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CLOSED MAPS AND SPACES WITH ZERO-DIMENSIONAL REMAINDERS

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ABSTRACT. A O-space is a completely regular Hausdorff space possessing a compactification with zero-dimensional remainder. It is well known that any O-space X possesses a maximum compactification F_0X having this property. The following question is considered: if f: $X \to Y$ is a closed map, and X, Y are O-spaces, under what conditions on X, Y and/or f will f extend to $g \in C(F_0X, F_0Y)$? It is proved that if Y is rimcompact, then it is (necessary and) sufficient that for any distinct pair of points $y, z \in Y$, $Cl_{F_0X}f^{\leftarrow}(y) \cap Cl_{F_0X}f^{\leftarrow}(z) = \emptyset$. This result is used to show that if i) X is a realcompact or metacompact O-space and Y is a rimcompact space in which the set of q-points has discrete complement, or if ii) X is a metacompact O-space or a locally compact realcompact space, and Y is a rimcompact k-space, then any closed map from X into Y extends to a map from F_0X into F_0Y.

1. Introduction and known results. A 0-space is a completely regular Hausdorff space possessing a compactification with zero-dimensional remainder. Such a compactification will be called zero-dimensional at infinity (denoted by O.I.). Any 0-space X possesses a maximum O.I. compactification ([**11**]) which we denote by F_0X. (A discussion of the standard partial ordering on the compactifications of X appears below.)

Various researchers have considered the following question. If X, Y are 0-spaces, and f: $X \to Y$ is a closed map, under what conditions on X, Y and/or f will f extend to $g \in C(F_0X, F_0Y)$? Recall that a space is rimcompact if it has a basis of open sets with compact boundaries ([**9**]). Any rimcompact space is a 0-space; the converse is not true ([**17**]). In Lemma 1 of [**5**] it is shown that if X is rimcompact, $f \in C(X, [0, 1])$, and the set $\{y \in [0, 1]: f^{\leftarrow}(y)$ contains a compact set K such that $X \backslash K$ can be written as $U \cup V$, where U, V are π-open in X and $U \subset f^{\leftarrow}[0, y]$, while $V \subset f^{\leftarrow}[y, 1]\}$ is dense in $[0, 1]$, then f extends to $g \in C[F_0X, [0, 1])$. An argument in the proof of Theorem 3 of [**15**] shows that if f: $X \to Y$ is closed, X and Y are rimcompact and $\mathrm{bd}_X f^{\leftarrow}(y)$ is compact for each $y \in Y$,

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then f extends to $g \in C(F_0X, F_0Y)$. This result is used to prove Theorem 5 of **[14]** which states that if $f\colon X \to Y$ is a closed map, and if X and Y are locally compact and paracompact, then f extends to $g \in C(F_0X, F_0Y)$. In Theorem 4 of **[16]**, it is shown that the paracompactness of X and Y can be weakened to metacompactness. The method of proof depends heavily on the local compactness of X.

We show in § 2 that if X is a 0-space and Y is rimcompact, then a closed map $f\colon X \to Y$ entends to $g \in C(F_0X, F_0Y)$ if and only if for any distinct pair of points $y, z \in Y$, $C\ell_{F_0X} f^{\leftarrow}(y) \cap C\ell_{F_0X} f^{\leftarrow}(z) = \varnothing$. In §3, we apply a corollary of this result to several classes of spaces. In particular, we show that if X is a metacompact 0-space, Y is a rimcompact quotient space of a locally compact space, and $f\colon X \to Y$ is a closed map, then f extends to $g \in C(F_0X, F_0Y)$. Thus we remove the requirement of Theorem 4 of **[16]** that X be locally compact; in fact X need not even be rimcompact. Note that since the closed image of a metacompact space is metacompact, Y is necessarily metacompact.)

In the remainder of this section, we present our notation and terminology and some known results. All spaces are assumed to be completely regular and Hausdorff. The notions used from set theory are standard. The symbol ω_α is used to denote the α'th infinite cardinal. For any set X, $|X|$ denotes the cardinality of X. A map is a continuous surjection. A function $f\colon X \to Y$ is closed if whenever F is a closed subset of X, $f[F]$ is a closed subset of Y. A closed function $f\colon X \to Y$ is perfect if for each $y \in Y$, $f^{\leftarrow}(y)$ is compact. If $A \subset X$, then the boundary of A in X, denoted by $\mathrm{bd}_X A$, is defined to be the set $C\ell_X A \cap C\ell_X(X/A)$. Following the terminology of **[13]** and **[17]**, we say that an open set U of X is π-open in X if $\mathrm{bd}_X U$ is compact. The intersection and union of finitely many π-open sets are π-open, as is the complement of the closure of a π-open set. A space is zero-dimensional (written 0-dimensional) if it has a basis of closed-and-open (denoted clopen) sets.

The family $\mathscr{K}(X)$ of (equivalence classes of) compactifications of X is partially ordered in the usual way: $JX \leqq KX$ if there is a map $f\colon KX \to JX$ such that $f(x) = x$ for all $x \in X$; KX is equivalent to JX if f is a homeomorphism. For background information on compactifications the reader is referred to [3] or [7]. The maximum element of $\mathscr{K}(X)$, the Stone-Cech compactification of X, is denoted by βX. In the sequel, if $KX \in \mathscr{K}(X)$, the natural map from βX into KX is denoted by Kf. If $f\colon X \to Y$ is a map, then the natural map from βX into βY extending f will be denoted by f^β.

