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# SOME CHOQUET THEOREMS*

JOSÉ ANTONIO VILLASANA O.

ABSTRACT. A geometric and equivalent version of a fundamental analytic compact Choquet theorem is proved. This new theorem is then strengthened and, furthermore, an analogous non-compact version of this last geometric theorem is demonstrated. These theorems introduce a new point of view. Other results are proved as well.

**1. Introduction.** Fundamental to a good part of Choquet's representation theory are the Choquet-Bishop-de Leeuw and Choquet-Meyer theorems, since the former ensures the existence of representing boundary measures in the important geometric compact case, and the latter gives necessary and sufficient conditions for the uniqueness of such representation. More precisely, these theorems state that:

THEOREM 1.1. (CHOQUET-BISHOP-DE LEEUW). *Let $K$ be a compact convex subset of a (real or complex) locally convex topological vector space. Then every point of $K$ is the barycenter of a maximal probability measure on $K$.*

THEOREM 1.2. (CHOQUET-MEYER). *Every point of $K$ is the barycenter of a unique maximal probability measure on $K \Leftrightarrow K$ is a simplex.*

If $E$ denotes the locally convex topological vector space that contains $K$, and $E^*$ its topological dual, then the point $x \in K$ is said to be the barycenter of $\mu$ (a probability measure on $K$) provided $x$ is the resultant of $\mu$ (written $x = r(\mu)$); i.e., $\ell(x) = \int_K \ell|_K d\mu \ \forall \ell \in E^*$ [12]. Denote $\{\rho \in \mathbf{R}: \rho \geqq 0\}$ by $\mathbf{R}^+$; a convex subset $C$ of a real or complex vector space $V$ is said to be a simplex $\Leftrightarrow$ the proper cone $\mathbf{R}^+(C \times 1)$ is a lattice in the partial ordering which it induces on $V \times \mathbf{R}$. Probability and maximal measures are defined in the first of the next section definitions; almost all the notational conventions used in this paper are explained in the definitions of (and elsewhere in) the following section. More ample in-

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formation about Choquet theory can be found in **[1]**, **[3]** and **[13]**.

The Choquet-Bishop-de Leeuw and Choquet-Meyer theorems can be equivalently stated in the analytic form of the ensuing theorem.

THEOREM 1.3. *Let* $(X, \tau)$ *be a compact Hausdorff topological space,* $A \subset C_{\mathbf{R}}(X)$ *a vector subspace that separates points and contains the constants. Then* $\forall\, \Lambda \in A^*$ *there exists* $\mu \in M_{\mathbf{R}}(X)$ *such that:*

(i) $\mu$ *is an A-boundary measure;*

(ii) $\mu$ *represents* $\Lambda$;

(iii) $\| \mu \| = \| \Lambda \|$; *and*

(iv) $\mu \in M_1^+(X) \Leftrightarrow \Lambda \in S_A$, *the state space of A.*

*Uniqueness holds for* $A \Leftrightarrow S_A$ *is a simplex.*

Remember that, by definition, $S_A = \{\Lambda \in A^*\colon \Lambda(1) = 1 = \| \Lambda \|\}$ and $\mu$ represents $\Lambda \Leftrightarrow \Lambda(f) = \int_X f d\mu \;\forall\, f \in A$. By the Banach-Alaoglu theorem, the convex set $U_{A^*} = \{\Lambda \in A^*\colon \| \Lambda \| \leqq 1\}$, the closed unit ball of $A^*$, is $w^*$-compact ($\| \;\|$ denotes here the usual norm of $A^*$). Let $\phi_A\colon X \to A^*$ denote the evaluation map (i.e. $\phi_A(x)\colon {}^{A \to \mathbf{R}}_{f \to f(x)}$, for every $x \in X$), which is (in this case) an embedding of $X$ into $A^*$. Notice that $\phi_A(X) \subset S_A \subset U_{A^*}$. Thus, the measure $\mu \in M_{\mathbf{R}}(X)$ is said to be an $A$-boundary measure $\Leftrightarrow$ $\phi_A(\mu) = \mu \circ (\phi_A \mid^{U_{A^*}})^{-1} \in M_{\mathbf{R}}^b(U_{A^*})$. Uniqueness holds for $A \Leftrightarrow \forall\, \Lambda \in A^*$ there exists a unique $A$-boundary measure $\mu$ that represents $\Lambda$ and such that $\|\mu\| = |\mu|(X) = \|\Lambda\|$, where $|\mu|$ denotes the total variation (measure) of $\mu$. Notice that $\mu$ represents $\Lambda \Leftrightarrow r(\phi_A(\mu)) = \Lambda$ in $A^*$ with its $w^*$-topology.

Theorem 1.3 can, in turn, be extended (modifying accordingly the preceding definitions) in order to include both the real and the complex cases, and the case $1 \notin A$, too. For stating this theorem a couple of additional definitions are necessary, namely (1) definition 2.3 of the next section and (2) the convex subset $C$ of a real or complex vector space is said to be a simplexoid $\Leftrightarrow$ every proper (and non-void) extremal subset of $C$ is a simplex.

