ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 16, Number 2, Spring 1986

ON BOUNDARY VALUES OF SOLUTIONS OF A QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATION OF ELLIPTIC TYPE

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Introduction. In this article we study traces of generalized solutions of quasi-linear elliptic equations. We obtain a sufficient condition for a solution in $W^{1,2}_{\text{loc}}(Q)$ to have an L^2-trace on the boundary. The results are then applied to establish an existence theorem for the Dirichlet problem. The arguments which we give here are based partially on the references **[2]** and **[4]**.

The outline of this paper is as follows. §1 contains preliminary work. §2 deals with the problem of traces for solutions in $W^{1,2}_{\text{loc}}(Q)$. The main result here is Theorem 1, which justifies the approach to the Dirichlet problem adopted in §4. In §3 we derive an energy estimate for solutions of the Dirichlet problem with L^2-boundary data.

1. Preliminaries. Consider the quasi-linear elliptic equation of the form

$$-\sum_{i,j=1}^{n} D_i(a_{ij}(x, u)D_j u) + b(x, u, Du) = 0 \tag{1}$$

in a bounded domain $Q \subset R_n$ with the boundary ∂Q of the class C^2, $Du = (D_1 u, \ldots, D_n u)$, $D_i u = \partial u/\partial x_i$.

Throughout this paper we make the following assumptions.

(A) There is a positive constant γ such that

$$\gamma^{-1}|\xi|^2 \leqq \sum_{i,j=1}^{n} a_{ij}(x, u)\xi_i\xi_j \leqq \gamma|\xi|^2,$$

for all $\xi \in R_n$ and $(x, u) \in Q \times (-\infty, \infty)$; moreover, $a_{ij}(x, u)$ are uniformly continuous on $\bar{Q} \times (-\infty, \infty)$ and, for every $u \in (-\infty, \infty)$, $a_{ij}(\cdot, u) \in C^1(\bar{Q})$ $(i, j = 1, \ldots, n)$, and there exists a positive constant K such that $|D_i a_{ij}(x, u)| \leqq K$, for all $(x, u) \in Q \times (-\infty, \infty)$, $a_{ij} = a_{ji}$ $(i, j = 1, \ldots, n)$.

(B) The function $b(x, u, s)$ is defined for $(x, u, s) \in Q \times R_{n+1}$, $s = (s_1, \ldots, s_n)$, and satisfies the Carathéodory condition:

(i) for a.e. $x \in Q$, $b(x, \cdot, \cdot)$ is a continuous function on R_{n+1}; and

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Received by the editors on July 27, 1984.

(ii) for every fixed $(u, s) \in R_{n+1}$, $b(\cdot, u, s)$ is a measurable function on Q. Moreover we assume that

$$|b(x, u, s)| \leqq f(x) + L(|u| + |s|),$$

for all $(x, u, s) \in Q \times R_{n+1}$, where L is a positive constant and f is a non-negative measurable function on Q such that

$$\int_Q f(x)^2 r(x)^\theta dx < \infty,$$

where $2 \leqq \theta < 3$, $r(x) = \operatorname{dist}(x, \partial Q)$.

It is well-known that under assumption (B), $b(x, u(x), s(x))$ is a measurable function of $x \in Q$, where $(u(x), s(x))$ is a measurable vector function on Q and

$$b(x, \cdot, \cdot): L^1_{\mathrm{loc}}(Q)^{n+1} \to L^1_{\mathrm{loc}}(Q)$$

is continuous.

In this paper we use the notion of a weak (generalized) solution of (1) involving Sobolev spaces $W^{1,2}_{\mathrm{loc}}(Q)$, $W^{2,2}(Q)$ and $\mathring{W}^{1,2}(Q)$ (for the definition of these spaces see [6] or [7]).

A function $u(x)$ is said to be a weak solution of the equation (1) if $u \in W^{1,2}_{\mathrm{loc}}(Q)$ and u satisfies

$$\int_Q \left(\sum_{i,j=1}^n a_{ij}(x, u) D_i u D_j v + b(x, u, Du) v \right) dx = 0, \tag{2}$$

for every $v \in W^{1,2}(Q)$ with compact support in Q.

It follows from the regularity of the boundary ∂Q that there exists a number $\delta_0 > 0$ such that, for $\delta \in (0, \delta_0]$, the domain

$$Q_\delta = Q \cap \{x; \min_{y \in \partial Q} |x - y| > \delta\} \qquad \text{with the boundary } \partial Q_\delta,$$

possesses the property that to each $x_0 \in \partial Q$ there is a unique point $x_\delta(x_0) \in \partial Q_\delta$ such that $x_\delta(x_0) = x_0 - \delta\nu(x_0)$, where $\nu(x_0)$ is the outward normal to ∂Q at x_0. The inverse mapping to $x_0 \to x_\delta(x_0)$ is given by the formula $x_0 = x_\delta + \delta\nu(x_\delta)$, where $\nu_\delta(x_\delta)$ is the outward normal to ∂Q_δ at x_δ.