A map $f\colon X \to Y$ is a WZ-map if $C\ell_{\beta X} f^{\leftarrow}(y) = f^{\beta\leftarrow}(y)$ for each $y \in Y$. Theorems 1.1, 1.2 and 1.3 of **[10]** show that a closed map is a WZ-map, and that the converse is true if either X is normal, or $\mathrm{bd}_X f^{\leftarrow}(y)$ is compact for each $y \in Y$.

The following is an easy consequence of 3.2.1 of [*6*].

PROPOSITION 1.1 (Taimanov's theorem): *Let KX and KY be compactifications of X and Y respectively, and let f be a map from X into Y. There is a map* $f': KX \to KY$ *such that* $f'|_X = f$ *if and only if, for* $A, B \subset Y$, $C\ell_{KY} A \cap C\ell_{KY} B = \emptyset$ *implies* $C\ell_{KX} f^{\leftarrow}[A] \cap C\ell_{KX} f^{\leftarrow}[B] = \emptyset$.

The next result follows from 1.5 of **[8]**.

PROPOSITION 1.2 *Let X, Y, KX, KY and f be as in* 1.1. *If f is perfect, and if f′ exists, then* $f'[KX \backslash X] = KY \backslash Y$.

We often call $KX \backslash X$ the remainder of KX. For any space X, the residue of X (denoted by $R(X)$) is the set of points at which X is not locally compact. If KX is any compactification of X, then $C\ell_{KX}(KX \backslash X) = R(X) \cup (KX \backslash X)$.

If U is an open subset of X, and $KX \in \mathscr{K}(X)$, then $Ex_{KX} U$ is defined to be $KX \backslash C\ell_{KX}(X \backslash U)$. The set $Ex_{KX} U$ is often called the extension of U in KX. It is an easy exercise to verify that if W is open in KX, then $W \subset Ex_{KX}(W \cap X)$. Hence if U is any open subset of X, then $Ex_{KX} U$ is the largest open subset of KX whose intersection with X is the set U. The collection $\{Ex_{KX} U$: U is an open subset of $X\}$ of open sets of KX is easily seen to be a basis for the topology of KX.

A compactification KX of X is a perfect compactification of X if for each open subset U of X, $C\ell_{KX}(bd_X U) = \mathrm{bd}_{KX}(Ex_{KX} U)$. According to the corollary to Lemma 1 of **[17]**, βX is a perfect compactification of X. If X is a 0-space, then $F_0 X$ is the minimum perfect compactification of X (**[11]**).

The equivalence of (i), (ii), (iii) and (iv) of the following proposition appears in Theorems 1 and 2 and Lemma 1 of **[17]**.

PROPOSITION 1.3 *Let* $KX \in \mathscr{K}(X)$. *The following are equivalent.*

(i) *KX is a perfect compactification of X.*

(ii) *If U and V are disjoint open sets of X, then* $Ex_{KX}(U \cup V) = Ex_{KX} U \cup Ex_{KX} V$.

(iii) *For each* $p \in KX$, $(Kf)^{\leftarrow}(p)$ *is a connected subset of* βX.

(iv) *If A and B are disjoint subsets of X, then* $C\ell_{KX} A \cap C\ell_{KX} B = \emptyset$ *if and only if* $C\ell_{KX} bd_X A \cap C\ell_{KX} \mathrm{bd}_X B = \emptyset$.

If X is rimcompact, then the maximum *O.I.* compactification of X is called the Freudenthal compactification of X, and is denoted by FX. If X is rimcompact, and $A, B \subseteq X$, then $C\ell_{FX} A \cap C\ell_{FX} B = \emptyset$ if and only if A, B are contained in disjoint π-open subsets of X ([9]). If X is 0-dimensional, then $FX = \beta_0 X$, where $\beta_0 X$ is the maximum 0-dimensional compactification of X.

2. Extending maps into rimcompact spaces. Suppose that f: $X \to Y$ is a map, and that KX, KY are compactifications of X and Y respectively.

According to 1.1, f extends to $g \in C(KX, KY)$ if and only if for $C, D \subset Y$, $C\ell_{KY}C \cap C\ell_{KY} D = \emptyset$ implies $C\ell_{KX}f^{\leftarrow}[C] \cap C\ell_{KX}f^{\leftarrow}[D] = \emptyset$. suppose that Y is rimcompact, that KX is a perfect compactification of X and that $f\colon X \to Y$ is a WZ-map. The following result states that to show that f extends to $g \in C(KX, FY)$, it suffices to show that $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}f^{\leftarrow}(z) = \emptyset$, where y and z are distinct points of Y.

THEOREM 2.1 *Suppose that Y is rimcompact, and that f is a WZ-map from a space X into Y. If KX is a perfect compactification of X, then the following are equivalent.*

(i) *f extends to $g \in C(KX, FY)$.*

(ii) *For any distinct pair of points $y, z \in Y$, $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}f^{\leftarrow}(z) = \emptyset$.*

PROOF. Clearly, (i) implies (ii).