THEOREM **1.4**. [**14**; 2.2, 3.2 & 3.3]. *Let* $(X, \tau)$ *be a compact Hausdorff topological space,* $A \subset C_{\mathbf{F}}(X)$ *a vector subspace that separates points. Then* $\forall\, \Lambda \in A^*$ *there exists* $\mu \in M_{\mathbf{F}}(X)$ *such that:*

(i) $\mu$ *is an A-boundary measure;*

(ii) $\mu$ *represents* $\Lambda$; *and*

(iii) $\|\mu\| = \|\Lambda\|$.

*Uniqueness holds modulo* $\approx_A$ *for* $A \Leftrightarrow U_{A^*}$ *is a simplexoid.*

In the present paper an equivalent geometric version of the above fundamental analytic theorem is demonstrated (Theorem 3.1). This new theorem is then sharpened and proved in a different and simpler way (Theorem 3.2), and its existence part is shown to be equivalent to the

Choquet-Bishop-de Leeuw theorem; its uniqueness part is equivalent to a Choquet-Meyer-type characterization theorem (Theorem 3.3). Subsequently an analogous non-compact version of Theorem 3.2 is demonstrated making use of very similar methods (Theorem 3.4); the uniqueness part of this theorem is analogously equivalent to a Choquet-Meyer-type theorem (Theorem 3.6). These theorems, which constitute the paper's most important results and appear in the third section, introduce a new point of view. §3 is preceded by a section composed of necessary, but for the most part, essentially known material. The only proposition of the fourth and last section, studies some properties of compact absolutely convex sets and their affine spans, that are relevant to Choquet theory.

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**2. Some necessary definitions and results.** Gathered in this section are, for convenience, definitions and lemmas which shall be refered to later, and which are commented upon briefly, if at all.

DEFINITION 2.1. Let $(X, \tau)$ be a topological space, and denote by $\mathbf{F}$ (as it shall always denote throughout this paper) either $\mathbf{R}$ or C. Then define

(i) $C_{\mathbf{F}}(X) = \{f:X \to \mathbf{F} | f \text{ is continuous and bounded}\}$

$B_{\mathbf{F}}(X) = \{f:X \to \mathbf{F} | f \text{ is Borel and bounded}\}$

(ii) $M_{\mathbf{F}}(X) = \{\mu:B_X \to \mathbf{F} | \mu \text{ is a regular finite measure}\}$ ($B_X$ denotes the $\sigma$-algebra generated by $\tau$, i.e., the Borel subsets of $X$)

$M^+(X) = \{\mu \in M_{\mathbf{R}}(X)\colon \mu \geqq 0\}$

$M_1^+(X) = \{\mu \in M^+(X) : \|\mu\| = 1\}$, the convex set of probability measures on $X$.

$m_{\mathbf{F}}(X) = \{\mu:B_X \to \mathbf{F} \,|\, \mu \text{ is finitely additive and finite}\}$

(iii) In the case that $\chi = K$, where $K$ is a compact convex subset of a locally convex topological vector space $E$, and $\tau$ is the topology induced by that of $E$, then define

$\partial_e K = \{x \in K:x \text{ is an extreme point of } K\}$, the extreme boundary of $K$.

$A_{\mathbf{F}}(K) = \{f \in C_{\mathbf{F}}(K) : f \text{ is affine}\}$

Choquet's partial ordering $\prec$ on $M^+(K)$: if $\mu, \nu \in M^+(K)$, then $\mu \prec \nu \Leftrightarrow \int_K f d\mu \leqq \int_K f d\nu$, for every continuous (bounded) convex function $f$ on $K$.

$\mu \in M^+(K)$ is said to be a maximal measure $\Leftrightarrow$ $\mu$ is maximal in Choquet's partial ordering. Denote by $M_m(K)$ the cone of maximal measures on $K$.

$\lambda \in M_{\mathbf{F}}(K)$ is said to be a boundary measure $\Leftrightarrow |\lambda| \in M_m(K)$. Denote by $M_{\mathbf{F}}^{b}(K)$ the vector subspace of boundary measures on $K$.

(iv) In the case that $(X, \tau)$ is metrizable, then define

${}_tM_{\mathbf{F}}(X) = \{\mu \in M_{\mathbf{F}}(X)\colon \mu$ has $\sigma$-compact support$\}$, the vector subspace of tight measures on $X$.

${}_tM^{+}(X) = M^{+}(X) \cap {}_tM_{\mathbf{R}}(X)$

${}_tM_1^{+}(X) = M_1^{+}(X) \cap {}_tM^{+}(X)$

In particular, if $X = K$, where $K$ is a closed bounded convex subset of a Banach space $E$, and if $\tau$ is the topology induced by that of $E$, then define analogously Choquet's partial ordering $\prec$ on ${}_tM^{+}(K)$ [**2**; 2.9 and note on page 176], and analogously denote by ${}_tM_m(K)$ the cone of maximal tight mesures on $K$.