Let x_δ denote an arbitrary point of ∂Q_δ. For fixed $\delta \in [0, \delta_0]$

$$A_\varepsilon = \partial Q_\delta \cap \{x; |x - x_\delta| < \varepsilon\},$$
$$B_\varepsilon = \{x; x = \tilde{x}_\delta + \delta\nu_\delta(\tilde{x}_\delta), \tilde{x}_\delta \in A_\varepsilon\},$$

and

$$\frac{dS_\delta}{dS_0}(x_\delta) = \lim_{\varepsilon \to 0} \frac{|A_\varepsilon|}{|B_\varepsilon|},$$

where $|A|$ denotes the $n-1$ dimensional Hausdorff measure of a set A. Mikhailov [10] proved that there is a positive number γ_0 such that

$$\gamma_0^{-2} \leqq \frac{dS_\delta}{dS_0} \leqq \gamma_0^2 \tag{3}$$

and

$$\lim_{\delta\to 0} \frac{dS_\delta}{dS_0} = 1 \tag{4}$$

uniformly with respect to $x_0 \in \partial Q$.

According to Lemma 1 in [5 p. 382], the distance $r(x)$ belongs to $C^2(\bar{Q} - Q_{\delta_0})$ if δ_0 is sufficiently small. Denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties: $\rho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_0}$; $\rho(x) \geqq (3/4)\delta_0$ in Q_{δ_0}; $\gamma_1^{-1} r(x) \leqq \rho(x) \leqq \gamma_1 r(x)$ in Q for some positive constant γ_1; $\partial Q_\delta = \{x;\ \rho(x) = \delta\}$ for $\delta \in (0, \delta_0]$; and, finally, $\partial Q = \{x;\ \rho(x) = 0\}$.

We will use the surface integrals

$$M_u(\delta) = \int_{\partial Q} \sum_{i,j=1}^{n} \int_0^{u(x_\delta(x))} a_{ij}(x_\delta(x), s) s ds D_i\rho(x_\delta(x)) D_j\rho(x_\delta(x)) dS_x$$

and

$$N_u(\delta) = \int_{\partial Q_\delta} \sum_{i,j=1}^{n} \int_0^{u(x)} a_{ij}(x, s) s ds D_i\rho(x) D_j\rho(x) dS_x,$$

where $u \in W^{1,2}_{\text{loc}}(Q)$ and the values $u(x_\delta(x))$ on ∂Q and $u(x)$ on ∂Q_δ are understood in the sense of traces (see [6, Chapter 6]). It follows from Lemma 4 in [2] that $M_u(\delta)$ and $N_u(\delta)$ are absolutely continuous in $[\delta_1, \delta_0]$, for every $0 < \delta_1 < \delta_0$.

LEMMA 1. *Let u be a weak solution of* (1) *belonging to* $W^{1,2}_{\text{loc}}(Q)$. *Then the following conditions are equivalent*:

(i) $N_u(\delta)$ *is u bounded function on* $(0, \delta_0]$;

(ii) $\int_Q |Du(x)|^2 r(x) dx < \infty$; *and*

(iii) $\lim_{\delta\to 0} M_u(\delta) < \infty$.

PROOF. Let $0 < \delta < \delta_0$ and set

$$v(x) = \begin{cases} u(x)(\rho(x) - \delta), & \text{for } x \in Q_\delta, \\ 0, & \text{for } x \in Q - Q_\delta. \end{cases}$$

Using v as a test function in (2), we obtain

$$\begin{aligned} &\int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}(x, u) D_i u D_j u(\rho - \delta) dx + \int_{Q_\delta} \sum_{i,j=1}^{n} a_{ij}(x, u) D_i u \cdot u \cdot D_j \rho dx \\ &\quad + \int_{Q_\delta} b(x, u, Du) u(\rho - \delta) dx = 0. \end{aligned} \tag{5}$$

Let us denote the second and third integral in (5) by I and J respectively. Using Green's theorem, we obtain

$$I=\int_{Q_\delta}\sum_{i,j=1}^n D_i\left(\int_0^{u(x)} a_{ij}(x,s)sds\right)D_j\rho dx-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} D_ia_{ij}(x,s)sdsD_j\rho dx$$
$$=-\int_{\partial Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)sdsD_i\rho D_j\rho dS-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)sdsD_{ij}\rho dx$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} D_ia_{ij}(x,s)sdsD_j\rho dx.$$

It follows from (A) that there exists a positive constant C_1, independent of δ, such that

$$|I|\leqq C_1\left(N_u(\delta)+\int_{Q_\delta}u^2dx\right). \tag{6}$$

By the Young and Hölder inequalities, we have

$$\begin{aligned}|J|\leqq\frac{\gamma^{-1}}{2}\int_{Q_\delta}|Du|^2(\rho-\delta)dx+C_2\Big(\int_{Q_\delta}u^2(\rho-\delta)dx\\+\int_{Q_\delta}u^2(\rho-\delta)^{-\alpha}dx+\int_{Q_\delta}f(\rho-\delta)^\theta dx\Big),\end{aligned} \tag{7}$$

where $\alpha=\theta-2$ and C_2 is a positive constant independent of δ. Combining (5), (6) and (7), we obtain

$$\int_{Q_\delta}|Du|^2(\rho-\delta)dx\leqq C_3\Big(N_u(\delta)+\int_{Q_\delta}u^2(\rho-\delta)dx+\int_{Q_\delta}u^2(\rho-\delta)^{-\alpha}dx$$
$$+\int_{Q_\delta}u^2dx+\int_{Q_\delta}f(\rho-\delta)^\theta dx\Big).$$

Now, if $N_u(\delta)$ is bounded on $(0,\delta_0]$, then, by Lemma 5 in **[2]**, for every $0\leqq\mu<1$, there exists a positive constant C such that

$$\int_{Q_\delta}|u(x)|^2(\rho(x)-\delta)^{-\mu}dx\leqq C,$$

for every $\delta\in(0,\delta_0/2]$. Consequently, the implication (i) $\Rightarrow$ (ii) follows from the monotone convergence theorem.