(ii) implies (i). We wish to show that if $C, D \subset Y$ and $C\ell_{FY}C \cap C\ell_{FY}D = \emptyset$, then $C\ell_{KX}f^{\leftarrow}[C] \cap C\ell_{KX}f^{\leftarrow}[D] = \emptyset$. Recall that if Y is rimcompact, then $C\ell_{FY}C \cap C\ell_{FY}D = \emptyset$ if and only if C and D are contained in π-open sets of Y whose closures in Y are disjoint. It then suffices to show that if C and D are disjoint closed subsets of Y with compact boundaries in Y, then $C\ell_{KX}f^{\leftarrow}[C] \cap C\ell_{KX}f^{\leftarrow}[D] = \emptyset$.

We claim that (ii) implies the following statement: if C is a closed subset of Y with compact boundary, and $y \in Y\backslash C$, then $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}f^{\leftarrow}[C] = \emptyset$. If $y \in Y\backslash C$, then $y \notin \mathrm{bd}_Y C$. Hence if $z \in \mathrm{bd}_Y C$, i) implies that $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}f^{\leftarrow}(z) = \emptyset$. Then there is an open set $U(z)$ of X such that $C\ell_{KX}f^{\leftarrow}(z) \subset Ex_{KX}U(z)$, while $C\ell_{KX}U(z) \cap C\ell_{KX}f^{\leftarrow}(y) = \emptyset$. As $(Kf)^{\leftarrow}[Ex_{KX}U(z)] \subset Ex_{\beta X}U(z)$, it follows that $C\ell_{\beta X}f^{\leftarrow}(z) \subset Ex_{\beta X}U(z)$. Since f is a WZ-map, $C\ell_{\beta X}f^{\leftarrow}(z) = (f^{\beta})^{\leftarrow}(z)$. The map f^{β} is closed, hence there is an open set $V(z)$ of βY such that $(f^{\beta})^{\leftarrow}(z) \subset (f^{\beta})^{\leftarrow}[V(z)] \subset Ex_{\beta X}U(z)$. Let $W(z) = V(z) \cap Y$. Then $f^{\leftarrow}(z) \subset f^{\leftarrow}[W(z)] \subset U(z)$, and so $f^{\leftarrow}[\mathrm{bd}_Y C] \subset \bigcup\{f^{\leftarrow}[W(z)]: z \in \mathrm{bd}_Y C\}$. It follows that $\mathrm{bd}_Y C \subset \bigcup\{W(z): z \in \mathrm{bd}_Y C\}$. As $\mathrm{bd}_Y C$ is compact, there is a finite subset $\{z_1, z_2, \ldots, z_n\} \subset \mathrm{bd}_Y C$ such that $\mathrm{bd}_Y C \subset \bigcup\{W(z_i): 1 \leqq i \leqq n\}$. Then $f^{\leftarrow}[\mathrm{bd}_Y C] \subset \bigcup\{f[W^{\leftarrow}(z_i)]: 1 \leqq i \leqq n\} \subset \bigcup\{U(z): 1 \leqq i \leqq n\}$. Since $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}U(z_i) = \emptyset$, $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}(\bigcup\{U(z_i): 1 \leqq i \leqq n\}) = \emptyset$. As $\mathrm{bd}_X f^{\leftarrow}[C] \subset f^{\leftarrow}[\mathrm{bd}_Y C]$, $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}\mathrm{bd}_X f^{\leftarrow}[C] = \emptyset$. It then follows from 1.3 that $C\ell_{KX}f^{\leftarrow}(y) \cap C\ell_{KX}f^{\leftarrow}[C] = \emptyset$, and the claim is proved.

Suppose then that C and D are disjoint closed subsets of Y whose boundaries are compact. If $p \in \mathrm{bd}_Y D$, then $p \notin C$, hence $C\ell_{KX}f^{\leftarrow}(p) \cap C\ell_{KX}f^{\leftarrow}[C] = \emptyset$. Then there is an open set $U_1(p)$ of X such that $C\ell_{KX}f^{\leftarrow}(p) \subset Ex_{KX}U_1(p)$, and $f^{\leftarrow}[C] \cap C\ell_{KX}U_1(p) = \emptyset$. From an argument essentially identical to that in the preceding paragraph, where $f^{\leftarrow}(y)$ is replaced by $f^{\leftarrow}[C]$, it follows that $C\ell_{KX}\mathrm{bd}_X f^{\leftarrow}[D] \cap C\ell_{KX}f^{\leftarrow}[C] = \emptyset$.

Thus by 1.3, $Cl_{KX}f^{\leftarrow}[D] \cap Cl_{KX}f^{\leftarrow}[C] = \emptyset$, and the theorem is proved.

Example 4.4 of [4] illustrates that in 2.1 X, Y and f can be chosen so that X is not rimcompact, Y is rimcompact, and f is perfect (and therefore extends to $g \in C(F_0X, FY)$).

DEFINITION 2.2 Let $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ be a collection of open sets of X. A subset F of X is $\mathcal{U}$-compact if there exists a finite subset A' of A such that $F \subset \bigcup\{U_\alpha: \alpha \in A'\}$.