Remark and Notation. Remember that $B_{\mathbf{F}}(X)$ with the supremum norm $\| \ \|_\infty$ and $m_{\mathbf{F}}(X)$ with the norm $\mu \to |\mu|(X)$, are Banach spaces; that $B_{\mathbf{F}}(X)^*$ normed in the usual way is also a Banach space, and that the linear transformation

$$\begin{array}{lll} \Lambda_X\colon m_{\mathbf{F}}(X) \to B_{\mathbf{F}}(X)^*, & \text{where } \Lambda_\mu\colon B_{\mathbf{F}}(X) \to \mathbf{F} \\ \qquad \mu \quad \to \quad \Lambda_\mu & \qquad\qquad f \to \int_X f d\mu \end{array}$$

is an isomorphism and also an isometry [**17**; 7.9-A]; that is, $m_{\mathbf{F}}(X)$ and $B_{\mathbf{F}}(X)^*$ are congruent. Moreover, if $(X, \tau)$ is a compact Hausdorff topological space and if $M_{\mathbf{F}}(X)$ and $C_{\mathbf{F}}(X)^*$ are likewise normed, then they are congruent, too (this is the Riesz representation theorem [**6**; IV.6.3]).

Definition 2.2. [**8**; §4]. Let $K$ be a compact absolutely convex (i.e., $K$ is convex and balanced) subset of a locally convex topological vector space over $\mathbf{F}$, $V$ a vector space over $\mathbf{F}$, and denote $\{\alpha \in \mathbf{F}\colon |\alpha| = 1\}$ by $T_{\mathbf{F}}$. Let $\theta_{\mathbf{F}}\colon BT_{\mathbf{F}} \to \mathbf{R}$ be Haar's probability measure on $T_{\mathbf{F}}$, and define $\sigma_\alpha\colon {K \to K \atop x \to \alpha x}$, for every $\alpha \in T_{\mathbf{F}}$.

(i) $f\colon K \to V$ is said to be $T_{\mathbf{F}}$-homogeneous $\Leftrightarrow f(\alpha x) = \alpha f(x)$ for every $\alpha \in T_{\mathbf{F}}$, and every $x \in K$.

$\mu \in m_{\mathbf{F}}(K)$ is said to be $T_{\mathbf{F}}$-homogeneous $\Leftrightarrow \mu \circ \sigma_\alpha^{-1} = \alpha\mu$ for every $\alpha \in T_{\mathbf{F}}$.

Denote by $C_{\mathbf{F}}^{\mathrm{hom}}(K)$, $B_{\mathbf{F}}^{\mathrm{hom}}(K)$, $m_{\mathbf{F}}^{\mathrm{hom}}(K)$ and $M_{\mathbf{F}}^{\mathrm{hom}}(K)$ the corresponding vector subspaces of $T_{\mathbf{F}}$-homogeneous functions and measures.

(ii) Define, for every $f \in B_{\mathbf{F}}(K)$,

$$\begin{array}{l} \hom_{T_{\mathbf{F}}}(f)\colon K \to \mathbf{F} \\ \qquad\qquad x \to \int_{T_{\mathbf{F}}} \alpha^{-1} f(\alpha x) d\theta_{\mathbf{F}}(\alpha) \end{array}$$

(iii) The linear transformation $\hom_{T_{\mathbf{F}}}\colon B_{\mathbf{F}}(K) \to B_{\mathbf{F}}(K)$ is a projection of $B_{\mathbf{F}}(K)$ onto $B_{\mathbf{F}}^{\mathrm{hom}}(K)$, and its adjoint

$$\begin{array}{lll} (\hom_{T_{\mathbf{F}}})^*\colon m_{\mathbf{F}}(K) \to & m_{\mathbf{F}}(K) \\ \qquad\qquad \mu & \to \Lambda_K^{-1}(\Lambda_\mu \circ \hom_{T_{\mathbf{F}}}) \end{array}$$

is a projection of $m_{\mathbf{F}}(K)$ onto $m_{\mathbf{F}}^{\mathrm{hom}}(K)$.

DEFINITION 2.3. Let $K$ be a compact absolutely convex subset of the locally convex topological vector space $E$ over $\mathbf{F}$, and let $\mu, \nu \in M_{\mathbf{F}}(K)$. Define $\mu \approx_K \nu \Leftrightarrow (\mathrm{hom}_{T_{\mathbf{F}}})^*(\mu) = (\mathrm{hom}_{T_{\mathbf{F}}})^*(\nu)$. It is said that there exists a unique $\mu$ modulo $\approx_K$ that satisfies a condition $C \Leftrightarrow [\nu$ satisfies $C \Rightarrow \nu \approx_K \mu]$.

Let $(X, \tau)$ be a compact Hausdorff topological space, $A \subset C_{\mathbf{F}}(X)$ a vector subspace that separates points, and let $\lambda, \upsilon \in M_{\mathbf{F}}(X)$. Define $\lambda \approx_A \upsilon \Leftrightarrow \phi_A(\lambda) \approx_{U_{A^*}} \phi_A(\upsilon)$. It is said that there exists a unique $\lambda$ modulo $\approx_A$ that satisfies a condition $C \Leftrightarrow [\upsilon$ satisfies $C \Rightarrow \upsilon \approx_A \lambda]$. Uniqueness holds modulo $\approx_A$ for $A \Leftrightarrow \forall\, \Lambda \in A^*$ there exists a unique $A$-boundary measure $\lambda \in M_{\mathbf{F}}(X)$ modulo $\approx_A$, such that $\lambda$ represents $\Lambda$ and $\|\lambda\| = \|\Lambda\|$.

