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WREATH PRODUCTS INDEXED BY PARTIALLY ORDERED SETS

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1. Introduction. In his classic paper Wreath powers and characteristically simple groups [**6**], Philip Hall gave a generalized definition of the wreath product of an infinite number of permutation groups indexed by a totally ordered set. Until this paper, wreath products had usually been described as ascending unions of finite wreath products. Hall's generalized definition allows one to construct more elaborate wreath products and, in fact, he constructs various characteristically simple groups using his definition.

Hall's definition has itself been generalized by Holland [**7**] and later by Dixon and Fournelle [**1, 2**] (see also McCleary [**10**]). As with the standard wreath product $A \wr B$ where one speaks of unrestricted and restricted wreath products, Holland's wreath products are unrestricted while those of Dixon and Fournelle are restricted. Both, however, consider the situation in which the indexing set is partially ordered rather than totally ordered.

The purpose of this paper is to extend some of the main results of Hall's paper [**6**] to the generalized (restricted) wreath products of [**1, 2**]. In particular, we let

$$W = \operatorname*{Wr}_{\lambda \in \Lambda} G_\lambda$$

be the wreath product of the permutation groups G_λ indexed by the partially ordered set Λ. We say that Λ is *upwardly directed* if for all λ, $\mu \in \Lambda$, there is a $\tau \in \Lambda$ such that $\lambda \leqq \tau$ and $\mu \leqq \tau$. Similarly, we way that Λ is downwardly directed if for all λ, $\mu \in \Lambda$, there is some $\tau \in \Lambda$ such that $\tau \leqq \lambda$ and $\tau \leqq \mu$. We have the following analogues of Theorems A and B of [**6**].

THEOREM 1. *If Λ is upwardly directed and has no maximal element and if each G_λ is non-trivial and transitive, then $W' = W''$ and any normal subgroup of W' is normal in W.*

THEOREM 2. *If Λ is downwardly directed and has no minimal element and*

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each G_λ is non-trivial and transitive, then W has no non-trivial finitely generated subnormal subgroups.

While the proofs of Theorems 1 and 2 are almost direct transcriptions of Hall's Theorems A and B, our analogue of his Theorem C has a more difficult proof (which may be found in **[1]**). Specifically, we have:

THEOREM 3. *Suppose $\mathfrak{X}$ is a class of groups which is closed with respect to forming (a)direct powers and (b)split extensions.*
(i) *If Λ is finite and $G_\lambda \in \mathfrak{X}$ for all $\lambda \in \Lambda$ then $W \in \mathfrak{X}$.*
(ii) *If $G_\lambda \in L\mathfrak{X}$, for all $\lambda \in \Lambda$, then $W \in L\mathfrak{X}$.*

Theorem D of Hall's paper involves automorphisms of wreath powers induced by order automorphisms of the indexing set Λ. We will prove:

THEOREM 4. *If θ is an order automorphism of Λ and if for each $\lambda \in \Lambda$ there is a permutation group isomorphism $\theta_\lambda : G_\lambda \to G_{\lambda\theta}$ which satisfies a certain compatibility condition, then there is an automorphism of W which restricts on the factor G_λ to θ_λ.*

The exact compatibility condition of Theorem 4 is stated in section 4. There are two extreme cases here which are of interest. First, suppose that each G_λ is isomorphic to a given group G. Then we may take θ to be any order automorphism and each θ_λ to be (essentially) the identity. The induced automorphism is the one discussed by Hall in **[6]**. Next, we may take θ to be the identity and let θ_λ be any permutation automorphism of G_λ. The induced automorphism of W then restricts to θ_λ on the factor G_λ (cf. Houghton [8]).
We also prove:

THEOREM 5. *Let $W = \mathrm{Wr}_{\lambda \in \Lambda}\, G_\lambda$ where each G_λ is finite.*
(i) *If $\Gamma_\lambda = \{\mu \in \Lambda : \mu > \lambda\}$ is finite for all $\lambda \in \Lambda$, then W is residually finite.*
(ii) *If $U_\lambda = \{\mu \in \Lambda : \mu > \lambda \text{ or } \mu < \lambda\}$ is finite for all $\lambda \in \Lambda$, then W is an FC-group.*
By FC-group we mean, of course, a group in which each element has only finitely many conjugates.