THEOREM 2.3 *Let f: $X \to Y$ be a WZ-map, where X is 0-space and Y is rimcompact. Suppose that for any open cover $\mathcal{U}$ of X, $Y(\mathcal{U})$ is a discrete subspace of Y, where $Y(\mathcal{U}) = \{y \in Y$: $\mathrm{bd}_X f^{\leftarrow}(y)$ is not $\mathcal{U}$-compact$\}$. If either f is closed, or X is rimcompact, then f extends to $g \in C(F_0X, FY)$.*

PROOF. According to 2.1, it suffices to show that if y and z are distinct points of Y then $Cl_{F_0X}f^{\leftarrow}(y) \cap Cl_{F_0X}f^{\leftarrow}(z) = \emptyset$. Choose $y, z \in Y$ such that $y \neq z$. Let V be an open sebset of Y such that $y \in V$, while $z \notin Cl_Y V$. Then $f^{\leftarrow}(y) \subset f^{\leftarrow}[V]$, and $f^{\leftarrow}(z) \cap f^{\leftarrow}[Cl_Y V] = \emptyset$. We define an open cover $\mathcal{U}$ of X in the following way. If $x \in f^{\leftarrow}[Cl_Y V]$, then $x \notin Cl_{F_0X}f_{\leftarrow}(z)$, so there is an open set $U(x)$ of X such that $x \in U(x)$, and $Cl_{F_0X}f^{\leftarrow}(z) \cap Cl_{F_0X}U(x) = \emptyset$. If $x \in X\backslash f^{\leftarrow}[Cl_Y V]$, then $x \notin Cl_{F_0X}f^{\leftarrow}(y)$, so there is an open set $V(x)$ of X such that $x \in V(x)$ and $Cl_{F_0X}f^{\leftarrow}(y) \cap Cl_{F_0X}V(x) = \emptyset$. We define $\mathcal{U} = \{U(x)$: $x \in f^{\leftarrow}[Cl_Y V]\} \cup \{V(x)$: $x \notin f^{\leftarrow}[Cl_Y V]\}$, which is an open cover of X. Note that $f^{\leftarrow}(y) \cap [\bigcup\{V(x)$: $x \notin f^{\leftarrow}[Cl_Y V]\}] = \emptyset = f^{\leftarrow}(z) \cap [\bigcup\{U(x): x \in f^{\leftarrow}[Cl_Y V]\}]$.

Let $Y(\mathcal{U}) = \{w \in Y$: $\mathrm{bd}_X f^{\leftarrow}(w)$ is not $\mathcal{U}$-compact$\}$. If $y \notin Y(\mathcal{U})$, then $\mathrm{bd}_X f^{\leftarrow}(y) \subset \bigcup\{U(x_i): 1 \leqq i \leqq n\}$ for some finite set $\{x_1, x_2, \ldots, x_n\} \subset f^{\leftarrow}[Cl_Y V]$. Since $Cl_{F_0X}f^{\leftarrow}(z) \cap Cl_{F_0X}(\bigcup\{U(x_i)$: $1 \leqq i \leqq n\}) = \emptyset$, it follows from 1.3 that $Cl_{F_0X}f^{\leftarrow}(y) \cap Cl_{F_0X}f^{\leftarrow}(z) = \emptyset$, and the theorem is proved. Now suppose that $y \in Y(\mathcal{U})$.

By assumption $Y(\mathcal{U})$ is a discrete subset of Y, hence there is a π-open set W of Y such that $y \in W \subset V$, and $Cl_Y W \cap Y(\mathcal{U}) = \{y\}$. If $p \in \mathrm{bd}_Y W$, then $p \notin Y(\mathcal{U})$ so there is an open set $U'(p)$ of X which is a finite union of elements of $\mathcal{U}$ such that $\mathrm{bd}_X f^{\leftarrow}(p) \subset U'(p)$ and $Cl_{F_0X}U'(p) \cap Cl_{F_0X}f^{\leftarrow}(z) = \emptyset$. It follows from 1.3, the choice of $\mathcal{U}$ and the fact that F_0X is a perfect compactification of X, that there is an open set $W(p)$ of X such that $f^{\leftarrow}(p) \subset W(p)$ and $Cl_{F_0X}W(p) \cap Cl_{F_0X}f^{\leftarrow}(z) = \emptyset$. If X is rimcompact, $W(p)$ can be chosen to be a π-open subset of X.

We claim that there is an open set $W'(p)$ of Y such that $f^{\leftarrow}(p) \subset f^{\leftarrow}[W'(p)] \subset W(p)$. This is obvious if f is a closed map. Suppose that X is rimcompact, and that $W(p)$ is π-open in X. Since f is a WZ-map, an easy computation shows that $(f^\beta)^{\leftarrow}(p) = Cl_{\beta X}f^{\leftarrow}(p) \subset Ex_{\beta X}W(p)$. Since

f^β is a closed map, we can again find the desired open set $W'(p)$ of Y, and the claim is true.

