

**A GENERALIZATION OF A RESULT OF A. MEIR
 FOR NON-DECREASING SEQUENCES**

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1. In [3], the following result is given.

THEOREM A. *Let $0 \leq p_1 \leq p_2 \leq \dots \leq p_n$ and $0 = a_0 \leq a_1 \leq \dots \leq a_n$, satisfying $a_i - a_{i-1} \leq p_i$ ($i = 1, 2, \dots, n$). If $r \geq 1$ and $s + 1 \geq 2(r + 1)$, then*

$$(1.1) \quad \left((s + 1) \sum_{i=1}^{n-1} a_i^s \frac{p_i + p_{i+1}}{2} \right)^{1/(s+1)} \leq \left((r + 1) \sum_{i=1}^{n-1} a_i^r \frac{p_i + p_{i+1}}{2} \right)^{1/(r+1)}.$$

In this paper we shall prove an inequality which is stronger than inequality (1.1). Also, we show a generalization of Theorem A.

THEOREM 1. *Let $0 \leq p_1 \leq p_2 \leq \dots \leq p_n$ and $0 = a_0 \leq a_1 \leq \dots \leq a_n$, satisfying $a_i - a_{i-1} \leq p_i$ ($i = 1, 2, \dots, n$). If $r \geq 1$ and $s + 1 \geq 2(r + 1)$, then*

$$(1.2) \quad (s + 1) \sum_{i=1}^{n-1} a_i^s \frac{p_i + p_{i+1}}{2} + \frac{(s + 1)(s - r)}{8} \sum_{i=1}^{n-1} (p_{i+1}^2 - p_i^2) a_i^{s-1} \leq \left((r + 1) \sum_{i=1}^{n-1} a_i^r \frac{p_i + p_{i+1}}{2} \right)^{(s+1)/(r+1)}.$$

2. PROOF. Since $x \mapsto x^r$ ($r \geq 1$) is a convex function on $[0, \infty)$, the inequality

$$\sum_{i=1}^j \int_{a_{i-1}}^{a_i} x^r dx \leq \sum_{i=1}^j (a_i - a_{i-1}) \frac{a_i^r + a_{i-1}^r}{2} \quad (1 \leq j \leq n),$$

holds, wherefrom, according to the condition $a_i - a_{i-1} \leq p_i$, we obtain

$$(2.1) \quad \frac{1}{r + 1} a_j^{r+1} \leq \sum_{i=1}^j p_i \frac{a_i^r + a_{i-1}^r}{2}.$$

For $q_j = \sum_{i=1}^j a_{i0}^r \frac{p_i + p_{i+1}}{2}$, the inequality (2.1) becomes

$$\frac{1}{r + 1} a_j^{r+1} \leq q_j - \frac{1}{2} p_{j+1} a_j^r = q_{j-1} + \frac{1}{2} p_j a_j^r,$$

Received by the editors on July 23, 1984.

i.e.,

$$(2.2) \quad a_j^{r+1} \leq (r + 1) c_j,$$

where $c_j = q_j - p_{j+1} a_j^r/2 = q_{j-1} + p_j a_j^r/2$.

If we take $k = (s + 1)/(r + 1)$, the inequality (2.2) becomes

$$(2.3) \quad a_j^{s-r} \leq (r + 1)^{k-1} c_j^{k-1}.$$

Note that $q_{j-1} \leq c_j \leq q_j$ ($j = 1, \dots, n$). Using a generalization of Hadamard's integral inequality for convex functions, which is proved in [2], we find that the inequality

$$c_j^{k-1} + (k - 1) c_j^{k-2} \left(\frac{q_j + q_{j-1}}{2} - c_j \right) \leq \frac{q_j^k - q_{j-1}^k}{k(q_j - q_{j-1})},$$

i.e.,

$$k \frac{p_j + p_{j+1}}{2} a_j^r c_j^{k-1} + k(k - 1) \frac{p_{j+1}^2 - p_j^2}{8} a_j^{2r} c_j^{k-2} \leq q_j^k - q_{j-1}^k$$

is valid. Whence, after summing for $j = 1, \dots, n - 1$ and using (2.2) and (2.3), we obtain the inequality (1.2).

Since

$$(s + 1)(s - r) \sum_{i=1}^{n-1} (p_{i+1}^2 - p_i^2) a_i^{s-1} \geq 0,$$

we conclude that the inequality (1.2) is stronger than inequality (1.1).

For $p_1 = \dots = p_n = 1$, the inequality (1.2) is reduce to the inequality proved in [1].

3. Similarly, as in Theorem 1 (also, see [4]), the following result can be proved.

THEOREM 2. *Let f and g be differentiable functions on $[0, \infty)$ satisfying $f(0) = f'(0) = g(0) = g'(0) = 0$. Suppose that f and g are convex and increasing on $[0, \infty)$. Set $h(x) = g(f(x))$. Then for any finite sequences (a_i) , (p_i) such that $0 = a_0 \leq a_1 \leq \dots \leq a_n$ and $0 \leq p_1 \leq \dots \leq p_n$, which satisfy $a_i - a_{i-1} \leq p_i$ ($i = 1, \dots, n$), we have*

$$(3.1) \quad h^{-1} \left(\sum_{i=1}^{n-1} \frac{p_i + p_{i+1}}{2} h(a_i) \right) \leq f^{-1} \left(\sum_{i=1}^{n-1} \frac{p_i + p_{i+1}}{2} f(a_i) \right).$$

COROLLARY. *For $h(x) = x^{s+1}$ and $f(x) = x^{r+1}$ the Meir's result is obtained from the inequality (3.1).*

EXAMPLE. Functions $f(x) = x^2$ and $g(x) = x^3 e^x$ satisfy the conditions of the above theorem. This shows that potential functions are not the only ones which satisfy the conditions of the Theorem 2.

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