To prove (ii) $\Rightarrow$ (iii), note that

$$N_u(\delta)=\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju(\rho-\delta)dx-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)sdsD_{ij}\rho dx$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} D_ia_{ij}(x,s)sdsD_j\rho dx+\int_{Q_\delta}b(x,u,Du)u(\rho-\delta)dx.$$

Now, by Lemma 6 in **[2]**, the condition (ii) implies that, for every $0\leqq\mu<1$, there exists a positive constant C, independent of δ, such that

$$\int_{Q_\delta} u(x)^2(\rho(x) - \delta)^{-\mu} dx \leqq C,$$

for $\delta \in (0, \delta_0/2]$. Thus, using the estimates from the step (i) $\Rightarrow$ (ii), we conclude that $\lim_{\delta\to 0} N_u(\delta)$ exists by the dominated and monotone convergence theorem. On the other hand, let

$$v(x) = \sum_{i,j=1}^{n} \int_0^{u(x)} a_{ij}(x, s) s ds D_i\rho(x) D_j\rho(x).$$

Then

$$N_u(\delta) - M_u(\delta) = \int_{\partial Q_\delta} v(x) dS_\delta - \int_{\partial Q} v(x_\delta(x)) dS = \int_{\partial Q} v(x_\delta(x))\left(\frac{dS_\delta}{dS_0} - 1\right) dS_0.$$

By (4), $dS_\delta/dS_0 \to 0$ uniformly as $\delta \to 0$, and, consequently, $\lim_{\delta\to 0} M_u(\delta)$ exists.

Finally, (iii) $\Rightarrow$ (i) follows from the proof of (ii) $\Rightarrow$ (iii).

2. Traces in $L^2(\partial Q)$. Our next objective is to prove that u has a trace on ∂Q in $L^2(\partial Q)$; that is, $u(x_\delta)$ converges in $L^2(\partial Q)$ as $\delta \to 0$. To do this we first show that

$$\int_0^{u(x_\delta)} \sum_{i,j=1}^{n} a_{ij}(x_\delta, s) ds D_i\rho(x_\delta) D_i\rho(x_\delta)$$

converges strongly in $L^2(\partial Q)$ to some function ζ.

LEMMA 2. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of* (1). *Assume that one of conditions* (i), (ii) *or* (iii) *holds. Then there is a function $\zeta \in L^2(\partial Q)$ such that*

$$\begin{aligned} &\lim_{\delta\to 0} \int_{\partial Q} \int_0^{u(x_\delta)} \sum_{i,j=1}^{n} a_{ij}(x_\delta, s) ds D_i\rho(x_\delta) D_j\rho(x_\delta) \Psi(x) dS_x \\ &\qquad = \int_{\partial Q} \zeta(x) \Psi(x) dS_x, \end{aligned} \tag{8}$$

for each $\Psi \in L^2(\partial Q)$.

PROOF. By assumption (A), the condition (iii) implies the boundedness of

$$\left\| \int_0^{u(x_\delta(\cdot))} \sum_{i,j=1}^{n} a_{ij}(x_\delta(\cdot), s) ds D_i\rho(x_\delta(\cdot)) D_j\rho(x_\delta)(\cdot) \right\|_{L^2(\partial Q)}$$

on $(0, \delta_0]$. Hence, there exists a sequence $\delta_\nu \to 0$ and a function ζ such that

$$\lim_{\nu\to 0} \int_{\partial Q} \int_0^{u(x_{\delta_\nu})} \sum_{i,j=1}^{n} a_{ij}(x_{\delta_\nu}, s) ds D_i\rho(x_{\delta_\nu}) D_j\rho(x_{\delta_\nu}) \Psi(x) dS_x = \int_{\partial Q} \zeta(x) \Psi(x) dS_x,$$

for each $\Psi \in L^2(\partial Q)$. To prove (8) it suffices to show that the function $\bar{G}$, defined on $(0, \delta_0]$ by

$$\bar{G}(\delta) = \int_{\partial Q_\delta} \int_0^{u(x)} \sum_{i,j=1}^n a_{ij}(x, s) ds D_i\rho(x) D_j\rho(x) \Psi(x) dS_\delta$$

has a continuous extension to $[0, \delta_0]$, for each $\Psi \in C^1(\bar{Q})$.