Then $f^\leftarrow[\mathrm{bd}_Y W] \subset \bigcup\{f^\leftarrow[W'(p)]: p \in \mathrm{bd}_Y W\}$, so $\mathrm{bd}_Y W \subset \bigcup\{W'(p): p \in \mathrm{bd}_Y W\}$. Since $\mathrm{bd}_Y W$ is compact, there is a finite set $\{p_1, p_2, \ldots, p_n\} \subset \mathrm{bd}_Y W$ such that $\mathrm{bd}_Y W \subset \bigcup\{W'(p_i): 1 \leqq i \leqq n\}$. Then $f^\leftarrow[\mathrm{bd}_Y W] \subset \bigcup\{f^\leftarrow[W'(p_i)]: 1 \leqq i \leqq n\} \subset \bigcup\{W(p_i): 1 \leqq i \leqq n\}$. As $C\ell_{F_0X} f^\leftarrow(z) \cap C\ell_{F_0X} W(p_i) = \varnothing$, and $\mathrm{bd}_X f^\leftarrow[W] \subset f^\leftarrow[\mathrm{bd}_Y W]$ it follows that $C\ell_{F_0X} f^\leftarrow(z) \cap C\ell_{F_0X}\mathrm{bd}_X f^\leftarrow[W] = \varnothing$. Thus by 1.3 $C\ell_{F_0X} f^\leftarrow(z) \cap C\ell_{F_0X} f^\leftarrow[W] = \varnothing$. Since $f^\leftarrow(y) \subset f^\leftarrow[W]$, $C\ell_{F_0X} f^\leftarrow(z) \cap C\ell_{F_0X} f^\leftarrow(y) = \varnothing$ and the theorem is proved.

The next result is a special case of 2.3.

Corollary. 2.4 *Suppose that f: $X \to Y$ is a closed map, where X is a* 0-*space and Y is rimcompact. Let $Y_0 = \{y \in Y$: $\mathrm{bd}_X f^\leftarrow(y)$ is not compact$\}$. If Y_0 is a discrete subspace of Y, then f extends to $g \in C(F_0X, FY)$.*

As mentioned in our summary of known results, it is shown in **[15]** that if X is rimcompact and the set Y_0 defined in 2.4 is empty, then the conclusions of 2.4 hold.

3. Applications to several classes of spaces. We now apply the results of the previous section to several classes of spaces.

A space X is a k-space if a subset F of X is closed if and only $F \cap K$ is compact for each compact subset K of X. It is well known that a space X is a k-space if and only if X is the quotient of a locally compact space, and that any first countable space is a k-space.

The following are 1.3 of **[1]** and 7.2 (d) of **[10]** respectively.

PROPOSITION. 3.1 *Suppose that Y is a k-space, and that f is a closed map from a space X into Y. If $\mathcal{U}$ is any point-finite open cover of X, and $Y(\mathcal{U}) = \{y \in Y$: $f^\leftarrow(y)$ is not $\mathcal{U}$-compact$\}$, then $Y(\mathcal{U})$ is a closed discrete subspace of Y.*

PROPOSITION. 3.2 *Suppose that X is locally compact and realcompact and that f is a closed map from X into a space Y. If $Y_0 = \{y \in Y$: $f^\leftarrow(y)$ is not compact$\}$, then Y_0 is a closed discrete subspace of Y.*

We point out that although the normality of X is included as a hypothesis in 7.2 of **[10]**, it is not required in the proof of 7.2 (d).

The following shows that the requirement that X be locally compact in Theorem 4 of **[16]** is not necessary.

THEOREM. 3.3 *Suppose that Y is a rimcompact k-space, and that X is either* (i) *locally compact and realcompact, or* (ii) *a metacompact O-space. If f: $X \to Y$ is a closed map, then f extends to $g \in C(F_0X, FY)$.*

PROOF. In the case where X is realcompact and locally compact, the theorem follows immediately from 2.4 and 3.2.

If X is a metacompact 0-space, and $\mathscr{U}$ is any open cover of X, choose $\mathscr{V}$ to be a point-finite open refinement of $\mathscr{U}$. Clearly $Y(\mathscr{U}) \subset Y(\mathscr{V})$, where $Y(\mathscr{U})$ and $Y(\mathscr{V})$ are as in 3.1. The theorem then follows from 2.4 and 3.1.

We now consider maps into q-spaces. If $x \in X$, then x is a q-point of X if there exists a sequence $\{N_i\}_{i\in\mathbf{N}}$ of neighborhoods of x such that if $x_i \in N_i$, for $i \in \mathbf{N}$, and $i \neq j$ implies that $x_i \neq x_j$, then the set $\{x_i: i \in \mathbf{N}\}$ has an accumulation point in X. A space X is a q-space if every point of X is a q-point of X.

Clearly any first countable or locally countably compact space is a q-space. An example of a countably compact space which is not a k-space is outlined in 1.10 of [**2**]. The following example shows that a k-space need not be a q-space.

