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WHY SECOND ORDER PARABOLIC SYSTEMS?

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0. Introduction. A natural phenomenon is envisaged, describable by a set of functions

$$\rho_i(x, t), \qquad 1 \leqq i \leqq m,$$

subject to some evolutionary law. Here, x is interpreted as a space variable and the $\rho_i(x, t)$ as the concentration of "species" $i = 1, \ldots, m$ at x and at time t. We ask for certain "first experiments" which permit us to conclude that the evolutionary law governing the envisaged phenomenon is a system of partial differential equations of parabolic type independent of the initial distribution $\rho(x, 0)$. These "first experiments" do not necessarily have to be real experiments, but may be any source of information.

We shall, in fact, provide a set of general properties listed below as $A_1, A_2, \ldots$, which in a purely mathematical way imply that the $\rho_i(x, t)$ solve a system of equations of the form

$$(*) \qquad \frac{\partial}{\partial t}\rho_j(x, t) = \sum_{i,k,\ell} a^{i,k}_{j,\ell}(x)\frac{\partial^2}{\partial x_i \partial x_k}\rho_\ell(x, t) + \sum_{i,\ell} b^i_{j,\ell}(x)\frac{\partial}{\partial x_i}\rho_\ell(x, t)$$
$$+ F_j(x, \rho(x, t)) \; i, k \in \{1, 2, \ldots, n\}, j, \ell \in \{1, 2, \ldots, m\}.$$

Some of the properties $A_1, A_2, \ldots$ are in fact necessary for a process to satisfy such a system of equations. A particularly simple property considered is

$$A_6: \quad \text{If } \rho_i(x, 0) \text{ is nonnegative for all } x \text{ and } i, \text{ then } \rho_i(x, t) \text{ is nonnegative for all } x, i \text{ and } t \geqq 0.$$

In compiling our set of assumptions $A_1, A_2, \ldots$, we have tried to make them as simple and as few in number as possible, as well as being subject to actual verification by measurement.

Once the form of the evolutionary law governing the envisaged process is determined to be (*), one can try to find the coefficients

$$a^{i,k}_{j,\ell}, b^i_{j,\ell} \cdots$$

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for any concrete process by a set of "secondary experiments". In fact, these coefficients are determined by the values of $\partial/\partial t|_{t=0}\, \rho_j(x, t)$, for a finite set of polynomials $\rho_j(x, 0)$ of degree $\leqq 2$. For example, if, for some fixed n, i and k, we set

$$\rho_n(y, 0) = (y_i - x_i)(y_k - x_k) \text{ and } \rho_\ell = 0, \qquad \text{for } \ell \neq n,$$

then $\partial/\partial t|_{t=0}\, \rho_j(x, t) = a^{i;k}_{j;n} + F_j(x, 0, \ldots, 0)$; and if $\sigma_j(y, 0)$ is constant, for all j, then

$$\frac{\partial}{\partial t}\bigg|_{t=0} \sigma_j(x, t) = F_j(x, \sigma_1(x, 0), \ldots, \sigma_m(x, 0)).$$

For a qualitative mathematical analysis of (*), however, it will in general not be necessary to know the values of $a^{i;k}_{j,\ell}(x)$, $b^i_{j,\ell}(x)$, ...

The motivation for our somewhat unusual approach is the fact that a great many different processes, occurring in most physical and biological sciences, have been modeled by such parabolic systems. The mathematical analysis of such systems often uses only very general abstract properties and, for reasons of economy, tries to minimize them. It is therefore natural to attempt to unify the model building aspect associated with them, using a systematic approach which employs the same "axiomatic" methods which the mathematician uses when analysing the final model.

We now list the "axioms" to be used in this paper and comment on them. Let Ω—the reactor—be an open (bounded) subset of $\mathbf{R}^n$. At each time t, any of a fixed number m of species is present with a concentration $\rho_i(x)$ $(x \in \Omega,\ i = 1, 2, \ldots, m)$. The possible concentration vectors $\rho = (\rho_1, \ldots, \rho_m)$ form a set C of continuous functions $\rho\colon \Omega \to \mathbf{R}^m$. C is ordered componentwise, i.e.,$\rho \leqq \sigma$ if for all i, x $\rho_i(x) \leqq \sigma_i(x)$. C is supposed to satisfy certain properties (see §1) which are satisfied, for example, by the cone of all non-negative continuous functions $f\colon \Omega \to \mathbf{R}^m$ with compact support. The following "axioms" will be used.

(A_1). (DETERMINISM). There is a semigroup P_t acting on C such that the concentration $\rho(\cdot, t + h)$ at time $t + h$ is given by $P_h\rho(\cdot, t)$.

COMMENT. (A_1) states that the concentration at time t uniquely determines the concentration at any time $t + h$ in the future.