From (2), taking

$$v(x) = \begin{cases} \Psi(x)\,(\rho(x) - \delta), & \text{for } x \in Q_\delta \\ 0, & \text{for } x \in Q - Q_\delta \end{cases}$$

as a test function, we have

$$\begin{aligned} \bar{G}(\delta) = & \int_{Q_\delta} \sum_{i,j=1}^n a_{ij}(x, u) D_i u D_j \Psi(\rho - \delta) dx - \int_{Q_\delta} \sum_{i,j=1}^n \int_0^u a_{ij}(x, s) ds D_{ij}\rho \Psi dx \\ & - \int_{Q_\delta} \sum_{i,j=1}^n \int_0^u a_{ij}(x, s) ds D_i\Psi D_j\rho dx - \int_{Q_\delta} \sum_{i,j=1}^n \int_0^u D_i a_{ij}(x, s) ds D_j\rho\Psi dx \\ & + \int_{Q_\delta} b(x, u, Du)\Psi(\rho - \delta) dx. \end{aligned}$$

The integrand on the right is dominated by

$$\text{Const}\,(|Du|^2\rho + u^2 + f^2\rho^\theta + u^2\rho^{-\alpha}\psi^2 + |D\psi|^2),$$

which belongs to $L^1(Q)$, where $\alpha = \theta - 2$ and Const. is independent of δ. The result follows.

In order to prove the convergence in the norm we use the following function.

For $\delta \in (0, \delta_0]$ we define the mapping $x^\delta \colon \bar{Q} \to \bar{Q}_\delta$ by

$$x^\delta(x) = \begin{cases} x, & \text{for } x \in Q_\delta \\ y_\delta(x) + \frac{1}{2}(x - y_\delta)(x) & \text{for } x \in Q - Q_\delta, \end{cases}$$

where $y_\delta(x)$ denotes the closest point on ∂Q_δ to x. Thus, $x^\delta(x) = x$, for each $x \in Q_\delta$, and $x^\delta(x) = x_{\delta/2}(x)$, for $x \in \partial Q$. Moreover, $\rho(x^\delta) \geqq \delta/2$ and x^δ is uniformly Lipschitz continuous. Note that if $u \in W^{1,2}_{\text{loc}}(Q)$, then $u(x^\delta) \in W^{1,2}(Q)$.

Let $\Psi \in W^{1,2}(Q)$. As in the proof of Lemma 1 we find that

$$\begin{aligned} \int_{\partial Q} \zeta(x)\,\Psi(x)\,dS_x = & - \int_Q \sum_{i,j=1}^n \int_0^{u(x)} a_{ij}(x, s)\,ds\,\Psi(x)\,D_{ij}\rho(x)\,dx \\ & - \int_Q \sum_{i,j=1}^n \int_0^{u(x)} a_{ij}(x, s)\,ds\,D_i\,\Psi(x) D_j\rho(x)\,dx \\ & - \int_Q \sum_{i,j=1}^n \int_0^{u(x)} D_i a_{ij}(x, s)\,ds\,\Psi(x)\,D_j\rho(x)\,dx \\ & + \int_Q \sum_{i,j=1}^n a_{ij}(x, u)\,D_i u D_j \Psi\,\rho dx + \int_Q b(x, u, Du)\Psi\,\rho dx. \end{aligned} \tag{9}$$

LEMMA 3. *Let $u \in W^{1,2}_{\text{loc}}(Q)$ be a solution of* (1) *such that one of the conditions* (i), (ii) *or* (iii) *holds. Then there is a function $\zeta \in L^2(\partial Q)$ such that*

$$\int_0^{u(x_\delta)} \sum_{i,j=1}^n a_{ij}(x_\delta, s)\, ds\, D_i\rho(x_\delta)\, D_i\rho(x_\delta)$$

converges to ζ in $L^2(\partial Q)$ as $\delta \to 0$.

PROOF. Using Lemma 2 and (9), we find that

$$\int_{\partial Q} \zeta(x)\, \Psi(x)\, dS_x = F(\Psi),$$

for all $\Psi \in W^{1,2}(Q)$. Let

$$\Psi(x) = \int_0^{u(x)} \sum_{i,j=1}^n a_{ij}(x, s)\, ds\, D_i\rho(x)\, D_j\rho(x).$$

As $\Psi(x^\delta) \in W^{1,2}(Q)$, we have

$$\int_{\partial Q} \zeta(x) \int_0^{u(x^\delta)} \sum_{i,j=1}^n a_{ij}(x^\delta, s)\, ds\, D_i\rho(x^\delta) D_j\rho(x^\delta)\, dS_x$$
$$= \int_{Q \to Q_\delta} F(\Psi(x^\delta(x)) dx + \int_{Q_\delta} F(\Psi(x))\, dx,$$

since $x^\delta(x) = x$, for every $x \in Q_\delta$. We show that

$$\lim_{\delta \to 0} \int_{Q - Q_\delta} F(\Psi(x^\delta)) dx = 0 \tag{10}$$

and that

$$\lim_{\delta \to 0} \int_{Q_\delta} F(\Psi(x)) dx = \lim_{\delta \to 0} \int_{\partial Q} \left(\int_0^{u(x^\delta)} \sum_{i,j=1}^n a_{ij}(x^\delta, s) ds\, D_i\rho(x^\delta) D_j\rho(x^\delta) \right)^2 dS, \tag{11}$$

so that

$$\int_{\partial Q} \zeta^2\, dS_x = \lim_{\delta \to 0} \int_{\partial Q} \zeta(x) \int_0^{u(x^\delta)} \sum_{i,j=1}^n a_{ij}(x^\delta, s) ds\, D_i\rho(x^\delta) D_j\rho(x^\delta)\, dS_x$$
$$= \lim_{\delta \to 0} \int_{\partial Q} \Psi(x^\delta(x))^2 dS_x$$

as $x^\delta(x) = x_{\delta/2}(x)$ on ∂Q. The result will follow from the uniform convexity of $L^2(\partial Q)$.