REMARKS. The above definition of the equivalence relation $\approx_A$ is precisely the conclusion of Proposition 3.5 of [**14**]. On the other hand, the equivalence relations $\approx_K$ and $\approx_A$ do coincide in the following especial, and relevant, case, i.e., $X = K$, $\tau$ is the topology induced by that of $E$, and $A = E^*|_K = \{\ell|_K : \ell \in E^*\}$ or $A = A_{\mathbf{F}}(K) \cap C_{\mathbf{F}}^{\mathrm{hom}}(K)$, since in this case $\phi_A|^{U_{A^*}}$ is an affine, $T_{\mathbf{F}}$-homogeneous homeomorphism. This also implies that, for all $x \in K$, $\phi_A(x)$ is represented by the $A$-boundary measure $\mu \in M_{\mathbf{F}}(K) \Leftrightarrow x = r(\mu)$ and $\mu \in M_{\mathbf{F}}^{b}(K)$.

DEFINITION 2.4. [**18**; definition 3.2]. Let $C$ be a subset of the vector space $V$ over $\mathbf{F}$, and suppose that $C$ is star-shaped relative to $0 \in C$ (that is, for all $x \in C$ and for all $0 < \rho < 1$ it is true that $\rho x \in C$). Then define $p_C$, the generalized Minkowski functional of $C$ as

$$\begin{aligned} p_C\colon V &\to [0, +\infty] \\ x &\to \inf\{\rho > 0 : x \in \rho C\}. \end{aligned}$$

LEMMA 2.5. *Let $C$ be an absolutely convex subset of the vector space $V$ over $\mathbf{F}$, $V'$ another vector space over $\mathbf{F}$, $a\colon C \to V'$ affine and $T_{\mathbf{F}}$-homogeneous, and let $C' = a(C)$. Then*

$$(p_{C'} \circ a)^{-1}(1) \subset p_C^{-1}(1).$$

*If, in particular, $a$ is injective, then*

$$(p_{C'} \circ a)^{-1}(1) = p_C^{-1}(1).$$

The proof of this lemma is an immediate consequence of the definitions and shall, therefore, be omitted.

DEFINITION 2.6. Let $V$ denote a vector space over $\mathbf{F}$. Define the affine span of $D \subset V$, aff$(D)$, as follows:

$$\mathrm{aff}(D) = \{\sum_{i=1}^{n} \alpha_i v_i : \alpha_i \in \mathbf{F},\ v_i \in D,\ \sum_{i=1}^{n} \alpha_i = 1\}.$$

REMARKS. Notice, first, that if $D$ is convex and $\mathbf{F} = \mathbf{R}$, then $\mathrm{aff}(D) = \{\alpha_1 v_1 - \alpha_2 v_2 : \alpha_1, \alpha_2 \geqq 0 \text{ and } \alpha_1 - \alpha_2 = 1;\ v_1, v_2 \in D\}$. Second, if $D$ is absolutely convex, then it necessarily follows that $\mathrm{aff}(D) = \mathbf{F}D = \mathbf{R}^+D$, and, thus, in this case $p_D|_{\mathrm{aff}(D)}$ is a seminorm.

DEFINITION 2.7. [**11**; §2]. Let $(X, \tau)$ be a compact Hausdorff topological space, $A \subset C_{\mathbf{F}}(X)$ a separating vector subspace, and let $\pi_1: T_{\mathbf{F}} \times X \to T_{\mathbf{F}}$; $\pi_2: T_{\mathbf{F}} \times X \to X$ be the canonical projections. Define

(i) $\Phi_A = \pi_1(\phi_A \circ \pi_2): T_{\mathbf{F}} \times X \to A^*$,

(ii) $\mathrm{L}: B_{\mathbf{F}}(X) \to B_{\mathbf{F}}(T_{\mathbf{F}} \times X)$,
$\qquad f \to \pi_1(f \circ \pi_2)$

(iii) $\mathrm{L}^*: m_{\mathbf{F}}(T_{\mathbf{F}} \times X) \to m_{\mathbf{F}}(X)$, the adjoint of L.
$\qquad \mu \to \Lambda_X^{-1}(\Lambda_\mu \circ \mathrm{L})$

REMARK AND NOTATION. $\Phi_A$ always has a Borel right inverse $\psi_A$: $T_{\mathbf{F}}\, \phi_A(X) \to (T_{\mathbf{F}} \times X)$ [**9**; 7.2].

LEMMA 2.8. *Let $(X, d)$ be a complete metric space and let* $(\mathrm{Lip}_{\mathbf{R}}(X), \|\ \|_{\mathrm{Lip}})$ *denote the Banach space of all bounded (continuous) Lipschitz real valued Functions on $X$; that is*

$$\mathrm{Lip}_{\mathbf{R}}(X) = \{f \in C_{\mathbf{R}}(X) : \|f\|_{\mathrm{Lip}} < +\infty\},$$

*where*

$$\|f\|_{\mathrm{Lip}} = \max\ \{\|f\|_\infty, \sup\{|f(x) - f(y)|/d(x, y) : x, y \in X;\ x \neq y\}\}.$$

*Then*

(i) *The linear transformation*

$$\Lambda_X|_{{}_tM_{\mathbf{R}}(X)} : {}_tM_{\mathbf{R}}(X) \to \mathrm{Lip}_{\mathbf{R}}(X)^*$$
$$\mu \quad \to \quad \Lambda_\mu$$

*is a monomorphism and $\|\Lambda_\mu\| = \|\mu\|$, for all $\mu \in {}_tM^+(X)$.*

(ii) *$\Lambda_X({}_tM_1^+(X))$ is a closed and bounded convex subset of* $\mathrm{Lip}_{\mathbf{R}}(X)^*$.