EXAMPLE. 3.4 Let X be the quotient space $\mathbf{R}/\{\mathbf{N}\}$, where $\mathbf{R}$ and $\mathbf{N}$ denote the real and natural numbers respectively. Since X is the quotient of a locally compact space, X is a k-space. We show that $\{\mathbf{N}\}$ is not a q-point of X. Let $\{U_n: n \in \mathbf{N}\}$ be a sequence of open neighbourhoods of $\mathbf{N}$ in X. For each $n \in \mathbf{N}$, let V_n be an open interval of the form $(n - r_n, n + r_n)$ which is countained in U_n and $0 < r_n < 1$. If $s_n = (n + r_n)/2$, for each n, then $s_n \in U_n$, and $s_n \neq s_m$ if $n \neq m$, but $\{s_n: n \in \mathbf{N}\}$ has no accumulation point in X.

A subset F f a space X is relatively pseudocompact in X if for each $f \in C(X)$, f is bounded on F. Following the terminology of [**10**], we say that a subset F of X has property (*) if $\inf\{f(x): x \in F\} > 0$ for each $f \in C(X)$ which is positive on F. It is pointed out in [**10**] that a pseudocompact subset of X has property (*), and that a subset with property (*) is relatively pseudocompact.

DEFINITION. 3.5 *A* subset F of a space X has property (**) if for any point-finite collection $\mathscr{U} = \{U_\alpha: \alpha \in A\}$ of open sets of X covering F, there is a finite subset A' of A such that $F \subset \bigcup\{Cl_X U_\alpha: \alpha \in A'\}$.

LEMMA. 3.6 *If a subset F of a space X is pseudocompact, then F has property* (**). *If F has property* (**) *then F has property* (*).

PROOF. The proof of the first statement follows from 9.3 of [**7**]. Now suppose that F has property (**). Let $f \in C(X)$ such that f is positive on F. If $g = f \wedge 1$, then g is positive on F, and $\inf\{g(x): x \in F\} \leqq \inf\{f(x): x \in F\}$. For $n \in \mathbf{N}$ let $U(n) = g^{\leftarrow}[(1/(n + 2), 1/n)]$. Then $\{U(n); n \in \mathbf{N}\}$ is a point-finite collection of open sets of X which covers F. Since F has property (**), there is a finite subset $\{n_1, n_2, \ldots, n_m\}$ of $\mathbf{N}$ such that $F \subset$

$\bigcup\{C\ell_X U(n_i)\colon 1 \leqq i \leqq m\}$. If $m' = \max\{n_1, n_2, \ldots, n_m\}$, then $F \subset g^{\leftarrow}[[1/(m'+2), 1]]$, hence $\inf\{f(x)\colon x \in F\} \geqq \inf\{g(x)\colon x \in F\} > 0$. Thus F has property (*).

It is shown in 2.1 of **[12]** that if $f\colon X \to Y$ is a closed map, and $y \in Y$ is a q-point of Y, then $\mathrm{bd}_X f^{\leftarrow}(y)$ is relatively pseudocompact. It follows from 3.6, and the remarks preceding 3.5, that the next result generalizes this fact.

PROPOSITION 3.7 *Suppose that $f\colon X \to Y$ is a closed map. If $y \in Y$ is a q-point of Y, then* $\mathrm{bd}_X f^{\leftarrow}(y)$ *has property* (**).

PROOF. Let $\mathscr{U} = \{U_\alpha\colon \alpha \in A\}$ be a point-finite collection of open sets of X covering $\mathrm{bd}_X f(y)$. Since f is a closed map, and $f^{\leftarrow}(y) \subset \mathrm{int}_X f^{\leftarrow}(y) \cup (\bigcup\{U_\alpha\colon \alpha \in A\})$, there is an open set V of Y such that $f^{\leftarrow}(y) \subset f^{\leftarrow}[C\ell_X V] \subset \mathrm{int}_X f^{\leftarrow}(y) \cup (\bigcup\{U_\alpha\colon \alpha \in A\})$. Let $\{N_i\}_{i \in \mathbf{N}}$ be a sequence of open neighbourhoods of y in Y witnessing the fact that y is a q-point of Y. If $M_i = N_i \cap V$, then $\{M_i\colon i \in \mathbf{N}\}$ witnesses the fact that y is a q-point of Y. Suppose that for any finite subset A' of A, $\mathrm{bd}_X f^{\leftarrow}(y) \not\subset \bigcup\{C\ell_X A_\alpha\colon \alpha \in A'\}$. We will construct inductively a closed discrete set $\{x_i\colon i \in \mathbf{N}\}$ of X such that $x_i \in f^{\leftarrow}[M_i]$ for each i, and $f(x_i) = f(x_j)$ implies that $i = j$. Let $x_1 \in \mathrm{bd}_X f^{\leftarrow}(y)$. Suppose we have chosen x_i, for $i < n$, such that $x_i \in f^{\leftarrow}[M_i]$ and $f(x_i) \neq f(x_j)$ if $i \neq j$. Let $A_n = \{\alpha \in A\colon x_i \in U_\alpha, 1 \leqq i \leqq n\}$. Since $\mathscr{U}$ is a point-finite collection of subsets of X, $|A_n| < \omega$. Hence $\mathrm{bd}_X f^{\leftarrow}(y) \not\subset \bigcup\{C\ell_X U_\alpha\colon \alpha \in A_n\}$. Let $V_n = [f^{\leftarrow}[M_n] \cap (X \setminus \bigcup\{C\ell_X U_\alpha\colon \alpha \in A_n\})] \setminus [f^{\leftarrow}[f(x_2), f(x_3), \ldots, f(x_n)]]$. Since $f(x_i) \neq y$ if $i > 1$ by our inductive hypothesis, V_n is a non-empty open subset of X which intersects $\mathrm{bd}_X f^{\leftarrow}(y)$. Hence there is a point $x_n \in V_n \setminus f^{\leftarrow}(y)$. Clearly $f(x_n) \neq f(x_i)$ for $i < n$.