Setting

$$v(x) = \begin{cases} \displaystyle\int_0^{u(x)} \sum_{i,j=1}^n a_{ij}(x, s) ds\, D_i\rho(x)\, D_j\rho(x)(\rho(x) - \delta), & \text{on } Q_\delta, \\ 0, & \text{on } Q - Q_\delta \end{cases}$$

in equation (2), we obtain

$$\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}(x,u)D_iu\,D_j\Big(\int_0^{u(x)}\sum_{\ell,k=1}^n a_{\ell k}(x,s)ds\,D_\ell\rho\,D_k\rho\Big)(\rho-\delta)\,dx$$
$$+\int_{Q_\delta} b(x,u,Du)\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds(\rho-\delta)dx$$
$$=-\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}(x,u)D_iu\int_0^{u(x)}\sum_{\ell,k=1}^n a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\,D_j\rho\,dx.$$

Hence,

$$\lim_{\delta\to 0}\int_{Q_\delta} F(\Psi(x))dx$$
$$=\lim_{\delta\to 0}\Big(-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)ds\Big(\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell,k}(x,s)dsD_\ell\rho\,D_k\rho\Big)D_{ij}\rho\,dx$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} D_ia_{ij}(x,s)ds\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\,D_j\rho\,dx$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)ds\,D_i\Big(\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)ds\,D_\ell\rho\,D_k\rho\Big)D_j\rho\,dx$$
$$+\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}(x,u)\,D_iu\,D_j\Big(\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\Big)(\rho-\delta)\,dx$$
$$+\int_{Q_\delta} b(x,u,Du)\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho(\rho-\delta)\,dx\Big)$$
$$=\lim_{\delta\to 0}\Big(-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)ds\Big(\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\Big)D_{ij}\rho\,dx$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} D_ia_{ij}(x,s)\,ds\Big(\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\Big)D_j\rho\,dx$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)\,ds\,D_i\Big(\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\Big)D_j\rho\,ds$$
$$-\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}(x,u)\,D_iu\Big(\int_0^{u(x)}\sum_{\ell,k=1}^n a_{\ell k}(x,s)\,ds\,D_\ell\rho\,D_k\rho\Big)D_j\rho\,dx\Big)$$
$$=\lim_{\delta\to 0}\int_{\partial Q_\delta}-\sum_{i,j=1}^n D_i\Big(\int_0^{u(x)} a_{ij}(x,s)\,ds\,D_j\rho\sum_{\ell,k=1}^n\int_0^{u(x)} a_{\ell k}(x,s)D_\ell\rho\,D_k\rho\Big)dS$$
$$=\lim_{\delta\to 0}\int_{\partial Q_\delta}\Big(\sum_{i,j=1}^n\int_0^{u(x)} a_{ij}(x,s)\,ds\,D_i\rho\,D_j\rho\Big)^2 dS.$$

Now it remains to prove (10). Note that

$$|F(\Psi(x^\delta))|\leqq \text{Const}\,(|u(x)||u(x^\delta)|+|u(x)||Du(x^\delta)|+|Du(x)||Du(x^\delta)|\rho(x)$$
$$+|Du(x)||u(x^\delta)|\rho(x)+f(x)|u(x^\delta)|\rho(x)).$$

By an argument similar to that used in the proof of Theorem 5 in **[3]** (see also the proof of Theorem 4 in **[2]**), one can easily show (11).

Now we are in a position to formulate our main result of this section.

THEOREM 1. *Let* $u \in W^{1,2}_{\mathrm{loc}}(Q)$ *be a solution of* (1) *such that one of the conditions* (i), (ii) *or* (iii) *holds. Then there exists a function* $\phi \in L^2\,(\partial Q)$ *such that*

$$\lim_{\delta\to 0} u(x_\delta) = \phi \qquad \text{in } L^2\,(\partial Q).$$

PROOF. By Lemma 3, there exists $\zeta \in L^2\,(\partial Q)$ such that

$$\lim_{\delta\to 0}\int_0^{u(x_\delta)} \sum_{i,j=1}^n a_{ij}(x_\delta, s)\, ds\, D_i\rho(x_\delta)\, D_j\rho(x_\delta) = \zeta \qquad \text{in } L^2\,(\partial Q),$$

Now, note that

$$\left\{\int_{\partial Q}\left(\int_0^{u(x_\delta)} \sum_{i,j=1}^n a_{ij}(x, s)\, ds\, D_i\rho(x)\, D_j\rho(x) - \zeta\right)^2 dS_x\right\}^{1/2}$$

$$\leqq \left\{\int_{\partial Q}\left(\int_0^{u(x_\delta)} \sum_{i,j=1}^n (a_{ij}(x, s) - a_{ij}(x_\delta, s))\, ds\, D_i\rho(x)\, D_j\rho(x)\right)^2 dS_x\right\}^{1/2}$$

$$+ \left\{\int_{\partial Q}\left(\int_0^{u(x_\delta)} \sum_{i,j=1}^n a_{ij}(x_\delta, s)\, ds\, (D_i\rho(x)\, D_j\rho(x) - D_i\rho(x_\delta) D_j\rho(x_\delta))\right)^2 dS_x\right\}^{1/2}$$

$$+ \left\{\int_{\partial Q}\left(\int_0^{u(x_\delta)} \sum_{i,j=1}^n a_{ij}\,(x_\delta, s)\, ds\, D_i\rho(x_\delta)\, D_j\rho(x_\delta) - \zeta(x)\right)^2 dS_x\right\}.$$