(iii) *If $\{\mu_\alpha\}_{\alpha \in \Omega}$ is a net in ${}_tM^+(X)$ and $\mu \in {}_tM^+(X)$, then $\Lambda_{\mu_\alpha} \to \Lambda_\mu$ in the norm of* $\mathrm{Lip}_{\mathbf{R}}(X)^* \Leftrightarrow \int_X f d\mu_\alpha \to \int_X f d\mu$, *for all $f \in C_{\mathbf{R}}(X)$.*

For the proof of the first part of i) consult [**5**; Lemma 6]; for the proof of the second part, notice that, for every $\mu \in {}_tM_{\mathbf{R}}(X)$, it necessarily ensues that $\|\Lambda_\mu\| \leqq \|\mu\|$ since $\mu$ represents $\Lambda_\mu$. But if $\mu$ happens to be positive, then $|\Lambda_\mu(1)| = \|\mu\| \Rightarrow \|\Lambda_\mu\| \geqq \|\mu\|$ in this case. For the proof of ii) and iii) consult [**5**; Theorems 9 and 18]; see also [**7**; 1.2] and [**16**; §1].

## 3. Theorems.

THEOREM 3.1. *Let $K$ be a compact absolutely convex subset of the locally convex topological vector space $E$ over* $\mathbf{F}$, *and let $x \in K$. Then there exists $\mu \in M_{\mathbf{F}}^b(K)$ such that $r(\mu) = x$ and $\|\mu\| = p_K(x)$.*

*Such representation is unique modulo* $\approx_K$ $\forall\, x \in K \Leftrightarrow K$ *is a simplexoid.*

PROOF. Let $A$ denote either $E^*|_K$ or $A_{\mathbf{F}}(K) \cap C_{\mathbf{F}}^{\mathrm{hom}}(K)$ throughout the proof. Since $K$ is necessarily bounded, it follows that $p_K|_{\mathrm{aff}(K)}$ is a norm, which implies that $p_K(y)^{-1} y \in p_K^{-1}(1) \subset K$, for every $y \in K\backslash\{0\}$. Thus, $\phi_A$ is an isometry onto $U_{A^*}$ as a consequence of Lemma 2.5 (i.e., $\|\phi_A(x)\| = p_K(x)$ $\forall x \in K$). By Theorem 1.4, there exists an $A$-boundary measure $\mu \in M_{\mathbf{F}}(K)$ that represents $\phi_A(x)$ and such that $\|\mu\| = \|\phi_A(x)\|$. But it was remarked after definition 2.3 that $\phi_A(x)$ is represented by the $A$-boundary measure $\mu \Leftrightarrow x = r(\mu)$ and $\mu \in M_{\mathbf{F}}^{\natural}(K)$. Thus, the first part of the proof is complete.

For every $x \in K$, let $M_x$ be $\{\mu \in M_{\mathbf{F}}^{\natural}(K)\colon r(\mu) = x \text{ and } \|\mu\| = p_K(x)\}$. Since $\phi_A|^{U_{A^*}}$ is in this case an affine, $T_{\mathbf{F}}$-homogeneous homeomorphism, and an isometry as well (in the above mentioned sense), and since $\approx_K = \approx_A$, it necessarily follows that, on the one hand $(M_x)/\approx_K$ contains a single point $\forall\, x \in K \Leftrightarrow \forall\, \Lambda \in U_{A^*}$ there exists a unique $A$-boundary measure $\mu \in M_{\mathbf{F}}(K)$ modulo $\approx_A$ such that $\mu$ represents $\Lambda$ and $\|\mu\| = \|\Lambda\|$; and on the other hand, that $U_{A^*}$ is a simplexoid $\Leftrightarrow K$ is a simplexoid. By Theorem 1.4, it is known that uniqueness holds modulo $\approx_A$ for $A \Leftrightarrow U_{A^*}$ is a simplexoid. But it is obvious that uniqueness holds modulo $\approx_A$ for $A \Leftrightarrow \forall\, \Lambda \in U_{A^*}$ there exists a unique $A$-boundary measure $\mu \in M_{\mathbf{F}}(K)$ modulo $\approx_A$ such that $\mu$ represents $\Lambda$ and $\|\mu\| = \|\Lambda\|$. Thus, it can be concluded that $(M_x)/\approx_K$ contains a single point $\forall\, x \in K \Leftrightarrow K$ is a simplexoid.

