

To this effect, by (4.7) and the Cauchy inequality,

$$(7.5)\quad \begin{aligned}({}^1\delta^{\alpha}_{p,2^q}(x))^2 &= \frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\left(\sum_{i=1}^{m}\sum_{k=0}^{2^q}\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}\, ia_{ik}\phi_{ik}(x)\right)^2\\ &\leqq \frac{q+2}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{t=-2}^{q-1}\left(\sum_{i=1}^{m}\sum_{k=2^t+1}^{2^{t+1}}\frac{A^{\alpha-1}_{m-1}}{\alpha A^{\alpha}_m}\, ia_{ik}\phi_{ik}(x)\right)^2,\end{aligned}$$

with the agreement concerning 2^t, for $t = -2$ and -1, that we made after (7.2). Setting

$$F^2(x) = \sum_{p=0}^{\infty}\sum_{t=-2}^{\infty}\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\left(\sum_{i=1}^{m}\sum_{k=2^t+1}^{2^{t+1}}\frac{A^{\alpha-1}_{m-i}}{\alpha A^{\alpha}_m}\, ia_{ik}\phi_{ik}(x)\right)^2,$$

we have, among other things,

$$(7.6)\quad {}^1\delta^{\alpha}_{p,2^q}(x) \leqq F(x)\,(q+2)^{1/2}.$$

Also, if we prove $F(x) \in L^2$, then B. Levi's theorem and (7.5) imply (7.4), in the case when $p \to \infty$. But this is the case, as term-wise integration shows:

$$\begin{aligned}\alpha^2\int F^2(x)d\mu(x) &= \sum_{p=0}^{\infty}\sum_{t=-2}^{\infty}\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{i=1}^{m}\sum_{k=2^t+1}^{2^{t+1}}\left(\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\right)^2 i^2a^2_{ik}\\ &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\sum_{i=1}^{m}\left(\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\right)^2 i^2a^2_{ik}\\ &\leqq 2\sum_{m=2}^{\infty}\sum_{k=0}^{\infty}\sum_{i=1}^{m}\left(\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\right)^2 ia^2_{ik}\\ &\leqq 2\sum_{i=1}^{\infty}\sum_{k=0}^{\infty} ia^2_{ik}\sum_{m=i}^{\infty}\left(\frac{A^{\alpha-1}_{m-i}}{A^{\alpha}_m}\right)^2 < \infty,\end{aligned}$$

the last series being finite due to (4.8) and (1.2).

In the case when $q \to \infty$, (7.4) is an immediate consequence of (7.6).

Step 2. Given $n \geqq 3$, there exists an integer $q \geqq 1$ such that $2^q < n \leqq 2^{q+1}$. Since

$$ {}^1\delta^\alpha_{pn}(x) \leqq {}^1\delta^\alpha_{p,2^q}(x) + \left\{\frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} \left(\sum_{i=1}^{m} \sum_{k=2^q+1}^{n} \frac{A^{\alpha-1}_{m-i}}{\alpha A^\alpha_m} i a_{ik}\phi_{ik}(x)\right)^2\right\}^{1/2} $$

(cf. the first equality in (7.5)), we can estimate

$$ \max_{2^q<n\leqq 2^{q+1}} {}^1\delta^\alpha_{pn}(x) \leqq {}^1\delta^\alpha_{p,2^q}(x) + M^\alpha_{pq}(x), \tag{7.7} $$

where

$$ (M^\alpha_{pq}(x))^2 = \frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} \max_{2^q<n\leqq 2^{q+1}} \left(\sum_{i=1}^{m} \sum_{k=2^q+1}^{n} \frac{A^{\alpha-1}_{m-i}}{\alpha A^\alpha_m} i a_{ik}\phi_{ik}(x)\right)^2. $$

Applying the Rademacher-Menšov inequality (see [**1**, p. 79] or [**2**, Theorem 3]) separately for each fixed m, we get

$$ \alpha^2 \int (M^\alpha_{pq}(x))^2\, d\mu(x) \leqq \frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} (\log 2^{q+1})^2 \sum_{i=1}^{m} \sum_{k=2^q+1}^{2^{q+1}} \left(\frac{A^{\alpha-1}_{m-i}}{A^\alpha_m}\right)^2 i^2 a^2_{ik}. $$