We claim that $\{x_i\colon i \in \mathbf{N}\}$ is a closed discrete subspace of X. Since $\{x_i\}_{i \in \mathbf{N}} \subset C\ell_X f^{\leftarrow}[V] \setminus \mathrm{int}_X f^{\leftarrow}(y)$, $C\ell_X\{x_i\colon i \in \mathbf{N}\} \subset C\ell_X f^{\leftarrow}[V] \setminus \mathrm{int}_X f^{\leftarrow}(y) \subset \bigcup\{U_\alpha\colon \alpha \in A\}$. If $x \in C\ell_X f^{\leftarrow}[V] \setminus \mathrm{int}_X f^{\leftarrow}(y)$, then $x \in U_\alpha$ for some $\alpha \in A$. If $x_j \in U_\alpha$, then $\alpha \in A_j$ and $x_m \notin U_\alpha$ for $m > j$. So, $U_\alpha \cap \{x_i\colon i \in \mathbf{N}\} \subset \{x_1, x_2, \ldots, x_j\}$, and the claim is proved.

Since f is a closed map, every subset of $f[\{x_i\colon i \in \mathbf{N}\}]$ is a closed discrete subset of Y. This contradicts the fact that since $f(x_i) \in N_i$, and $f(x_i) \neq f(x_j)$ if $i \neq j$, $\{f(x_i)\colon i \in \mathbf{N}\}$ has an accumulation point in Y. Thus $\mathrm{bd}_X f^{\leftarrow}(y) \subset \bigcup\{C\ell_X U_\alpha\colon \alpha \in A'\}$ for some finite subset A' of A, and so $\mathrm{bd}_X f^{\leftarrow}(y)$ has property (**).

We now have the following.

THEOREM. 3.8 *Suppose that Y is rimcompact, and that the set Y_0 of non q-points of Y is discrete in Y. If $f\colon X \to Y$ is closed, where X is* (i) *a metacompact* 0-*space or* ii) *a realcompact* 0-*space, then f extends to $g \in C(F_0 X, FY)$.*

PROOF. If $y \notin Y_0$, then by 3.7, $\mathrm{bd}_X f^{\leftarrow}(y)$ has property (**). If X is realcompact, then it follows from 3.6 and the remarks preceding 3.5 that $\mathrm{bd}_X f^{\leftarrow}(y)$ is compact, since any relatively pseudocompact subset of a realcompact space is compact by 8E.1 of [7].

We show that if X is metacompact, and $\mathrm{bd}_X f^{\leftarrow}(y)$ has property (**), then $\mathrm{bd}_X f^{\leftarrow}(y)$ is compact. According to 17B.1, 17K.2 and 17K.3 of **[18]**, it suffices to show that if $\mathscr{V} = \{V_\alpha : \alpha \in A\}$ is a collection of open sets of X such that $\mathrm{bd}_X f^{\leftarrow}(y) \subset \bigcup\{V_\alpha : \alpha \in A\}$, then there is a finite subcollection of $\mathscr{V}$ whose closures cover $\mathrm{bd}_X f^{\leftarrow}(y)$. Let $\mathscr{V}$ be such a collection. Then $\mathscr{V}' = \mathscr{V} \cup \{\mathrm{int}_X f^{\leftarrow}(y), X \backslash f^{\leftarrow}(y)\}$ is an open cover of X. Choose $\mathscr{W}$ to be a point-finite open refinement of $\mathscr{V}'$. Then $\mathscr{U} = \{w \in \mathscr{W} : W \cap \mathrm{bd}_X f^{\leftarrow}(y) \neq \varnothing\}$ is a point-finite refinement of $\mathscr{V}$ which covers $\mathrm{bd}_X f^{\leftarrow}(y)$. Since $\mathrm{bd}_X f^{\leftarrow}(y)$ has property (**), there is a finite subcollection of $\mathscr{U}$ whose closures cover $\mathrm{bd}_X f^{\leftarrow}(y)$. Since $\mathscr{U}$ refines $\mathscr{V}$, there is a finite subcollection of $\mathscr{V}$ whose closures cover $\mathrm{bd}_X f^{\leftarrow}(y)$. Thus $\mathrm{bd}_X f^{\leftarrow}(y)$ is compact.

Then if X is either realcompact or metacompact, and $y \notin Y_0$, $\mathrm{bd}_X f^{\leftarrow}(y)$ is compact. It follows that if $Y_1 = \{y \in Y : \mathrm{bd}_X f^{\leftarrow}(y)$ is not compact$\}$, then $Y_1 \subset Y_0$, hence Y_1 is a discrete subspace of X. Thus by 2.4 f extends to $g \in C(F_0X, FY)$.