Theorem 3.1 is equivalent to Theorem 1.4. Indeed, since Theorem 3.1 was proved with the aid of Theorem 1.4, it is plain that the latter implies the former. Suppose now that Theorem 3.1 holds and let $(X, \tau)$ and $A \subset C_{\mathbf{F}}(X)$ be as in the statement of Theorem 1.4. It is clear that it is sufficient to consider only the case where $\Lambda \in A^*$ is such that $\|\Lambda\| = 1$; by Theorem 3.1, there exists $\lambda \in M_{\mathbf{F}}^{\natural}(U_{A^*})$ such that $\|\lambda\| = 1$ and $r(\lambda) = \Lambda$ in $A^*$ with its $w^*$-topology. Then, the measure $\mu = \mathrm{L}^*(\lambda \circ \phi_A^{-1}) \in M_{\mathbf{F}}(X)$ can be proved to be an $A$-boundary measure that represents $\Lambda$ and that satisfies the equality $\|\mu\| = 1$, in the same way it is done in the demonstration of the existence part of Theorem 1.4 employing the Choquet-Bishop-de Leeuw theorem instead [**9**; 7.3]. In regard to the uniqueness part of the theorem, notice that uniqueness holds modulo $\approx_A$ for $A \Leftrightarrow \forall\, \Lambda \in U_{A^*}$ there exists a unique $A$-boundary measure $\mu \in M_{\mathbf{F}}(X)$ modulo $\approx_A$, such that $\mu$ represents $\Lambda$ and $\|\mu\| = \|\Lambda\| \Leftrightarrow \forall\, \Lambda \in U_{A^*}$ there exists a unique $\lambda \in M_{\mathbf{F}}^{\natural}(U_{A^*})$ modulo $\approx_{U_{A^*}}$ such that $\|\lambda\| = \|\Lambda\|$ and $r(\lambda) = \Lambda$ in $A^*$ with its $w^*$-topology (for the proof of the last implication $\Rightarrow$ consult the demonstration of the necessity part of [**14**; 3.2]). But this last condition is equivalent to $U_{A^*}$ being a simplexoid, by Theorem 3.1. So, Theorem 3.1 implies Theorem 1.4.

Theorem 3.1 can be strengthened, making it even more similar to the Choquet-Bishop-de Leeuw and Choquet-Meyer theorems. This similarity is not merely superficial, but lies deeper since the Choquet-Bishop-de Leeuw theorem and the existence part of the ensuing theorem, are in fact equivalent, as it shall be later shown.

THEOREM 3.2. *Let $K$ be a compact absolutely convex subset of a locally convex topological vector space, and let $x \in K$. Then there exists $\mu \in M_m(K)$ such that $r(\mu) = x$ and $\|\mu\| = p_K(x)$.*

*Such representation is unique $\forall x \in K \Leftrightarrow K$ is a simplexoid.*

Only the existence part of this theorem shall be demonstrated, since its uniqueness part is equivalent to the following characterization theorem which was actually proved by Fuhr and Phelps [**9**; 3.10 & 3.11] in an only apparently more limited situation (see Proposition 4.1 of the next section), and therefore will not be demonstrated here.

THEOREM 3.3. *Let $K$ be a compact absolutely convex subset of a locally convex topological vector space. Then $K$ is a simplexoid $\Leftrightarrow \forall\, x \in p_K^{-1}(1)$ there exists a unique $\mu \in M_1^+(K) \cap M_m(K)$ such that $r(\mu) = x$.*

PROOF OF THE EXISTENCE PART OF THEOREM 3.2. If $x = 0$, then choose $\mu = 0$. If $x \neq 0$, then $p_K(x)^{-1} x \in p_K^{-1}(1) \subset K$ and, by the Choquet-Bishop-de Leeuw theorem, there exists $\nu \in M_1^+(K) \cap M_m(K)$ such that $r(\nu) = p_K(x)^{-1}x \Rightarrow \mu = p_K(x)\, \nu \in M_m(K)$ is such that $r(\mu) = x$ and $\|\mu\| = p_K(x)$.

The existence part of Theorem 3.2 implies the Choquet-Bishop-de Leeuw theorem and, thus, in view of the above proof, it can be inferred that the latter is equivalent to the former. Indeed, if $K$ is a compact convex subset of a locally convex topological vector space and $x \in K$, then let $A = A_{\mathbf{R}}(K)$. By Theorem 3.2, there exists $\lambda \in M_m(U_{A^*})$ such that $r(\lambda) = \phi_A(x)$ in $A^*$ with its $w^*$-topology and $\|\lambda\| = p_{U_{A^*}}(\phi_A(x)) = \|\phi_A(x)\| = 1$. Since $\phi_A(x) \in S_A$ and $S_A$ is a closed extremal subset of $U_{A^*}$, $\lambda$ is thus necessarily supported by $S_A$ and hence by $\phi_A(X)$, because of its maximality [**14**; §1] which implies that $\mu = \mathrm{L}^*(\lambda \circ \phi_A^{-1}) = \lambda \circ \phi_A \in M_1^+(K)$ is an $A$-boundary measure that represents $\phi_A(x) \Leftrightarrow r(\mu) = x$ and $\mu \in M_1^+(K) \cap M_m(K)$, which is the desired conclusion.

The point of view of Theorem 3.2 can serve as a model for similar theorems, like the following one. All these theorems can be easily extended in order to be applicable to translations of the appropriate absolutely convex subsets, and their conclusions applicable to all the points of their affine spans.