Let us briefly discuss situations where A_1 does not hold.

a) The number of particles (or members of a population) at time t in a unit volume at x which is counted by $\rho_i(x, t)$ may be the result of a stochastic movement of these individuals and as such a random quantity. In this case A_1 does not hold. However, if we replace $\rho_i(x, t)$ by the mean of ρ_i, $\sigma(x, t)$ over a large number of experiments σ, there is a good chance that A_1 may hold.

b) It may be that the future concentrations $\rho(x, t)$, for $t \geqq s$, depend not only on $\rho(\cdot, s)$ (the present) but also on $\rho(\cdot, r)$, for $r < s$ (the past). This happens quite naturally not only in biology (see **[2]**), but also in the study of anorganic materials which have a "memory" **[8]**, and models for heat conduction in such materials have been constructed **[12]**.

c) There may be external influences (like a time-dependent magnetic field) such that the concentration $\rho(\cdot, t+h)$ depend not only on $\rho(\cdot, t)$ and h but also on t; hence, $\rho(\cdot, t+h) = P_{t+h,t}\,\rho(\cdot, t)$. Instead of the semigroup property

$$P_{s+t} = P_s \circ P_t$$

we would find

$$P_{t,s} \circ P_{r,q} = P_{t,q} \qquad (t \geqq s \geqq r \geqq q \geqq 0).$$

The effect of such a double dependence on time would be that we get time dependent coefficients in (∗), for example, $a^{ik}_{j,\ell}(x, t)$ instead of $a^{ik}_{j,\ell}(x)$. The considerations of this paper may be generalized to cover such systems.

(A_2). (SMOOTHNESS). $P_t\rho(x)$ depends in a differentiable way on t, x, ρ.

A_2 will be made precise when needed. We don't bother with the physical meaning of A_2 except to note that often a discrete model (with respect to t, x) will be more appropriate than a continuous one.

(A_3). (LOCALITY). Let $\rho_1, \rho_2 \in C$ be twice continuously differentiable and equal to each other in a neighborhood of $x \in \Omega$. Then

$$\lim_{t \downarrow 0} \frac{1}{t}(p_t\rho_1(x) - P_t\,\rho_2(x)) = 0.$$

COMMENT. In general, we would expect A_3 to be violated if $P_t\,\rho(x)$, for $t \geqq s$, depends on $P_r\,\rho$, for $r < s$, and not only on $P_s\,\rho$. In such a case a disturbance at a finite distance from x at time t may travel in time $s - r$ to x and influence the rate of change of ρ at (x, s). However, A_3 may be violated even when A_1, A_2 hold, $m = 1$ and the P_t are linear. In fact, this situation is well understood by Markov-process theory which (in terms of a precise mathematical model) provides the interpretation of A_3 that $\rho(x, t)$ is the concentration at (x, t) of a diffusing particle which moves along a continuous path in Ω. This interpretation explains why, in concrete situations, A_3 may be expected to hold if A_1 and A_2 hold.

On the otherhand there are Markov processes P_t with essentially no continuous paths. See **[1]** for the equations which, in such a case, replace (∗).

(A_4). (LINEARITY). $\rho \to P_t\,\rho$ is a linear mapping for any $t \geqq 0$.

COMMENT. If the change of $\rho(x, t)$ is due only to diffusion of non-interacting particles we expect A_4 to hold. Suppose, however, that particles react with each other according to some mass action kinetics. Then we expect A_4 to be violated. In that case we may consider the derivative

$$D_u P_t = Q_t$$

of P_t at some concentration (which we will take to be an equilibrium in this paper) which is linear by definition.

We now slightly generalize A_4 to take into account reactions.

(A_5). (SEMILINEARITY). There is a linear operator L and a function F on $\Omega \times \mathbf{R}^m$ such that

$$\frac{d}{dt}\bigg|_{t=0} (P_t\, \rho)(x) = (L\rho)(x) + F(x, \rho(x)),$$

for any ρ which is twice continuously differentiable and such that $\rho = \sigma$ in a neighborhood of x, for some $\sigma \in C$.

COMMENT. Obviously, (A_5) is necessary for (*) to hold. To test A_5 (assuming A_1, A_2), one can proceed as follows. Given $\rho \in C$, $x \in \Omega$, put

$$\rho^x(y) = \rho(x), \; (y \in \Omega)$$

$$F(x, \rho(x)) = \frac{\partial}{\partial t}\bigg|_{t=0} (P_t \rho^x)(x).$$

Then, test the linearity of $\rho \to \partial/\partial t|_{t=0}\, P_t\, \rho(x) - F(x, \rho(x))$. That is, in order to eliminate the influence of reactions, we subtract the rate of change at the well-stirred concentration ρ^x.

A_5 admits the following interpretation. A particle of species i, starting at time zero at x, moves along a continuous path in Ω according to a linear difiusion law, as if there was no coupling between the different species until some time $\tau > 0$, when it reaches a point y, where it reacts with other particles. This reaction exclusively depends on $F(y, \rho(y, \tau))$. As a result, the particle vanishes or multiplies or changes into a different species j. Then it starts anew from y moving along in Ω according to the diffusion law characteristic for j and so on. This interpretation can be proved mathematically if F is linear. Note that our notion of semilinearity is more restrictive than that of Friedmann [5], for example.

(A_6). (POSITIVITY). If $\rho \geqq 0$, then, for all $t \geqq 0$, $P_t\, \rho \geqq 0$.