As applications of our generalized definition we mention several constructions. First, for any group G we let $\Lambda(G)$ be the finite non-trivial subgroups of G ordered by inclusion. Define

$$W(G) = \mathop{\mathrm{Wr}}_{H \in \Lambda(G)} G_H$$

where $G_H = H$ considered to be acting on itself by right multiplication. By Theorem 3, $W(G)$ is a locally finite group. In fact, if G is an infinite locally finite group then $W(G)$ is a locally finite group whose cardinality is the same as that of G and which satisfies the hypotheses of Theorem 1.

If in addition G contains an element of order p, then it will be shown elsewhere that $W(G)$ contains an isomorphic copy of $\underbrace{(\cdots(C_p \wr C_p) \wr \cdots \wr C_p)}_{n \text{ factors}}$ for all n. Hence, $W(G)$ contains a copy of every finite p-group(see for example Robinson **[12]** p. 41). If G has an element of order p for all primes p, then $W(G)$ has a copy of every finite p-group and is, in the terminology of Kegel and Wehrfritz [**9**, p. 122], enormous. Also, notice that automorphisms of G induce order automorphisms of $\Lambda(G)$ and isomorphisms of the finite subgroups of G satisfying the compatibility condition of Theorem 4. In fact, if G is periodic then Aut G acts faithfully on $W(G)$ (Corollary 4.3).

Similarly, for any group G we let $\Lambda^*(G)$ be the set of subgroups of finite index in G, ordered by inclusion. Next we set

$$W^*(G) = \underset{H \in \Lambda^*(G)}{\mathrm{Wr}} G_H$$

where $G_H = G/\mathrm{Core}_G(H)$ acting on itself by right multiplication. Again $W^*(G)$ is locally finite. By Theorem 5(i), $W(G)$ is residually finite. If G is residually finite it will also be shown that Inn G acts faithfully on $W^*(G)$ (Corollary 4.4). If G is infinite and residually finite, then $W^*(G)$ satisfies the hypotheses of Theorem 2.

Finally, in section 6 we prove the following.

THEOREM 6. *For every prime* p *there exists an infinite, locally finite, residually finite, FC-group which is neither Hopfian nor co-Hopfian. In addition, this group is of exponent* p^2, *nilpotent, solvable of derived length* 2, *and is indecomposible as a direct product of non-trivial subgroups.*

Note that the groups of Theorem 6 could be easily constructed if we dropped the requirement that they be indecomposible, for then we may just take an infinite direct power of an appropriate finite p-group. Also, the groups of Theorem 6 provide an interesting contrast to some recent work of Gupta and Sidki [**3**, **4**, **5**]. Specifically, for every prime p they construct a 2-generator p-group which is residually finite, contains an isomorphic copy of every finite p-group, and in which every proper quotient is finite.

2. Definitions and Background Results. Let Λ be a partially ordered set and let $\{(G_\lambda, X_\lambda) : \lambda \in \Lambda\}$ be a set of permutation groups where each group G_λ acts on the set X_λ. For each λ we single out an element $1_\lambda \in X_\lambda$. (In many cases G_λ will be acting on itself by right multiplication and then 1_λ will be chosen to be the identity element of $X_\lambda = G_\lambda$.) Now we let X be the restricted direct product of the sets X_λ, that is, X is the collection of all "vectors" $x = (x_\lambda)_{\lambda \in \Lambda}$ where $x_\lambda = 1_\lambda$ for all but finitely many λ.

For $x \in X$ and $\lambda \in \Lambda$ we define

$$x \equiv 1 \pmod{\lambda}$$

to mean that $x_\mu = 1_\mu$ for all $\mu > \lambda$. Now for $x \in X$ and $g \in G_\lambda$ we define $x\bar{g} = x$ if $x \not\equiv 1 \pmod{\lambda}$ and if $x \equiv 1 \pmod{\lambda}$ we set $x\bar{g} = y$ where

$$y_\mu = x_\mu, \text{ if } \mu \neq \lambda$$

and

$$y_\lambda = x_\lambda g.$$

The mapping $g \mapsto \bar{g}$ is an isomorphism from G_λ to a subgroup $\bar{G}_\lambda$ of $\mathrm{Sym}(X)$, the group of all permutations on X. We define the wreath product of the permutation groups (G_λ, X_λ) to be