This inequality combined with the uniform continuity of a_{ij} and $D_i\rho$ $D_j\rho$ yields that

$$\lim_{\delta\to 0}\int_0^{u(x_\delta)} \sum_{i,j=1}^n a_{ij}(x, s)\, ds\, D_i\rho(x)\, D_j\rho(x) = \zeta$$

in $L^2(\partial Q)$. Finally, let $0 < \delta_1 < \delta_2 < \delta_0$. It follows, from (A), that

$$\gamma^{-2}\|u(x_{\delta_2}) - u(x_{\delta_1})\|^2_{L^2} \leqq \int_{\partial Q}\left(\int_0^{u(x_{\delta 2})} \sum_{i,j=1}^n a_{ij}(x, s)\, ds\, D_i\rho(x)\, D_j\rho(x)\right.$$

$$\left. - \int_0^{u(x_{\delta 1})} \sum_{i,j=1}^n a_{ij}(x, s)\, ds\, D_i\rho(x)\, D_j\rho(x)\right)^2 dS_x.$$

and, consequently, $\{u(x_\delta)\}$ $0 < \delta \leqq \delta_0$ is a Cauchy sequence in $L^2(\partial Q)$ and the proof of Theorem 1 is thus completed.

3. The energy estimate. Consider the elliptic equation of the form

$$(1_\lambda) \quad - \sum_{i,j=1}^n D_j(a_{ij}(x, u)D_i u) + b(x, u, Du) + \lambda u = 0$$

in Q, where λ is a real parameter.

Theorem 1 suggests the following approach to the Dirichlet problem. Let $\phi \in L^2(\partial Q)$. A weak solution $u \in W^{1,2}_{\mathrm{loc}}(Q)$ of (1_λ) is a solution of the Dirichlet problem with the boundary condition

$$u(x) = \phi(x) \qquad \text{on } \partial Q, \tag{12}$$

if $\lim_{\delta\to 0}\int_{\partial Q}(u(x_\delta(x)) - \phi(x))^2\, dS_x = 0$.

We now establish the following energy estimate.

THEOREM 2. *Let $u \in W^{1,2}_{\mathrm{loc}}(Q)$ be a solution of the Dirichlet problem* (1_λ), (12). *Then there exist positive constants d, λ_0 and C, independent of u, such that*

$$\int_Q |Du(x)|^2\, r(x)dx + \sup_{0<\delta\leq d}\int_{\partial Q} u(x_\delta)^2\, dS_x + \lambda\int_{Q_\delta} u(x)^2\, dx$$
$$\leq C\Big(\int_{\partial Q}\phi(x)^2 dS_x + \int_Q f(x)^2 r(x)^\theta dx\Big),$$

for $\lambda \geq \lambda_0$.

PROOF. Let v be the test function introduced in the proof of Lemma 1. Thus, we have

$$\int_{\partial Q_\delta}\sum_{i,j=1}^n\int_0^u a_{ij}(x, s)\, sds\, D_i\rho\, D_j\rho\, dS_x$$
$$= -\int_{Q_\delta}\sum_{i,j=1}^n\int_0^u a_{ij}(x, s)\, sds\, D_{ij}\rho\, dx - \int_{Q_\delta}\int_0^u\sum_{i,j=1}^n D_i a_{ij}(x, s)\, sds\, D_j\rho\, dx$$
$$+\int_{Q_\delta}\sum_{i,j=1}^n a_{ij}(x, u)\, D_i u\, D_j u\,(\rho - \delta)\, dx + \int_{Q_\delta} b(x, u, Du)\, u(\rho - \delta)\, dx$$
$$+\int_{Q_\delta}\lambda u^2(\rho - \delta)\, dx = 0$$

As in the proof of Lemma 1 we find that there exists positive constants C_1 and C_2 independent of δ such that

$$\begin{aligned}\int_{\partial Q_\delta} u(x)^2\, dS_x \leq C_1\Big(&\int_{Q_\delta} u^2 dx + \int_{Q_\delta} u^2\rho^{-\alpha} dx + \lambda\int_{Q_\delta} u^2(\rho-\delta)\, dx\\ &+\int_{Q_\delta}|Du|^2(\rho - \delta)\, dx + \int_{Q_\delta} f^2\rho^\theta\, dx\Big)\end{aligned} \tag{13}$$

and

$$\begin{aligned}&\int_{Q_\delta}|Du|^2(\rho - \delta)\, dx + \lambda\int_{Q_\delta} u^2(\rho - \delta)\, dx\\ &\leq C_2\Big(\int_{\partial Q_\delta} u^2\, dS_x + \int_{Q_\delta} u^2\rho^{-\alpha}\, dx + \int_{Q_\delta} u^2\, dx + \int_Q f^2\rho^\theta\, dx\Big).\end{aligned} \tag{14}$$