COMMENT. If $\rho(x)$ is to be a concentration of particles at $(x, 0)$ and $P_t\rho(x)$ is to be the concentration at (x, t), then A_6 is evident. The linearization of P_t at a nonzero equilibrium, however, will in general fail to satisfy A_6. Now, a phenomenon which is intuitively related to a diffusion is that a common maximum of all concentrations ρ_j at a point x is flattened in the immediate future:

$$\left.\frac{d}{dt}\right|_{t=0} P_t\,\rho(x) \leqq 0.$$

We shall make use of this property only in case $u = (u_1, \ldots, u_m) = (\rho_1(x), \ldots, \rho_m(x))$ happens to be a positive equilibrium of P_t.

($A_7(u)$). (MAXIMUM PRINCIPLE FOR u). If $\rho \in C$ is twice continuously differentiable and $u_j = \rho_j(x)$ $(1 \leqq j \leqq m)$ is a local maximum of ρ_j at x, then

$$\left.\frac{d}{dt}\right|_{t=0} P_t\,\rho(x) \leqq 0.$$

The paper is organized as follows. In §1, (*) is derived with $F_j(x, \rho(x)) = \sum_\ell c_{j\ell}(x)\rho_\ell(x)$ from A_1, A_2, A_3, A_4, A_6. In addition, it is shown that $a^{ik}_{j,\ell} = b^i_{j,\ell} = 0$, for $j \neq \ell$; that is there is no coupling of the species except by the $c_{j,\ell}$ which are shown to be $\geqq 0$ for $j \neq \ell$. In §2, the same result is proved for the linearization of a process satisfying a modification of A_1, A_2, A_3, A_6. In §3, (*) is derived from A_1, A_2, A_3, A_5, A_6. In §4, (*) is shown to hold for the linearization at a constant strictly positive equilibrium u, using A_1, A_2, A_3, A_7 (u). Whereas, in §1, §2, and §3, we show that $a^{ik}_{j,\ell} = b^i_{j,\ell} = 0$, for $j \neq \ell$, there seems to be no reason why this should hold in the present situation. In fact, a (mathematical) example seems to show that coupling by second order terms (cross diffusion) may be possible. (See also **[8]**). There is, however, an additional algebraic structure imposed on the coefficients if the corresponding Cauchy problem is "correctly posed". In §5, we prove this classical observation in the setting of strongly continuous semigroups on Hilbert space with $\Omega = \mathbf{R}^n$ or Ω bounded. Our result is that, for any $x \in \Omega$ and any $y \in \mathbf{C}^m$, all of the eigenvalues of $(\sum_{i,k} a^{ik}_{j,\ell}(x) y_i y_k)$ have non-negative real part. Except for §5, all our arguments are completely elementary and well known for $m = 1$.

CONCLUSION. The widely used model (*) for the description of time dependent processes (as well as some additional structure of the coefficients) follows from the simple properties given above. With the possible exception of the smoothness, these properties have a physical meaning.

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1. Linear systems.

NOTATIONS. Throughout this paper, C, the set of possible "concentrations", will be a convex cone of continuous mappings from Ω to $\mathbf{R}^m$ such that:

(α) For any neighborhood V of a point $x \in \Omega$ and any $\rho \in C$, there is a continuous $\varphi\colon \Omega \to [0, 1[$ such that $\varphi(\Omega \backslash V) = \{0\}$, $\varphi = 1$ in a neighborhood of x and $\varphi \cdot \rho \in C$; and

(β) If $\rho \in C - C$ and $\rho \geqq 0$, then $\rho \in C$.

NOTE. If $l\colon C \to \mathbf{R}$ is linear and $l(f) \geqq 0$, for $f \in C$, then $l(f) \geqq l(g)$ if $f \geqq g, f, g \in C$, by (β).

Let $C^2(x)$ be the space of all $f\colon \Omega \to \mathbf{R}^n$ which are twice continuousl differentiable in a neighborhood of x. Write $f =_x g$ if $f = g$ in a neighborhood of x. Throughout this section, P_t will denote a family of operators on C satisfying A_1, A_2, A_3, A_4 A_6, of the introduction.

We now fix the meaning of A_2 for this section. For $x \in \Omega$, denote, by $D_p(x)$, the set of functions $g\colon \Omega \to \mathbf{R}^m$ such that there is $f \in C$ and $\beta \in \mathbf{R}^m$, with

$$\text{(i)} \qquad f =_x g;$$

$$\text{(ii)} \qquad \lim_{t \downarrow 0} \frac{1}{t}(P_t f(x) - f(x)) = \beta.$$

By (A_3), β does not depend on the choice of f if $g \in C^2(x)$; in this case, put $Ag(x) = \beta$. Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, $\Delta_y^i(x) = (x_i - y_i)$, and $\Delta_y^{i,k}(x) = (x_i - y_i)(x_k - y_k)$ ($j \in \{1, 2, \ldots, m\}$, i, $k \in \{1, 2, \ldots, n\}$).