$$W = \langle \bar{G}_\lambda \colon \lambda \in \Lambda \rangle$$

considered as a permutation group on the set X. W will be denoted by $\mathrm{Wr}_{\lambda\in\Lambda}(G_\lambda, X_\lambda)$ or by $\mathrm{Wr}_{\lambda\in\Lambda} G_\lambda$ when the sets X_λ are clearly understood. Also, if each G_λ is isomorphic with a single G we write $W = Wr\, G^\Lambda$ which is called the wreath power of G. Note that this is actually Hall's definition in [6]. We are merely allowing Λ to be partially ordered rather than totally ordered. As with his definition $\mathrm{Wr}_{\lambda\in\Lambda} G_\lambda$ is transitive if each G_λ is transitive, and in this case the choice of 1_λ is irrelevant.

By a *segment* of the partially ordered set Λ we mean a non-empty subset Γ such that for all $\lambda \in \Lambda$ exactly one of the following holds: either $\lambda \in \Gamma$, or $\lambda < \mu$ for all $\mu \in \Gamma$, or $\lambda > \mu$ for all $\mu \in \Gamma$, or λ is unrelated to μ for all $\mu \in \Gamma$. Now suppose that $\Lambda = \mathring{\bigcup}_{i\in I} \Gamma_i$ is a disjoint union of segments. We may partially order I by setting $i < j$ exactly when there is some $\lambda \in \Gamma_i$ and $\mu \in \Gamma_j$ with $\lambda < \mu$. If we let $X_{(i)}$ denote the restricted direct product of the sets X_λ for $\lambda \in \Gamma_i$ and $1_i = (1_\mu)_{\mu\in\Gamma_i}$ we have:

GENERALIZED SEGMENTATION LAW

$$\underset{\lambda\in\Lambda}{\mathrm{Wr}}\, (G_\lambda, X_\lambda) \simeq \underset{i\in I}{\mathrm{Wr}}\, (\underset{\lambda\in\Gamma_i}{\mathrm{Wr}}\, (G_\lambda, X_{(i)})), \text{ as permutation groups.}$$

Suppose now that $\Lambda = \Gamma_i \mathring{\cup} \Gamma_2$ where Γ_1 and Γ_2 are segments and $1 < 2$. If G_λ is transitive for $\lambda \in \Gamma_2$, then $W \simeq (\mathrm{Wr}_{\lambda\in\Gamma_1} G_\lambda) \wr (\mathrm{Wr}_{\lambda\in\Gamma_2} G_\lambda)$, where $A \wr B$ denotes the standard wreath product of A and B.

Another useful result is the following.

GENERAL EMBEDDING LEMMA. *Suppose that Γ is a subset of Λ with the partial ordering inherited from Λ. Then*

$$\underset{\gamma\in\Gamma}{\mathrm{Wr}}\, (G_\gamma, X_\gamma) \leq \underset{\lambda\in\Lambda}{\mathrm{Wr}}\, (G_\lambda, X_\lambda)$$

and this embedding is as permutation groups.

A more complete discussion of the results of this section may be found in [1].

3. Proofs of Theorems 1 and 2 To prove Theorem 1 we assume that $W = \mathrm{Wr}_{\lambda \in \Lambda} G_\lambda$ where each G_λ is non-trivial and transitive, and that Λ is upwardly directed with no maximal element. Now let $\xi \in W'$. Then there are elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Lambda$ such that $\xi \in L'$ where $L = \langle \bar{G}_{\lambda_i} : \mathrm{i} = 1, 2, \ldots, n \rangle$. Since Λ is upwardly directed and has no maximal element, there exist $\rho, \tau \in \Lambda$ such that $\lambda_i < \rho < \tau$ for all i. Let $K = \langle \bar{G}_\rho, \bar{G}_\tau \rangle$. Then $\langle L, K \rangle = L \wr K \leqq W$ by the General Segmentation Law and the General Embedding Lemma. It now follows from Lemma 5 of [6] that $L' \leqq (L \wr K)''$. (Note that Lemma 5 of [6] is due to B. H. Neumann [11], p. 60.) Hence, $\xi \in W''$ and therefore $W' = W''$.