THEOREM 3.4. *Let $K$ be a closed and bounded absolutely convex subset of a Banach space having the Radon-Nikodým Property* [**4**], *and let $x \in K$.*

*Then there exists $\mu \in {}_tM_m(K)$ such that $r(\mu) = x$ as a Bochner integral (i.e., $x = \int_K y\, d\mu(y)$ [4]) and $\|\mu\| = p_K(x)$.*
*Such representation is unique $\forall x \in K \Leftrightarrow K$ is a simplexoid.*

The proof of this theorem is so similar to that of Theorem 3.2, that it shall be altogether omitted; instead of the Choquet-Bishop-de Leeuw theorem, one employs the existence part of Theorem 3.5 below, and, instead of Theorem 3.3, one has the characterization Theorem 3.6, which is demonstrated.

THEOREM 3.5. (EDGAR-BOURGIN [7; 4.6], [2; 4.3]). *Let $K$ be a closed bounded convex subset of a Banach space having the Radon-Nikodým Property, and let $x \in K$. Then there exists $\mu \in {}_tM_1^+(K) \cap {}_tM_m(K)$ such that $r(\mu) = x$ as a Bochner integral.*
*Such representation is unique $\forall x \in K \Leftrightarrow K$ is a simplex.*

THEOREM 3.6. *Let $K$ be a closed and bounded absolutely convex subset of a Banach space $E$ having the Radon-Nikodým Property. Then $K$ is a simplexoid $\Leftrightarrow \forall x \in p_K^{-1}(1)$ there exists a unique $\mu \in {}_tM_1^+(K) \cap {}_tM_m(K)$ such that $r(\mu) = x$ as a Bochner integral.*

PROOF. Suppose that $K$ is a simplexoid. Let discrete measure here mean one that is supported by an at most countable set and let barycenter of a probability tight measure on $K$ mean its resultant as a Bochner integral [7; §2]. Now, if $x \in p_K^{-1}(1)$ is the barycenter of $\mu^j \in {}_tM_1^+(K) \cap {}_tM_m(K)$, where $j = 1, 2$; and if $S_x$ is the necessarily proper extremal subset of $K$ generated by $x$ (which, by hypothesis, is therefore a simplex), then the existence of nets $\{\mu_\alpha^j\}_{\alpha \in \Omega}$, of discrete tight probability measures on $K$ with barycenters equal to $x$ and such that $\Lambda_K(\mu_\alpha^j) \to \Lambda_K(\mu^j)$ in the norm of $\mathrm{Lip}_{\mathbf{R}}(K)^*$, can be asserted (employing Lemma 2.8, modify appropriately the known procedure [13; 9.6] in order to apply it to the countable case, bearing in mind that the $\mu^j$ have $\sigma$-compact support). The fact that $S_x$ is an extremal subset of $K$, $r: {}_tM_{\mathbf{R}}(K) \to E$ is a continuous linear transformation [7; §2], and $K$ is a closed absolutely convex subset of $E$ together imply that every such measure is supported by an at most countable subset of $S_x$; the fact that $S_x$ is a simplex implies, by the decomposition lemma [13; 9.1 (iii)], the existence of a third net $\{\mu_\alpha\}_{\alpha \in \Omega}$ of discrete tight probability measures on $K$ with barycenters equal to $x$, and such that $\mu_\alpha^j \prec \mu_\alpha \; \forall \alpha \in \Omega$. Since $\mathrm{Lip}_{\mathbf{R}}(K)^*$ is a first countable topological vector space, one can assert the existence of subnets $\{\mu_{\alpha_n}^j\}_{n \in \mathbf{N}}$ such that $\lim_{n \to \infty} \Lambda_K(\mu_{\alpha_n}^j) = \Lambda_K(\mu^j)$. Now, since $S_x$ is a simplex, the decomposition lemma implies that the sequence $\{\mu_{\alpha_n}\}_{n \in \mathbf{N}}$ can be considered as monotonically increasing with respect to Choquet's partial ordering. Consequently, since $E$ has the Radon-Nikodým Property, this implies [7; 4.4] that $\{\Lambda_K(\mu_{\alpha_n})\}_{n \in \mathbf{N}}$ is convergent; say $\lim_{n \to \infty} \Lambda_K(\mu_{\alpha_n}) = \Lambda_\mu$, where, necessarily,

$\mu \in {}_tM_1^+(K)$, by Lemma 2.8 (ii). This implies, as a consequence of (iii) of the same Lemma, that $\mu^j \prec \mu$. But, since the $\mu^j$ were supposed to be maximal by hypothesis, it can then be concluded that $\mu^j = \mu$.

For the proof of the converse statement, it is sufficient to remark [**9**; 3.11] that the convex set ${}_tM_1^+(K) \cap {}_tM_m(K)$ is a simplex even if $E$ fails to have the Radon-Nikodým Property [**2**; proof of 4.2].

QUESTIONS. First, is the existence part of Theorem 3.4 equivalent to the existence part of Theorem 3.5? Second, are there analytic and equivalent versions of Theorems 3.4 and 3.5?

**4. A related result.** Since the absolutely convex sets have played a distinguished role in this paper, it is pertinent to point out certain properties of some of them.