(A_2). (SMOOTHNESS). Let $P_+^2(y)$ be the set of $(\rho_1, \ldots, \rho_m)$ such that each ρ_j is a real polynomial in $x \in \mathbf{R}^n$ of degree $\leqq 2$ which is non negative in a neighborhood of y. Then $P_+^2(y) \subset D_p(y)$.

NOTATION. Since P_t is linear on C it extends in a unique way to a linear operator, again denoted by P_t, on $C - C = \langle C \rangle$. Also, A extends in a unique way to a linear operator on $D_p(y) - D_p(y) = \langle D_p(y) \rangle$. Put

$$c_j(y) = A(e_j)(y)$$
$$b_j^i(y) = A(\Delta_y^i e_j)(y)$$
$$a_j^{ik}(y) = \frac{1}{2} A(\Delta_y^{ik} e_j)(y).$$

Let a^{ik} be the matrix with columns a_j^{ik}. Let b^i be the matrix with columns b_j^i. And, let c be the matrix with columns c_j.

PROPOSITION 1. *Suppose $\rho \in C$ is twice continuously differentiable with respect to x. Then, for all $x \in X$, $\rho \in D_p(x)$ and*

$$A\rho(x) = \sum_{i,k} a^{ik}(x)\, \rho_{x_i, x_k}(x) + \sum b^i(x)\, \rho_{x_i}(x) + C(x)\rho(x).$$

PROOF. By Taylor's formula,

$$\rho(x) - \rho(y) = \sum_i \Delta_y^i(x)\,\rho_{x_i}(y) + \sum_{i,k}\frac{1}{2}\Delta_y^{ik}(x)\,\rho_{x_ix_k}(y) + r(x,y), \tag{1}$$
$$\text{with } r(x,y) = \circ(|x-y|^2).$$

Using (1) and A_2, we have $r(\cdot, y) = \rho + \sigma$, with $\sigma \in \langle D_p(y)\rangle$, and

$$A\sigma(y) = -\sum_{i,k} a^{ik}(y)\rho_{x_ix_k}(y) - \sum_i b^i(y)\rho_{x_i}(y) - c(y)\rho(y).$$

Therefore, it suffices to show

$$\lim_{t\downarrow 0}\frac{1}{t}P_t\,\tau(y) = 0,$$

for some $\tau \in C - C$ such that

$$\tau =_y r(\cdot, y).$$

By property (α) of C and A_2, there is $\psi \in C$ such that

$$0 \leqq \psi_j \text{ on } \Omega$$
$$\psi_j(x) = (x-y)^2$$

for all x in some neighborhood of y. For all $\varepsilon > 0$, there is a neighborhood V_ε of y such that

$$|r(\cdot, y)_j| \leqq \varepsilon\,\psi_j \text{ on } V_\varepsilon.$$

By property (α) of C, there is $\tau_\varepsilon \in C - C$ such that

$$\tau_\varepsilon = 0 \text{ on } \Omega\backslash V_\varepsilon$$
$$|\tau_{\varepsilon,j}| \leqq |r(\cdot, y)_j| \text{ on } \Omega$$
$$\tau_\varepsilon =_y r(\cdot, y).$$

Then $\tau_{\varepsilon,j} \leqq \varepsilon\,\psi_j$ on Ω and, by the preceding note, $P_t\tau_\varepsilon \leqq \varepsilon\,P_t\,\psi$. Hence,

$$\delta_\varepsilon = \limsup_{t\downarrow 0}\frac{1}{t}P_t\,\tau_\varepsilon(y) \leqq \varepsilon\limsup_{t\downarrow 0}\frac{1}{t}P_t\,\psi(y) = \varepsilon(A\,\psi(y)).$$

By A_3, δ_ε is independent of $\varepsilon > 0$. Hence, $\delta_\varepsilon \leqq 0$. Let c be a constant concentration such that $c < \rho(x)$. There is $\sigma \in C$ such that σ is twice continuously differentiable and $\sigma =_x c - \rho$. Replacing ρ by σ, we find $\delta_\varepsilon \geqq 0$. Hence, $\delta_\varepsilon = 0$, and the proof is completed.

REMARK. For $m = 1$, the preceding result, as well as the proof using Taylor's formula, is classical. See [3, I, Theorem 5.7], for example.

PROPOSITION 2. *For any* $y \in \Omega$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$:
(i) *the matrix* $\sum_{i,k} a^{ik}(y)\,\xi_i\xi_k$ *is diagonal with non-negative entries*;
(ii) *the matrix* $b^i(y)$ *is diagonal*;
(iii) $c_{\ell,j}(y) \geqq 0$, for $j \neq \ell$.

PROOF. Let $h(x) = (\sum_i \xi_i(x_i - y_i))^2$.