The rest of the proofs of Theorems 1 and 2 will be omitted since they follow in precisely the same manner, namely by the Genreal Segmentation Law and the General Embedding Lemma we reduce the proof to basic results about standard wreath products which may be found in section 3 of [6].

4. Automorphisms of Wreath Products Now suppose that θ is an order automorphism of Λ. Hence, θ^{-1} exists and $\lambda < \mu$ if and only if $\lambda\theta < \mu\theta$ if and only if $\lambda\theta^{-1} < \mu\theta^{-1}$. Suppose that for each λ we have

$$\theta_\lambda : G_\lambda \to G_{\lambda\theta}$$

$$\bar{\theta}_\lambda : X_\lambda \to X_{\lambda\theta}$$

where $(\theta_\lambda, \bar{\theta}_\lambda)$ is a permutation group isomorphism between (G_λ, X_λ) and $(G_{\lambda\theta}, X_{\lambda\theta})$. Thus,

$$(x_\lambda g_\lambda)\bar{\theta}_\lambda = (x_\lambda \bar{\theta}_\lambda)(g_\lambda \theta_\lambda)$$

for all $x \in X$ and all $g_\lambda \in G_\lambda$. Further, we assume that $1_\lambda \bar{\theta}_\lambda = 1_{\lambda\theta}$ for all λ. We then call $(\theta, \theta_\lambda, \bar{\theta}_\lambda)_{\lambda \in \Lambda}$ a system of automorphisms for $\mathrm{Wr}_{\lambda \in \Lambda} G_\lambda$, or simply a system. Corresponding to such a system we may define a function $\alpha_\theta : X \to X$ by setting

$$(x\alpha_\theta)_\mu = (x_{\mu\theta^{-1}})\bar{\theta}_{\mu\theta^{-1}}.$$

The following lemma has a purely computational proof which we will omit.

LEMMA 4.1. *Let* $(\theta, \theta_\lambda, \bar{\theta}_\lambda)_{\lambda \in \Lambda}$ *and* $(\phi, \phi_\lambda, \bar{\phi}_\lambda)_{\lambda \in \Lambda}$ *be systems giving rise to* α_θ *and* α_ϕ, *respectively. Then,*

(i) $\alpha_\theta \alpha_\phi$ *is induced by the system*

$$(\theta\phi, \theta_\lambda \phi_{\lambda\theta}, \bar{\theta}_\lambda \bar{\phi}_{\lambda\theta})_{\lambda \in \Lambda},$$

(ii) $(\alpha_\theta)^{-1} = \alpha_{\theta^{-1}}$ is induced by the system

$$(\theta^{-1}, (\theta_{\lambda\theta^{-1}})^{-1}, (\bar{\theta}_{\lambda\theta^{-1}})^{-1})_{\lambda\in\Lambda}, \text{ hence,}$$

$$(x\alpha_{\theta^{-1}})_\mu = x_{\mu\theta}(\bar{\theta}_\mu)^{-1},$$

and (iii) $x\,\alpha_{\theta^{-1}} \equiv 1 \pmod{\lambda}$ if and only if

$$x \equiv 1 \pmod{\lambda\theta}.$$

If we let A denote the set of all α_θ induced by systems $(\theta, \theta_\lambda, \bar{\theta}_\lambda)_{\lambda\in\Lambda}$, then it follows at once that A is a subgroup of Sym(X). We may now prove the main result of this section.

PROPOSITION 4.2. *A normalizes $W = \mathrm{Wr}_{\lambda\in\Lambda}G_\lambda$ in* Sym(X). *In particular, if $\alpha_\theta \in A$* and *$g_\lambda \in G_\lambda$, then $\alpha_{\theta^{-1}}\bar{g}_\lambda\alpha_\theta = \overline{g_\lambda\theta_\lambda}$, and hence $(G_\lambda)^{\alpha_\theta} = \bar{G}_{\lambda\theta}$*

PROOF. It suffices to prove that $\alpha_{\theta^{-1}}\bar{g}_\lambda\alpha_\theta = \overline{g_\lambda\theta_\lambda}$. Thus, 13t $x \in X$ and assume $x \equiv 1 \pmod{\lambda\theta}$. Then $y = x\,\alpha_{\theta^{-1}} \equiv 1 \pmod{\lambda}$. Let $z = y\bar{g}_\lambda$. Then we have.