Consequently, again by (4.8) and (1.2),

$$ \begin{aligned} &\sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{\alpha^2}{(q+1)^2} \int (M^\alpha_{pq}(x))^2\, d\mu(x) \\ &\leqq \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} \frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} \sum_{i=1}^{m} \sum_{k=2^q+1}^{2^{q+1}} \left(\frac{A^{\alpha-1}_{m-i}}{A^\alpha_m}\right)^2 i^2 a^2_{ik} \\ &= \sum_{p=0}^{\infty} \sum_{k=3}^{\infty} \frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} \sum_{i=1}^{m} \left(\frac{A^{\alpha-1}_{m-i}}{A^\alpha_m}\right)^2 i^2 a^2_{ik} \\ &\leqq 2 \sum_{m=2}^{\infty} \sum_{k=3}^{\infty} \sum_{i=1}^{m} \left(\frac{A^{\alpha-1}_{m-i}}{A^\alpha_m}\right)^2 i a^2_{ik} \\ &\leqq 2 \sum_{i=1}^{\infty} \sum_{k=3}^{\infty} i a^2_{ik} \sum_{m=i}^{\infty} \left(\frac{A^{\alpha-1}_{m-i}}{A^\alpha_m}\right)^2 < \infty. \end{aligned} $$

Hence B. Levi's theorem implies that

$$ M^\alpha_{pq}(x) = o_x\{q+1\} \quad \text{a.e.} \tag{7.8} $$

Combining (7.4), (7.7) and (7.8) we find (7.3) to be proved.

8. Proof of Theorem 4. By the triangle inequality,

$$\varepsilon^{\alpha\beta}_{MN}(x)$$
$$\leqq \left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}(\sigma^{\alpha-1,\beta-1}_{mn}(x)-\sigma^{\alpha-1,\beta}_{mn}(x)-\sigma^{\alpha,\beta-1}_{mn}(x)+\sigma^{\alpha\beta}_{mn}(x))^2\right\}^{1/2}$$
$$+\left\{\frac{1}{(M+1)(N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}(\sigma^{\alpha-1,\beta}_{mn}(x)-\sigma^{\alpha\beta}_{mn}(x))^2\right\}^{1/2}$$
$$+\left\{\frac{1}{(M+1))N+1)}\sum_{m=0}^{M}\sum_{n=0}^{N}[\sigma^{\alpha,\beta-1}_{mn}(x)-\sigma^{\alpha\beta}_{mn}(x)]^2\right\}^{1/2}$$
$$= {}^1\varepsilon^{\alpha\beta}_{MN}(x) + {}^2\varepsilon^{\alpha\beta}_{MN}(x) + {}^3\varepsilon^{\alpha\beta}_{MN}(x).$$

Thus, Theorem 4 will be proved through the following Propositions 1–3.

PROPOSITION 1. *If* $\alpha > 1/2$, $\beta > 1/2$ *and condition* (1.2) *is satisfied, then*

$$ {}^1\varepsilon^{\alpha\beta}_{MN}(x) = o_x\{1\} \quad \text{a.e.}$$

This proposition was proved in [5].

PROPOSITION 2. *If* $\alpha > 1/2$, $\beta > 0$ *and condition* (1.2) *is satisfied, then*

$$ {}^2\varepsilon^{\alpha\beta}_{MN}(x) = o_x\{\log\log(N+4)\} \quad \text{a.e.} \tag{8.1}$$

The next symmetric counterpart of Proposition 2 can be derived in a similar way.

PROPOSITION 3. *If* $\alpha > 0$, $\beta > 1/2$ *and conditon* (1.2) *is satisfied, then*

$$ {}^3\varepsilon^{\alpha\beta}_{MN}(x) = o_x\{\log\log(M+4)\} \quad \text{a.e.}$$

PROOF OF PROPOSITION 2. Since, again,

$$ {}^2\varepsilon^{\alpha\beta}_{MN}(x) \leqq 2\,{}^2\varepsilon^{\alpha\beta}_{2^p,2^q}(x),$$

for $2^{p-1} < M \leqq 2^p$ and $2^{q-1} < N \leqq 2^q$ with $p, q \geqq 0$ (excluding the cases $M = 0$ or $N = 0$ for the sake of simplicity in notation), instead of (8.1) it is sufficient to prove

$$ {}^2\varepsilon^{\alpha\beta}_{2^p,2^q}(x) = o_x\{\log(q+2)\} \quad \text{a.e.} \tag{8.2}$$

We can insert one more simplifying step. Clearly,

$$({}^2\varepsilon^{\alpha\beta}_{2^p,2^q}(x))^2$$
$$= \sum_{r=-2}^{p-1}\sum_{t=-2}^{q-1} 2^{r+t-p-q+4} \times \frac{1}{2^{r+2}\,2^{t+2}}\sum_{m=2^r+1}^{2^{r+1}}\sum_{n=2^t+1}^{2^{t+1}}(\sigma^{\alpha-1,\beta}_{mn}(x)-\sigma^{\alpha\beta}_{mn}(x))^2,$$

with the same convention concerning 2^r and 2^t for $r, t = -2$ and -1 as we made after (7.2). Thus, in order to prove (8.2) we have to prove

$$
\begin{aligned}
{}^4\varepsilon_{pq}^{\alpha\beta}(x) &= \left\{\frac{1}{2^p 2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}(\sigma_{mn}^{\alpha-1,\beta}(x)-\sigma_{mn}^{\alpha\beta}(x))^2\right\}^{1/2} \\
&= o_x\{\log(q+2)\} \quad \text{a.e.}
\end{aligned} \tag{8.3}
$$