By the boundary condition (12), $\int_{\partial Q_\delta} u^2\, dx$ is bounded on $(0, \delta_0]$ and, consequently, $\int_{Q_\delta} u^2\rho^{-\alpha}\, dx$ is bounded independently of δ [**2**, Lemma 5] Letting $\delta \to 0$ and using (12) and (4), we deduce from (14) that

$$\begin{aligned}(15)\qquad &\int_Q |Du|^2\,\rho dx + \lambda\int_Q u^2\,\rho dx\\ &\leqq C_2\Big(\int_{\partial Q}\phi^2\,dS_x + \int_Q u^2\,dx + \int_Q u^2\rho^{-\alpha}\,dx + \int_Q f^2\,\rho^\theta\,dx\Big).\end{aligned}$$

It follows from (13) and (15) that

$$\begin{aligned}(16)\qquad &\sup_{0<\delta\leqq d}\int_{\partial Q_\delta} u^2\,dS_\delta\\ &\leqq C_3\Big(\int_{\partial Q}\phi^2\,dS_x + \int_Q u^2\,dx + \int_Q u^2\,\rho^{-\alpha}\,dx + \int_Q f^2\,\rho^\theta\,dx\Big),\end{aligned}$$

where C_3 is a positive constant. Note that

$$(17)\qquad \int_Q u^2\,\rho^{-\mu}\,dx \leqq \frac{d^{1-\mu}}{1-\mu}\sup_{0<\delta\leqq d}\int_{\partial Q_\delta} u^2\,dS_\delta + \frac{1}{m_d^{1+\mu}}\int_Q u^2\,\rho dx,$$

where $m_d = \inf_{Q_d}\rho$. Taking λ sufficiently large and d sufficiently small, the result follows from (15), (16) and (17).

4. Application to the Dirichlet problem. In this section, we will apply some of the results of previous sections to establish an existence theorem of the Dirichlet problem for the quasilinear elliptic equation

$$(18)\qquad -\sum_{i,j=1}^n D_i(a_{ij}(x,u)D_ju) + c(x,u)u + \lambda u = f(x) \qquad \text{in } Q,$$

$$(19)\qquad u(x) = \phi(x) \qquad \text{on } \partial Q,$$

where $\phi \in L^2(\partial Q)$ and (19) is understood in the sense of the L^2-convergence (see (12)). The eigenvalue problem for (18) has been recently investigated by Boccardo [**1**].

In this section we adopt the assumption (A) and, moreover, we assume that the function $c(x, u)$ is bounded on $\bar{Q} \times R$ and satisfies the Carathéodory conditions. Finally, we assume that $f \in L^2_{\mathrm{loc}}(Q)$ and $\int_Q f(x)^2 r(x)^\theta dx < \infty$, where $2 \leqq \theta < 3$.

THEOREM 3. *Let $\phi \in L^2(\partial Q)$. There exists a positive constant λ_0 such that, for every $\lambda \geqq \lambda_0$, the Dirichlet problem admits a unique solution in $W^{1,2}_{\mathrm{loc}}(Q)$.*

PROOF. The proof is similar to that of Theorem 3 in [**4**]. Let $\{\phi_m\}$ be a sequence of functions in $C^1(\bar{Q})$ converging to ϕ in $L^2(\partial Q)$. Define

$$f_m(x) = \begin{cases} f(x), & \text{for } x \in Q_{1/m},\\ 0, & \text{for } x \in Q - Q_{1/m},\end{cases}$$

for m such that $1/m \leqq \delta_0$, and consider the Dirichlet problem

$$(18m) \qquad -\sum_{i,j=1}^{n} D_i(a_{ij}(x, u)D_j u) + c(x, u)u + \lambda u = f_m(x) \qquad \text{in } Q,$$

$$(19m) \qquad u(x) = \phi_m(x) \qquad \text{on } \partial Q$$

in $W^{1,2}(Q)$. It is easy to see that the Dirichlet form on $W^{1,2}(Q) \times W^{1,2}(Q)$, defined by

$$a(u, v) = \int_Q \Big(\sum_{i,j=1}^{n} a_{ij}(x, u)D_i u\, D_j v + c(x, u)uv + \lambda uv\Big)dx,$$

is coercive, that is

$$\lim_{\|v\|_{W^{1,2}} \to \infty} \frac{a(v, v)}{\|v\|_{W^{1,2}(Q)}} = \infty,$$

provided λ is sufficiently large, say $\lambda \geqq \bar{\lambda}$ [**8**, Theorem 2.8 p. 183]. Hence, for every m the Dirichlet problem (18_m), (19_m) admits a solution u_m in $W^{1,2}(Q)$, provided $\lambda \geqq \bar{\lambda}$. Here the boundary condition means that $u_m - \phi_m \in \mathring{W}^{1,2}(Q)$ which of course implies the boundary condition in the sense of the L^2-convergence. Let $\lambda_1 = \max(\bar{\lambda}, \lambda_0)$, where λ_0 denotes the constant from Theorem 2. By Theorem 2, for every $\lambda \geqq \lambda_1$, we have

$$\int_Q |Du_m|^2 \rho\, dx + \lambda \int_Q u_m^2 \rho\, dx + \sup_{0<\delta<d} \int_{\partial Q_\delta} u_m^2\, dx$$
$$\leqq C\Big(\int_{\partial Q} \phi_m^2\, dx + \int_Q f_m^2 r^\theta\, dx\Big),$$

for $m = 1, \ldots$. Let $\tilde{W}^{1,2}(Q)$ be a Sobolev space defined by

$$\tilde{W}^{1,2}(Q) = \Big\{u \in W^{1,2}_{\text{loc}}(Q); \int_Q |Du(x)|^2 r(x)\, dx + \int_Q u(x)^2\, dx < \infty\Big\}$$

equipped with the norm

$$\|u\|_{\tilde{W}^{1,2}} = \Big(\int_Q |Du(x)|^2 r(x)\, dx + \int_Q u(x)^2 dx\Big)^{1/2}.$$