By A_2, there is $\rho \in C$ such that

$$\rho =_y he_j$$

$$A\rho(y) = (A\ he_j)(y) = \lim_{t\downarrow 0} \frac{1}{t} P_t\, \rho(y).$$

Since $\rho(y) = 0$, we have, by proposition 1,

$$0 \leqq \frac{d}{dt}\Big|_{t=0} (P_t\, \rho(y))_\ell = (A\rho(y))_\ell = \sum_{i,k} a^{ik}_{\ell j}(\rho_{x_i x_k})_j(y) + \sum_i b^i_{\ell,j}(\rho_{x_i})_j(y) + (c_{\ell j}\, \rho_j)(y) = \sum_{i,k} a^{ik}_{\ell,j}(y)\, \xi_i \xi_k.$$

This proves the second part of (i). Now suppose $\ell \neq j$ and $\sum_{i,k} a^{ik}_{\ell,j}(y)\, \xi_i\, \xi_k > 0$. There is $\alpha > 0$ such that $0 > -\alpha \sum_{i,k} a^{ik}_{\ell,j}\, \xi_i\, \xi_k + c_{\ell,j}(y)$. By A_2, there is $\rho \in D_p(y)$ such that $\rho =_y (1 - \alpha\, h)e_j$. Hence, by proposition 1,

$$0 \leqq \lim_{t\downarrow 0} \frac{1}{t}(P_t\, \rho(y))_\ell = -\alpha \sum_{i,k} a^{ik}_{\ell,j}(y)\, \xi_i\, \xi_k + c_{\ell,j}(y) < 0.$$

This contradiction proves (i).

To prove (ii) and (iii), choose $\alpha > 0$ such that $\alpha - y_i > 0$, and put $h(x) = \alpha - x_i$. By A_2, there is $\rho \in D_p(y)$ such that $\rho =_y he_j$. Since $j \neq \ell$, we have $\rho_\ell(y) = 0$, and, by proposition 1,

$$0 \leqq \frac{d}{dt}\Big|_{t=0} (P_t\, \rho(y)) = -b^i_{\ell,j}(y) + c_{\ell,j}(y)\, (\alpha - y_i).$$

If we choose α very large, we find $c_{\ell,j} \geqq 0$. This proves (iii). If we then choose α such that $\alpha - y_i$ becomes very small, we find $-b^i_{j\ell}(y) \geqq 0$. Finally, if we do the same calculation with $h(x) = \alpha + x_i$ instead of $h(x) = \alpha - x_i$, we find $b^i_{j,\ell}(y) \geqq 0$. This proves (ii).

THEOREM 1. *Suppose $\rho_0 \in C$ and $\rho(x, t) = P_t\, \rho_0(x)$ is twice continuously differentiable with respect to x. Then $\rho(x, t)$ is right differentiable with respect to $t \geqq 0$ and*

$$(**) \qquad \frac{\partial}{\partial t^+} \rho_j = \sum_{i,k} a^{ik}_{j,j}(\rho_{x_i x_k})_j + \sum_i b^i_{j,j}(\rho_{x_i})_j + \sum_{k=1}^m c_{jk}\, \rho_k,$$

with $(a^{ik}_{j,j})_{i,k}$ positive semidefinite and

$$c_{j,k} \geqq 0, \qquad \text{for } j \neq k.$$

PROOF. Since $P_{t+h}\, \rho = P_t P_h\, \rho$, we have

$$\frac{1}{h}(\rho(x, t + h) - \rho(x, t)) = \frac{1}{h}(P_{h+t}\, \rho(x) - P_t\, \rho(x)) = \frac{1}{h}(P_h \sigma(x) - \sigma(x)),$$

for $\sigma = P_t\rho$. Hence, it suffices to consider $s = 0$. The assertion then follows from proposition 1.2.

NOTE. A system of type (**) is called a weakly coupled system of linear parabolic equations of order 2. According to **[11]**, such a system models diffusing particles with spontaneous decay from one species into another.

2. The linear approximation at zero. In this section, P_t will denote a family of operators satisfying A_1, A_6 of the introduction. Additional smoothness and locality properties will be stated below and required for proposition 3.

DEFINITION. Let D be a set of functions from Ω into some topological vector space E over $\mathbf{R}$. Call a function $h \to g(h)$ from $[0, 1]$ to D differentiable at 0 if

$$\lim_{h \downarrow 0} h^{-1}(g(h) - g(o)) = g'(o)$$

existse for all $x \in \Omega$, and put

$$D_0 g(x) = g'(x) \qquad (x \in \Omega).$$

Let T be the closure of the space $\langle D \rangle$ of D with respect to the pointwise convergence. We say that a mapping $L: D \to E$ is differentiable at $\rho \in D$, if there is a linear mapping L' from T to E such that, for all $g: [0, 1] \to D$ which are differentiable at 0 and such that $g(0) = \rho$, we have

$$\lim_{h \downarrow 0} \frac{1}{h}(Lg(h) - Lg(0))(x) = L'(D_0 g)(x) \qquad (x \in \Omega).$$

Put $D_\rho L = L'$. We now take $D = C$ and assume

A_2^0 (Smoothness).

(i) For all $t \geqq 0$, $P_t 0 = 0$, P_t is differentiable at 0; if $Q_t = D_0 P_t$,then Q_t maps C into C.

(ii) For any $x \in \Omega$, $P_+^2(x) \subset D_Q(x)$, where $D_Q(x)$ is the set of all f such that $d/dt|_{t=0}\ Q_t g(x)$ exists for some $g \in C$ such that $g =_x f$.