Using representation (4.7), we can split ${}^4\varepsilon_{pq}(x)$ into three parts as follows

$$
\begin{aligned}
{}^4\varepsilon_{pq}^{\alpha\beta}(x) \leqq & \left\{\frac{1}{2^p}\sum_{m=2^p+1}^{2^{p+1}}\left(\sum_{i=1}^{m}\sum_{k=0}^{2^q}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha} i a_{ik}\phi_{ik}(x)\right)^2\right\}^{1/2} \\
&+ \left\{\frac{1}{2^p 2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\left(\sum_{i=1}^{m}\sum_{k=2^q+1}^{n}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha} i a_{ik}\phi_{ik}(x)\right)^2\right\}^{1/2} \\
&+ \left\{\frac{1}{2^p 2^q}\sum_{m=2^p+1}^{2^{p+1}}\sum_{n=2^q+1}^{2^{q+1}}\left(\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{A_{m-i}^{\alpha-1}}{\alpha A_m^\alpha}\left(1-\frac{A_{n-k}^\beta}{A_n^\beta}\right) i a_{ik}\phi_{ik}(x)\right)^2\right\}^{1/2} \\
= & {}^1\delta_{p,2^q}^\alpha(x) + {}^2\delta_{pq}^\alpha(x) + {}^5\varepsilon_{pq}^{\alpha\beta}(x),
\end{aligned}
$$

where ${}^1\delta_{pn}^\alpha(x)$ was already defined in (7.3) (as to the representation of ${}^1\delta_{p,2^q}^\alpha(x)$, see the first equality in (7.5)), while ${}^2\delta_{pq}^\alpha(x)$ and ${}^5\varepsilon_{pq}^{\alpha\beta}(x)$ are just now defined. On the basis of this decomposition, the next three lemmas will complete the proof of (8.3) and that of Proposition 2.

LEMMA 3. *If $\alpha > 1/2$ and condition* (1.2) *is satisfied, then*

$$
{}^1\delta_{p,2^q}^\alpha(x) = o_x\{\log(q+2)\} \quad \text{a.e.} \tag{8.4}
$$

PROOF. The statement of Lemma 3 is an easy consequence of (7.3). To this effect, we set

$$
a_{it}^* = \left\{\sum_{k=2^t+1}^{2^{t+1}} a_{ik}^2\right\}^{1/2} \quad (i = 0, 1, \ldots; t = -2, -1, 0, \ldots)
$$

and

$$
\phi_{it}^*(x) = \begin{cases} \dfrac{1}{a_{it}^*}\displaystyle\sum_{k=2^t+1}^{2^{t+1}} a_{ik}\phi_{ik}(x) & \text{if } a_{it}^* \neq 0, \\ \phi_{i,2^{t+1}}(x) & \text{if } a_{it}^* = 0 \end{cases}
$$

(keeping in mind the convention made after (7.2)). Obviously, $\{\phi_{it}^*(x)\colon i = 0, 1, \ldots; t = -2, -1, 0, \ldots\}$ is an orthonormal system and, by (1.2),

$$
\sum_{i=0}^{\infty}\sum_{t=-2}^{\infty}(a_{it}^*)^2 < \infty.
$$

So we can apply Theorem 3 and obtain (7.3), which in this case says

$$
{}^1\Delta_{pq}^\alpha(x) = o_x\{\log(q+2)\} \quad \text{a.e.,} \tag{8.5}
$$

where

$$
{}^1\Delta^\alpha_{pq}(x) = \left\{\frac{1}{2^p} \sum_{m=2^p+1}^{2^{p+1}} \left(\sum_{i=1}^{m} \sum_{t=-2}^{q} \frac{A^{\alpha-1}_{m-i}}{\alpha A^\alpha_m} i a^*_{it}\phi_{it}(x)\right)^2\right\}^{1/2}
$$
$$
= {}^1\delta^\alpha_{p,\,2^q+1}(x).
$$

By this, (8.5) is equivalent to (8.4) to be proved.

LEMMA 4. *If $\alpha > 1/2$ and condition* (1.2) *is satisfied, then*

$$
{}^2\delta^\alpha_{pq}(x) = o_x\{1\} \quad \text{a.e.}
$$

LEMMA 5. *If $\alpha > 1/2$, $\beta > 0$ and condition* (1.2) *is satisfied, then*

$$
{}^5\varepsilon^{\alpha\beta}_{pq}(x) = o_x\{1\} \quad \text{a.e.}
$$

Both Lemma 4 and Lemma 5 were proved in **[5]**.

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