$$\begin{aligned}
(x\alpha_{\theta^{-1}}\bar{g}_\lambda\alpha_\theta)_\mu &= (z_{\mu\theta^{-1}})\bar{\theta}_{\mu\theta^{-1}} \\
&= (y\bar{g}_\lambda)_{\mu\theta^{-1}}\bar{\theta}_{\mu\theta^{-1}} = \begin{cases} y_{\mu\theta^{-1}}\bar{\theta}_{\mu\theta^{-1}}, & \mu\theta^{-1} \neq \lambda \\ y_{\mu\theta^{-1}}g_\lambda\,\bar{\theta}_{\mu\theta^{-1}}, & \mu\theta^{-1} = \lambda \end{cases} \\
&= \begin{cases} (y\alpha_\theta)_\mu, & \mu\theta^{-1} \neq \lambda \\ ((x\alpha_{\theta^{-1}})_\lambda g_\lambda)\bar{\theta}_{\mu\theta^{-1}}, & \mu\theta^{-1} = \lambda \end{cases} \\
&= \begin{cases} (x\alpha_{\theta^{-1}}\alpha_\theta)_\mu, & \mu\theta^{-1} \neq \lambda \\ (x_{\lambda\theta}(\bar{\theta}_\lambda)^{-1}g_\lambda)\bar{\theta}_\lambda, & \mu\theta^{-1} = \lambda \end{cases} \\
&= \begin{cases} x_\mu, & \mu \neq \lambda\theta \\ (x_{\lambda\theta})\,(g_\lambda\theta_\lambda), & \mu = \lambda\theta \end{cases} \\
&= (x\overline{g_\lambda\theta_\lambda})_\mu.
\end{aligned}$$

Now if $x \not\equiv 1 \pmod{\lambda\theta}$ we also have $(x\alpha_{\theta^{-1}}\bar{g}_\lambda\alpha_\theta)_\mu = (xg_\lambda\theta_\lambda)_\mu = x_\mu$ for all μ and this completes the proof.

As an immediate corollary we have:

THEOREM 4. *Let θ be an order automorphism of Λ, and for each $\lambda \in \Lambda$ let $(\theta_\lambda, \bar{\theta}_\lambda)$: $(G_\lambda, X_\lambda) \to (G_{\lambda\theta}, X_{\lambda\theta})$ be a permutation group isomorphism. Suppose also that $1_\lambda\bar{\theta}_\lambda = 1_{\lambda\theta}$ for all λ. Then there is an automorphism α of the wreath product* $\mathrm{Wr}_{\lambda\in\Lambda}G_\lambda$ *such that $(\bar{G}_\lambda)^\alpha = \bar{G}_{\lambda\theta}$.*

We also have two other interesting corollaries.

COROLLARY 4.3. *If G is periodic then* Aut *G acts faithfully on $W(G)$.*

PROOF. Recall that $W(G)$ is the wreath product of all of the finite subgroups of G indexed by inclusion. Clearly, automorphisms of G do induce systems for $W(G)$ and therefore Aut G does act on $W(G)$.

Now suppose that $1 \neq \alpha \in \operatorname{Aut} G$. Then there is some $x \in G$ such that $x\alpha \neq x$. Since $\langle x \rangle$ is finite it is a factor of $W(G)$ on which α acts non-trivially. Thus, Aut G acts faithfully on $W(G)$ and the proof is complete.

COROLLARY 4.4. *If G is residually finite then* Inn *G acts faithfully on $W^*(G)$.*

PROOF. Inner automorphisms of G induce systems for $W^*(G)$ and thus Inn G acts on $W^*(G)$. Suppose $x, g \in G$ and $x^g \neq x$. Suppose that conjugation by g induces the identity on $W^*(G)$. Then $x^{-1}x^g \equiv 1 \bmod N$ for all normal subgroups N of finite index in G. Since the intersection of these subgroups is trivial we have $x^{-1}x^g = 1$, which is impossible. Hence, non-trivial inner automorphisms of G induce non-trivial automorphisms of $W^*(G)$.