We now assume that Q_t is local, that is,

(A_3^0) For any $x \in \Omega$ and $\rho_1, \rho_2 \in C \cap C^2(x)$ such that $\rho_1 =_x \rho_2$, we have

$$\lim_{t \downarrow 0} \frac{1}{t} \lim_{h \downarrow 0} \frac{1}{h} (P_t h\rho_1(x) - P_t h\rho_2(x)) = 0.$$

PROPOSITION 3. *Q_t satisfies A_1, A_2, A_3, A_4, A_6 (and, hence, Theorem* 1 *is applicable).*

PROOF. Put $g(h) = h \cdot \rho$ with $\rho \in C$, $0 \leqq h \in \mathbf{R}$. Then $h \to P_s(g(h))$ is differentiable and

$$D_0 P_s \circ g = Q_s \rho.$$

Hence

$$Q_{t+s}\, \rho = \lim_{h \downarrow 0} \frac{1}{h} P_{t+s}\, g(h) = \lim_{h \downarrow 0} \frac{1}{h} P_t(P_s(g(h)) = Q_t(Q_s\, \rho)$$

and Q_t is a semigroup on C. If $\rho \in C$, then $Q_t\rho = \lim(1/h)P_t h\rho \geqq 0$. Whence A_1, A_6 is satisfied by Q_t. A_2, A_3, A_4 are obvious.

Remark. A_2^0 and A_3^0 follow from A_3 and A_2^1:

(i) For all $t \geqq 0$, $P_t 0 = 0$, P_t is differentiable at 0 and $Q_t = D_0 P_t$ maps C into C.

(ii) For all $x \in \Omega$ and $\rho \in P_+^2(x) \cup (C^2(x) \cap C)$, there is $g \in C$ with $\rho =_x g$ such that the following equation (makes sense and) holds.

$$\left.\frac{d}{dt}\right|_{t=0} Q_t g(x) = \lim_{t \downarrow 0} \frac{1}{h} \lim_{h \downarrow 0} \frac{1}{t}(P_t hg - hg)\,(x).$$

3. Semilinear Systems. In this section, P_t is a family of operators on C satisfying A_1, A_2^1, A_3, A_5, A_6. Thus, $P_t 0 = 0$, P_t is differentiable at 0 and $Q_t = D_0 P_t$ maps C into C. By §2, Q_t satisfies A_1, A_2, A_3, A_4, A_6 and, hence, by Theorem 1, $\rho(x, t) = Q_t\, \rho_0(x)$ is right differentiable, for $t \geqq 0$, if $\rho \in C$ and $\rho(\cdot, t)$ is twice continuously differentiable in x. Furthermore, if $B\rho(\cdot, t)\,(x) = \partial/\partial t^+\, \rho(x, t)$, we have

$$B\rho = \sum_{i,k} a^{ik}\, \rho_{x_i x_k} + \sum_i b^i\, \rho_{x_i} + c\rho$$

where

$$c_j(y) = B(e_j)\,(y)$$
$$b_j^i(y) = B(\Delta_y^i e_j)\,(y)$$
$$a_j^{ik}(y) = \frac{1}{2} B(\Delta_y^{ik} e_j)\,(y).$$

On the other hand, if $\rho \in C^2(x) \cap D_p(x)$, $A\rho(x)$ is defined by

$$A\rho(x) = \left.\frac{d}{dt}\right|_{t=0} P_t f(x),$$

for some $f \in C$ with $f =_x \rho$, and by A_2^1, then

$$B\rho = (D_0 A)\rho.$$

By A_5 there is a function F_0: $\Omega \times \mathbf{R}^m \to \mathbf{R}^m$ such that $\rho \to A\rho(x) - F_0(x, \rho(x))$ is linear on $C^2(x) \cap D_p(x)$. Therefore, $F_0(x, 0) = 0$ and

$$\lim_{h \downarrow 0} \frac{1}{h} [A(h\rho)(x) - F_0(x, h\rho(x)) - A(0)(x) - F_0(x, 0)]$$

$$= A\rho(x) - F_0(x, \rho(x))$$
$$= B\rho(x) - \lim_{h\downarrow 0} \frac{1}{h} F_0(x, h\rho(x)).$$

Put $F(x, y) = F_0(x, y) + c(x)y - \lim_{h\downarrow 0}(1/h)F_0(x, hy)$. Then

$$A\rho(x) = \sum_{i,k} a^{i,k}(x)\rho_{x_i x_k}(x) + \sum_i b^i(x)\rho_{x_i}(x) + F(x, \rho(x)),$$

and we gave

THEOREM 2. *Let $\rho_0 \in C$ and $\rho(x, t) = P_t\rho_0(x)$ be twice continuously differentiable with respect to x. Then $\rho(x, t)$ is right differentiable with respect to $t \geqq 0$ and*

$$\frac{\partial}{\partial t^+} \rho(x, t) = \sum_{i,k} a^{ik}(x) \rho_{x_i x_k}(x, t) + \sum_i b^i(x) \rho_{x_i}(x, t) + F(x, \rho(x)),$$