PROPOSITION 4.1. *Let $K$ be a compact absolutely convex subset of the locally convex topological vector space $E$ over $\mathbf{F}$, and let $A$ denote either $E^*|_K$ or $A_{\mathbf{F}}(K) \cap C_{\mathbf{F}}^{\mathrm{hom}}(K)$.*

(i) *If $K$ has more than one point, then $\partial_e K \subset p_K^{-1}(1)$ and, therefore, $p_K^{-1}(1)$ is a boundary for $A$.*

(ii) *The linear transformation*

$$\mathscr{L}_A\colon \mathrm{aff}(K) \to A^*$$
$$\alpha x \to \alpha\phi_A(x)$$

*(where $\alpha \in \mathbf{F}$ and $x \in K$) is an isomorphism and a homeomorphism as well, provided* aff$(K)$ *has the topology induced by the weak one of $E$ and $A^*$ has its $\{\ell_f : f \in E^*|_K\}$-topology (where $\ell_f$: $\substack{A^* \to \mathbf{F} \\ \Lambda \to \Lambda(f)}$).*

*$\mathscr{L}_A$ is also an isometry, provided* aff$(K)$ *is normed with $p_K|_{\mathrm{aff}(K)}$ and $A^*$ is normed as usual.*

PROOF: (i). Since $K$ has more than one point by hypothesis, then necessarily $\partial_e U_{A^*} \subset p_{U_{A^*}}^{-1}(1)$, and it has already been remarked that in this case $\phi_A|^{U_{A^*}}$ is an affine $T_{\mathbf{F}}$-homogeneous homeomorphism. Thus, by Lemma 2.5,

$$\partial_e K = \phi_A^{-1}(\partial_e U_{A^*}) \subset (p_{U_{A^*}} \circ \phi_A)^{-1}(1) = p_K^{-1}(1).$$

But $\phi_A^{-1}(\partial_e U_{A^*})$ happens to be precisely the Choquet boundary for $A$ which is known to be a boundary for $A$ [**3**; 29.5 & 29.6], and, thus, this part of the demonstration is seen to be complete.

(ii). It is easy to verify that $\mathscr{L}_A$ is well defined and is indeed an isomorphism. In the proof of Theorem 3.1 it is shown why it is an isometry. So let $f \in E^*|_K$ be arbitrary and let $\ell \in E^*$ be such that $\ell|_K = f$. It is then true that $\ell|_{\mathrm{aff}(K)} = \ell_f \circ \mathscr{L}_A$, which implies that if $U \subset \mathbf{F}$ is open, then

$$\mathscr{L}_A^{-1}(\ell_f^{-1}(U)) = \ell^{-1}(U) \cap \mathrm{aff}(K)$$

is open in the topology of aff($K$) induced by the weak one of $E \Rightarrow \mathscr{L}_A$ is continuous.

Now let $\ell \in E^*$ be arbitrary and let $f \in E^*|_K$ be such that $f = \ell|_K$. It is then true that $\ell|_{\mathrm{aff}(K)} \circ \mathscr{L}_A^{-1} = \ell_f$. This implies that if $U \subset \mathbf{F}$ is open, then

$$\mathscr{L}_A(\ell^{-1}(U) \cap \mathrm{aff}(K)) = \ell_f^{-1}(U)$$

is open in the $\{\ell_f : f \in E^*|_K\}$-topology of $A^* \Rightarrow \mathscr{L}_A$ is open.

REMARKS. Notice, first, that $(\mathrm{aff}(K), p_K|_{\mathrm{aff}(K)})$ is thus a Banach space. Second, $p_K^{-1}(1)$ happens to be the intrinsic algebraic boundary of $K$ ([**6**; V.1.8] and [**10**; §2C]), which is closed in $K \Leftrightarrow K$ is of finite dimension ([**15**; 3.11] and [**17**; 3.41-C.d]). Third, the relation between $K$ and $A$ is an especial one, analogous to that of a compact convex subset $K'$ of a real locally convex topological vector space $E'$, and $A' = (E')^*|_{K'} + \mathbf{R}$ or $A' = A_{\mathbf{R}}(K')$. For instance, it can be shown that $E^*|_K$ is a dense vector subspace of the (closed) vector subspace $A_{\mathbf{F}}(K) \cap C_{\mathbf{F}}^{\mathrm{hom}}(K)$ in $(C_{\mathbf{F}}(K), \|\ \|_\infty)$; compare with [**1**; I.1.5]. Also, using almost the same method of proof employed in (ii), it can be demonstrated that

$$\begin{aligned} \alpha_{A'}\colon \mathrm{aff}(K') &\to (A')^* \\ \alpha x - \beta y &\to \alpha\phi_{A'}(x) - \beta\phi_{A'}(y) \end{aligned}$$

(where $\alpha, \beta \geqq 0$ are such that $\alpha - \beta = 1$ and $x, y \in K'$) is an affine embedding of aff($K'$) onto $\ell_1^{-1}(1)$, provided aff($K'$) is given the topology induced by the weak one of $E'$ and $(A')^*$ has its $\{\ell_f : f \in (E')^*|_{K'} + \mathbf{R}\}$-topology; consult [**1**; II. §2].

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CRESTÓN 293 COL. J. DEL PEDREGAL, DEL. A. OBREGÓN 01900 MÉXICO D.F., MEXICO.