5. Residual Finiteness and Finite Conjugacy If $W = \mathrm{Wr}_{\lambda \in \Lambda} G_\lambda$ we let

$$H_\Gamma = \{g \in W\colon (xg)_\mu = x_\mu, \forall x \in X, \forall \mu \in \Gamma\}$$

where Γ is some subset of Λ. Clearly H_Γ is a subgroup of W. We say that Γ is a filter in Λ if $\lambda \in \Gamma$ and $\mu > \lambda$ implies that $\mu \in \Gamma$.

LEMMA 5.1. *If Γ is a filter in Λ then H_Γ is normal in W.*

PROOF. Since $H_\varnothing = W$ we may assume that Γ is non-empty. Let $g \in G_\lambda$ and $h \in H_\Gamma$. If $\lambda \notin \Gamma$ then $\bar{g} \in H_\Gamma$ since $\bar{g}$ only alters the λ-coordinate, if any, of any $x \in X$. Hence, $\bar{g}^{-1}h\bar{g} \in H_\Gamma$. Thus, we assume that $\lambda \in \Gamma$. First suppose that $x \not\equiv 1 \pmod{\lambda}$. Then $x\bar{g}^{-1}h\bar{g} = xh\bar{g} = xh$. Next suppose that $x \equiv 1 \pmod{\lambda}$. Then $x\,\bar{g}^{-1} \equiv 1 \pmod{\lambda}$. Hence, $x\bar{g}^{-1}h \equiv 1 \pmod{\lambda}$ and if $\mu \in \Gamma$, $\mu \neq \lambda$, then $(x\bar{g}^{-1}h\,\bar{g})_\mu = (x\bar{g}^{-1}\,h)_\mu = (x\bar{g}^{-1})_\mu = x_\mu$. Also, $(x\bar{g}^{-1}h\bar{g})_\lambda = x_\lambda$. Hence $\bar{g}^{-1}h\bar{g}$ fixes the μ-coordinate of x for all $\mu \in \Gamma$ and the proof is complete.

We have an immediate corollary.

COROLLARY 5.2. *If Γ is a filter in Λ, then W is a split extension of H_Γ by $\langle \bar{G}_\lambda\colon \lambda \in \Gamma \rangle$.*

We may now prove Theorem 5(i), for suppose that the filter $\Gamma_\lambda = \{\mu \in \Lambda : \mu > \lambda\}$ is finite for all $\lambda \in \Lambda$ and that each G_λ is finite. Then by Corollary 5.2 it follows that W/H_{Γ_λ} is finite. Furthermore, $\bigcap_{\lambda \in \Lambda} H_{\Gamma_\lambda} = H_\Lambda = 1$. Hence, W is residually finite.

COROLLARY 5.3. *For all groups G the group $W^*(G)$ is residually finite.*

We now will discuss conditions which insure that W is an FC-group, that is, a group in which each element has only finitely many conjugates. To do this we need two lemmata on conjugacy in wreath products.

LEMMA 5.4. *Let $g \in G_\lambda$ and $w \in W = \mathrm{Wr}_{\lambda \in \Lambda} G_\lambda$. Then for each $x \in X$ we have $\mu \leqq \lambda$ if $(x\bar{g}^w)_\mu \neq x_\mu$.*

PROOF. Let $w = \bar{g}_1\bar{g}_2 \cdots \bar{g}_n$ where $g_i \in G_{\lambda_i}$. Clearly the result holds for $n = 1$. Thus by induction we may assume $(x\bar{g}^w)_\mu \neq x_\mu$ only for $\mu \leqq \lambda$ and consider $(x\bar{g}^{w\bar{h}})_\mu$ where $h \in G_\mu$ and $\mu \nleq \lambda$. But since $(x\bar{h}^{-1}\bar{g}^w)_\mu = (x\bar{h}^{-1})_\mu$ it follows that $(x\bar{h}^{-1}\bar{g}^w\bar{h})_\mu = x_\mu$ and the result follows.

For $\lambda \in \Lambda$ we now define D_λ to be the set of all $\tau \in \Lambda$ such that either $\lambda \leqq \tau$, or $\tau \leqq \lambda$, or such that there is a $\mu \in \Lambda$ with $\mu < \tau$ and $\mu < \lambda$.