$F(x, 0) = 0$, $F(x, \cdot)$ is diffeentiable at 0 and

$$D_0F(x, \cdot)y = c(x)y.$$

REMARK. In order to test the semilinearity of the operator A put $\rho^x(y) = \rho(x)(x, y \in \Omega, \rho \in C)$ and, since $\rho^x \in C^2(x)$, $(A\rho^x)(x)$ is well defined. Let $(\tilde{A}\rho)(x) := (A\rho^x)(x)$. Now, $(\tilde{A}\rho)(x)$ is an element of $\mathbf{R}^m$ which depends only on $(x, \rho(x)) \in \Omega \times \mathbf{R}^m$. Therefore, $(\tilde{A}\rho)(x) = F(x, \rho(x))$ with a mapping $F: \Omega \times \mathbf{R}^m \to \mathbf{R}^m$ and A is semilinear if and only if $A - \tilde{A}$ is linear. A is more likely to be semilinear if A is differentiable at 0. (which it is, by $A_{\frac{1}{2}}$).

LEMMA. *Suppose A is differentiable at zero; that is,* $\lim_{h\downarrow 0}(1/h)\, A(h\rho)\, (x)$ *exists and is equal to $B\rho(x)$, for a linear operator B. Then A is semilinear if and only if*

$$A\rho(x) - \tilde{A}\rho(x) = \lim_{h\downarrow 0} \frac{1}{h} [A(h\rho)(x) - \tilde{A}(h\rho)(x)].$$

The straightforward verification is left to the reader.

4. The linear approximation at a nonzero constant equilibrium. In this section P_t will be a semigroup of (possibly nonlinear) operators on C and $u = (u_1, \ldots, u_m) \in C$ a concentration vector such that

$$0 < u_i \in \mathbf{R} \quad (1 \leqq i \leqq m)$$
$$P_t u = u \quad (t \geqq 0).$$

In addition, (P_t) is supposed to satisfy A_3, $A_7(u)$ and to be smooth in the following sense:

$(A_{\frac{4}{2}})$ (*Smoothness*).

(i) P_t is differentiable at u for any $t \geqq 0$ and $Q_t = D_u P_t$ maps C into $C - C$.

(ii) For any $x \in \Omega$ and any $f \in ((C^2(x) \cap C) \cup P^2_+(x))$, there is a $g \in C$ with $f =_x g$ and such that both sides of the following equality make sense and hold

$$\frac{d}{dt}\bigg|_{t=0} Q_t g(x) = \lim_{h \downarrow 0} \frac{1}{h} A(u + hg)(x)$$
$$= \lim_{h \downarrow 0} \frac{1}{h} \lim_{t \downarrow 0} \frac{1}{t} [P_t(u + hg) - (u + hg)].$$

THEOREM 3. *There are real valued functions $a^{ik}_{j,\ell}$, $b^i_{j,\ell}$, $c_{j,\ell}$ on Ω such that if for $\rho_0 \in C$ $\rho(x, t) = Q_t\rho_0(x)$ is twice continuously differentiable with respect to x, for all $t \geqq 0$, then, $\rho(x, t)$ is right differentiable with respect to $t \geqq 0$ and*

$$\frac{\partial}{\partial t^+} \rho = \sum_{i,k} a^{ik} \rho_{x_i x_k} + \sum_i b^i \rho_{x_i} + c\rho.$$

Furthermore for any $y \in \Omega$, $x \in \Omega$ the matrix $\sum a^{ik}(x) y_i y_k$ is positive semi-definite.

PROOF. There is a unique linear extension of Q_t to $C - C$, which again will be denoted by Q_t. Let us show that $(Q_t)_{t \geqq 0}$ is a semigroup (of linear operators on $C - C$). For this purpose let $g(h) = u + h\rho_0$ $\rho_0 \in C$, $0 < h \in \mathbf{R}$. Then g is differentiable at 0 and $(D_0(P_t \circ g) = D_u p_t)\rho_0 = (Q_t\rho_0$. Hence

$$Q_{s+t}\,\rho_0 = \lim_{h \downarrow 0} \frac{1}{h} [P_{s+t}\, g(h) - P_{s+t} u] = \lim_{h \downarrow 0} \frac{1}{h} [P_s P_t g(h) - P_t P_s\,(g(0)]$$
$$= Q_s D_{g(0)} P_t g'(0) = Q_s D_u P_t\, \rho_0 = Q_s Q_t \rho_0 (s,\ t \geqq 0,\ \rho \in C).$$

Hence $Q_{t+s}\,\rho = Q_t Q_s\,\rho$ for $\rho \in C$ and by linearity also for $\rho \in C - C$, $(s,\ t \geqq 0)$. Hence A_1 holds for Q_t with C replaced by $C - C = \langle C \rangle$. Note that $\langle C \rangle$ also has properties (α), (β) of §1. In order to investigate the locality of Q_t let $\rho_1 =_x \rho_2\, \rho_1$, $\rho_2 \in C^2(x) \cap C$. Then, by A^4_2 and A_3, there are $g_1, g_2 \in C$ with $\rho_i =_x g_i$ and such that

$$\frac{1}{h} A(u + hg_1)(x) = \frac{1}{h} A(u + hg_2)(x)$$

$$\frac{d}{dt}\bigg|_{t=0} Q_t g_1(x) = \lim_{h \downarrow 0} \frac{1}{h} A(hg_1 + u)(x) = \frac{d}{dt}\bigg|_{t=0} Q_t g_2(x).$$