By Theorem 4.11 in [**9**], $\tilde{W}^{1,2}(Q)$ is compactly embedded in $L^2(\partial Q)$. Consequently, we may assume that u_m converges to a function u in $L^2(Q)$ and a.e. in Q. On the other hand we may also assume that u_m converges weakly to u in $\tilde{W}^{1,2}(Q)$. It is easy to check that u is a solution of (18) in $\tilde{W}^{1,2}(Q)$. By Theorem 1, there exists a function $\zeta \in L^2(\partial Q)$ such that $\lim_{\delta \to 0} u(x_\delta) = \zeta$ in $L^2(\partial Q)$. To complete the proof we must check that $\phi = \zeta$ a.e. on ∂Q. Since, by (A),

$$\int_{\partial Q} \Big(\sum_{i,j=1}^{n} \int_0^{u(x_\delta)} a_{ij}(x, s)ds\, D_i\rho(x)D_j\rho(x) - \sum_{i,j=1}^{n} \int_0^{\zeta(x)} a_{ij}(x, s)ds\, D_i\rho(x)D_j\rho(x)\Big)^2$$
$$dS \leqq \gamma^2 \|u(x_\delta(\cdot)) - \zeta(\cdot)\|_{L^2(\partial Q)},$$

we have

$$\lim_{\delta\to 0}\int_0^{u(x_\delta)}\sum_{i,j=1}^n a_{ij}(x,s)\,ds\,D_i\rho(x)\,D_j\rho(x)$$
$$=\int_0^{\zeta(x)}\sum_{i,j=1}^n a_{ij}(x,s)\,ds\,D_i\rho(x)\,D_j\rho(x)$$

in $L^2(\partial Q)$. Let $\Psi\in C^1(\bar{Q})$. As in the proof of Lemma 2 we find that

$$\int_{\partial Q}\sum_{i,j=1}^n\int_0^{\zeta(x)} a_{ij}(x,s)\,ds\,D_i\rho\,D_j\rho\,\Psi\,dS_x$$
$$=-\int_Q\sum_{i,j=1}^n\int_0^u a_{ij}(x,s)\,ds\,D_{ij}\rho\,\Psi\,dx$$
$$-\int_Q\sum_{i,j=1}^n\int_0^u a_{ij}(x,s)\,ds\,D_i\Psi\,D_j\rho\,dx$$
$$-\int_Q\sum_{i,j=1}^n\int_0^u D_i a_{ij}(x,s)\,ds\,D_j\rho\,\Psi\,dx$$
$$+\int_Q\sum_{i,j=1}^n a_{ij}(x,u)D_iu\,D_j\Psi\,\rho\,dx+\int_Q c(x,u)u\,\Psi\,\rho\,dx-\int_Q f\,\Psi\,\rho\,dx.$$

Similarly, we have

$$\int_{\partial Q}\sum_{i,j=1}^n\int_0^{\phi_m} a_{ij}(x,s)\,ds\,D_i\rho\,D_j\rho\,\Psi\,dS$$
$$=-\int_Q\sum_{i,j=1}^n\int_0^{u_m} a_{ij}(x,s)\,ds\,D_{ij}\rho\,\Psi\,dx$$
$$-\int_Q\sum_{i,j=1}^n\int_0^{u_m} a_{ij}(x,s)\,dx\,D_i\Psi\,D_j\rho$$
$$-\int_Q\sum_{i,j=1}^n\int_0^{u_m} D_i a_{ij}(x,s)\,ds\,D_j\rho\,\Psi\,dx$$
$$+\int_Q\sum_{i,j=1}^n a_{ij}(x,u_m)\,D_iu_m\,D_j\Psi\rho\,dx+\int_Q c(x,u_m)\,u_m\Psi\rho\,dx-\int_Q f\,\Psi\rho\,dx.$$

By the previous part of the proof we obtain

$$\lim_{m\to\infty}\int_{\partial Q}\sum_{i,j=1}^n\int_0^{\phi_m} a_{ij}(x,s)\,ds\,D_i\rho\,D_j\rho\,\Psi\,dx$$
$$=\int_{\partial Q}\sum_{i,j=1}^n\int_0^{\zeta} a_{ij}(x,s)\,ds\,D_i\rho\,D_j\rho\,\Psi\,dx,$$

for any $\Psi\in C^1(\bar{Q})$. Since $\lim_{m\to\infty}\phi_m=\phi$ in $L^2(\partial Q)$ we deduce from the last relation that

$$\int_{\partial Q} \sum_{i,j=1}^{n} \int_0^{\phi} a_{ij}(x, s)\, ds\, D_i\rho\, D_j\rho\, \Psi\, dx$$
$$= \int_{\partial Q} \sum_{i,j=1}^{n} \int_0^{\zeta} a_{ij}(x, s)\, ds\, D_i\rho\, D_j\rho\, \Psi\, dx,$$

for all $\Psi \in C^1(\bar{Q})$ and, consequently, $\zeta = \phi$ a.e. on ∂Q.

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