LEMMA 5.5. *Let* $\lambda \in \Lambda$, $w \in \langle \bar{G}_\mu \colon \mu \in D_\lambda \rangle$, $g \in G_\lambda$, *and* $h \in G_\tau$ *where* $\tau \notin D_\lambda$. *Then* $[\bar{g}^w, \bar{h}] = 1$.

PROOF. Let $x \in X$. Suppose $x \not\equiv 1 \pmod{\tau}$. Then $x\bar{h}\bar{g}^w = x\bar{g}^w$. Now $(x\bar{g}^w)_\mu = x_\mu$ for $\mu \nleq \lambda$. Hence $x\bar{g}^w \not\equiv 1 \pmod{\tau}$ and $x\bar{g}^w\,\bar{h} = x\bar{g}^w$.

Next suppose $x \equiv 1 \pmod{\tau}$. Then for all $\mu > \tau$ we have $x_\mu = 1_\mu$, and $(x\bar{h})_\tau = x_\tau h$ and $(x\bar{h})_\rho = x_\rho$ for $\rho \neq \tau$. Since $\tau \notin D_\lambda$ there is no $\mu \in D_\lambda$ such that $\tau \geqq \mu$. In particular,

$$(x\bar{h}\bar{g}^w)_\tau = x_\tau h$$

and (5.1)

$$(x\bar{h}\bar{g}^w)_\mu = (x\bar{g}^w)_\mu, \text{ for } \mu \neq \tau.$$

Now consider $x\bar{g}^w\,\bar{h}$. By Lemma 5.4. we have $(x\bar{g}^w)_\mu = x_\mu$ if $\mu \nleq \lambda$. Suppose that $\mu > \tau$. Then $\mu \nleq \lambda$ and $(x\bar{g}^w)_\mu = x_\mu$. Thus, $x \equiv 1 \pmod{\tau}$ implies that $x\bar{g}^w \equiv 1 \pmod{\tau}$. Hence, $(x\bar{g}^w\bar{h})_\tau = (x\bar{g}^w)_\tau\, h = x_\tau h$ and $(x\bar{g}^w\bar{h})_\mu = (x\bar{g}^w)_\mu$ for $\mu \neq \tau$. Comparing this with (5.1) we see that $\bar{g}^w\bar{h} = \bar{h}\,\bar{g}^w$ and the proof is complete.

As a consequence of this last result if we wish to find all the conjugates of $\bar{g} \in \bar{G}_\lambda$ we need only conjugate $\bar{g}$ by elements of $\langle \bar{G}_\mu \colon \mu \in D_\lambda \rangle$. Now let $U_\lambda = \{\mu \in \Lambda \colon \mu \geqq \lambda \text{ or } \mu \leqq \lambda\}$. Note that if U_λ is finite for all $\lambda \in \Lambda$ then D_λ is finite for all $\lambda \in \Lambda$. Theorem 5 (ii) now is an immediate consequence.

6. An Example (Proof of Theorem 6) Let Ω be the set of integers with the following partial ordering: $2n - 1 > 2n$, $2n - 1 > 2n - 2$ for all n. Let $L = \mathrm{Wr}\, C_p^\Omega$ be the wreath power of a cyclic group of order p over Ω. Let $\theta \colon \Omega \to \Omega$ be the order automorphism $n \mapsto n + 2$. By Theorem 4, θ induces an automorphism α_θ of L which sends the nth factor to the $(n + 2)$th factor. Now let Λ be the set of positive integers with the ordering inherited from Ω and set $G = \mathrm{Wr}\, C_p^\Lambda$. Then $G \leqq L$ and α_θ restricts to a monomorphism of G. However, the first two factors of G are not in the image of this monomorphism. Hence, G is isomorphic with the proper subgroup $K = \mathrm{Wr}\, C_p^\Gamma$ where $\Gamma = \{n \in \Lambda \colon n \geqq 3\}$. Thus, G is not co-Hopfian. Also, Γ is a filter in Λ and by Corollary 5.2, G is the split extension of H_Γ by K. Consequently, G is isomorphic with the proper quotient K and G is not Hopfian. By Theorem 3, G is a locally finite p-group, and by

Theorem 5, G is both residually finite and an FC-group. It will be shown elsewhere that G is, in fact, of exponent p^2, solvable of derived length 2, and nilpotent, but is not a direct product of finite groups (see for example [2]).

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