There are examples of maps of rimcompact spaces which do not extend to maps of the respective Freundenthal compactifications. The following is Example 1 of **[16]**.

EXAMPLE. 3.9 Let $X = \omega_1 \times I$, and let $Y = I$, where I denotes the unit interval. Then X is locally compact and Y is compact. Let f be the projection map from X onto Y. Then f is an open map. Since ω_1 is countably compact, f is also closed. However FX is the one-point compactification of $\omega_1 \times I$. Clearly f does not extend to $g \in C(FX, I)$.

In fact, if X is any countably compact 0-space such that $\beta X \backslash X$ is not 0-dimensional, there exists $f \in C(X, [0, 1])$ such that f does not extend to $g \in C(FX, [0, 1])$. For any bounded continuous real-valued function on X is closed. Thus if X is not C^*-embedded in F_0X, (i.e., if $F_0X \neq \beta X$), there is a closed function from X into I which does not extend over F_0X. This is not true if "countably compact" is weakened to "pseudocompact".

A collection of infinite subsets of $\mathbf{N}$ is called almost disjoint if the intersection of two distinct members is finite. Zorn's lemma implies that there exists a maximal collection of almost disjoint infinite subsets of $\mathbf{N}$. In the following $\mathscr{R}$ will denote a maximal such collection. The following topology on $\mathbf{N} \cup \mathscr{R}$ is credited to Isbell in [7]. Each point of $\mathbf{N}$ is isolated, and $\lambda \in \mathscr{R}$ has as an open base $\{\{\lambda\} \cup (\lambda \backslash F) : F$ is a finite subset of $\mathbf{N}\}$. It is noted in 5I of [7] that such spaces $\mathbf{N} \cup \mathscr{R}$ are first countable, locally compact, 0-dimensional and pseudocompact.

THEOREM. 3.10 *Let f: $\mathbf{N} \cup \mathscr{R} \to Y$ be a closed map, where Y is any space. Then Y is 0-dimensional, and f extends to $g \in C(F(\mathbf{N} \cup \mathscr{R}), FY)$.*

PROOF. First note that if $y \in Y \backslash f[\mathscr{R}]$, then $f^{\leftarrow}(y)$ is a closed subset of $\mathbf{N} \cup \mathscr{R}$ contained in $\mathbf{N}$, hence $f^{\leftarrow}(y)$ is a finite subset of $\mathbf{N} \cup \mathscr{R}$ contained in $\mathbf{N}$. Then $f^{\leftarrow}(y)$ is open in $\mathbf{N} \cup \mathscr{R}$. Since f is a quotient map, y is isolated in Y, hence has a basis of clopen subsets of Y.

Since $\mathscr{R}$ is a closed discrete subset of $\mathbf{N} \cup \mathscr{R}$, and F is a closed map, $f[\mathscr{R}]$ is a closed discrete subset of Y. Suppose that $y \in f[\mathscr{R}]T\backslash$, where T is closed in Y. Then there is an open subset U of Y such that $y \in U$, $U \cap T = \emptyset$, and $U \cap f[\mathscr{R}] = \{y\}$. Choose V to be open in Y such that $y \in V \subset C\ell_Y V \subset U$. Then $\mathrm{bd}_Y V \subset Y \backslash f[\mathscr{R}]$. Since each point of $\mathrm{bd}_Y V$ is isolated in Y, $\mathrm{bd}_Y V$ is open in Y, hence $V = C\ell_Y V$ is clopen in Y. It follows that Y is 0-dimensional. Then f extends to $g \in C(\beta_0(\mathbf{N} \cup \mathscr{R}), \beta_0 Y)$. Since $\beta_0(\mathbf{N} \cup \mathscr{R}) = F(\mathbf{N} \cup \mathscr{R})$ and $\beta_0 Y = FY$, the theorem follows.

DEFINITION 3.11 *A map f: $X \to Y$ is monotone if $f^{\leftarrow}(y)$ is connected for each $y \in Y$.*

The following is 4.7 of [**4**].

THEOREM 3.12 *Let f: $X \to Y$ be a monotone quotient map, and let KX, KY be perfect compactifications of X and Y respectively. If f extends to $g \in C(KX, KY)$, then g is monotone.*

COROLLARY 3.13 *Suppose that X is a 0-space and Y is rimcompact. If there is a perfect monotone map from X into Y, then $F_0X \backslash X$ is homeomorphic to $FY \backslash Y$.*

PROOF. Let f: $X \to Y$ be a perfect monotone map. Then f extends to $g \in C(F_0X, FY)$ by 2.1. Since f is perfect, $g^{\leftarrow}[FY \backslash Y] = F_0X \backslash X$. As f is monotone, it follows from 3.12 that $g^{\leftarrow}(y)$ is connected for each $y \in FY \backslash Y$. Since $F_0X \backslash X$ is 0-dimensional, and $g^{\leftarrow}(y) \subset F_0X \backslash X$, $|g^{\leftarrow}(y)| = 1$. Thus $g|_{F_0X\backslash X}$: $F_0X \backslash X \to FY \backslash Y$ is a closed continuous one-to-one map, hence g is a homeomorphism.

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