Now suppose $\rho_1, \rho_2 \in \langle C \rangle \cap C^2(x)$ and $\rho_1 =_x \rho_2$. Let $e = (1, 1, \ldots, 1)$. There is $0 < \varepsilon \in \mathbf{R}$ such that $\sigma_1 = \rho_1 + \varepsilon e$, $\sigma_2 = \rho_2 + \varepsilon e$ are non-negative on a neighborhood of x. By property (α) of C and the preceding argument $\partial/\partial t|_{t=0}\, Q_t\sigma_1(x)$ and $\partial/\partial t|_{t=0} Q_t\sigma_2(x)$ exist and are equal. By the same argument $\lim_{t\downarrow 0}(1/t)Q_t(\varepsilon e)\,(x)$ exists. Hence, by linearity, $\partial/\partial t_{t=0}\, Q_t\rho_1(x)$ and $\partial/\partial t|_{t=0}\, Q_t\rho_2(x)$ exist and are equal. Therefore A_3 holds for (Q_t) with C replaced by $\langle C\rangle$. For $x \in \Omega$, let $D_Q(x)$ be the set of all $f\colon \Omega \to \mathbf{R}^m$ such that $\beta = \lim_{t\downarrow 0}(1/t)\,(Q_t g(x) - g(x))$ exists for some $g \in \langle C\rangle$ with $g =_x f$. If $f \in C^2(x)$, then β does not depend on the choice of g, and we may define

$$Bf(x) = \beta.$$

Let $0 < \varepsilon \in \mathbf{R}$. Then by $A_2^4\, e_j$, $(\Delta_y^i + \varepsilon)e_j$, $(\Delta_y^{i,k} + \varepsilon)e_j$ belong to $D_P(x)$ and e_j, $\Delta_y^i\, e_j$, $\Delta_y^{i,k}\, e_j$ belong to $D_Q(x)$. Put

$$\begin{aligned} c_j(y) &= (Be_j)\,(y)\\ b_j^i(y) &= B(\Delta_y^i\, e_j)\,(y)\\ a_j^{ik}(y) &= \frac{1}{2} B(\Delta_y^{i,k}\, e_j)\,(y). \end{aligned}$$

Suppose $\rho \in D_Q \cap C^2(y)$. By Taylor's formula

$$\rho(x) = \rho(y) + \sum_i \Delta_y^i(x)\, \rho_{x_i}(y) + \sum_{i,k} \frac{1}{2} \Delta_y^{i,k}(x)\, \rho_{x_i x_k}(y) + r(x, y)$$

with $r(x, y) = \circ(|x - y|^2)$ and

$$B\rho(x) = \sum_{i,k} a^{ik}(y)\rho_{x_i x_k}(y) + \sum_i b^i(y)\rho_{x_i}(y) + c(y)\rho(y) + Br(\cdot, y)\,(x)$$

To complete the proof it suffices to show $Br(\cdot, y)\,(y) = 0$. For this purpose let $g(x) = |x - y|^2$ and for any $0 < \varepsilon, h \in \mathbf{R}$ $w = u + h(f - \varepsilon\, ge)$, $f(x) = r(x, y)$. Then $w \in D_Q(y)$. Since $u > 0$ there is a neighborhood V of y such that $w \geqq 0$ on V provided h is sufficiently small. Since u is constant and $f(x) = \circ(|x - y|^2)$, w has a local maximum at x_0. By A_2^4, $w \in D_p(y)$ and by $A_7(u)$

$$\frac{d}{dt}\bigg|_{t=0} P_t(u + h\sigma)\,(y) \leqq 0$$

for some $\sigma \in C - C$ with $\sigma =_y f - \varepsilon\, ge$. By A_2^4 we may assume $\sigma \in D_Q(y)$ and

$$B\sigma(y) = \lim_{h\downarrow 0} \frac{1}{h} \frac{d}{dt}\bigg|_{t=0} P_t(u + h\sigma)(y) \leqq 0.$$

Since also $\varepsilon\, ge \in D_Q(y)$ and B is linear

$$Bf(y) - \varepsilon\, B(ge)\,(y) \leqq 0 \quad (\varepsilon > 0) \text{ and hence}$$

$$Bf(y) \leqq 0.$$

Replacing ρ by $-\rho$ we find

$$Bf(y) \geqq 0$$

and hence

$$Bf(y) = 0.$$

The following example shows that the coefficient $a_{j\ell}^{ik}$ need not be zero for $j \neq \ell$ in contrast to the special situation of §1 – §3.

EXAMPLE 1. Consider the system of equations

$$(1)\qquad \begin{cases} V_{xx} = V_t & \\ W_{xx} + \alpha\, W\, V_{xx} = W_t & \text{on } \mathbf{R} \times \,]0, \infty[\\ V(x, 0) = f_1(x) & (x \in \mathbf{R}) \\ W(x, 0) = f_2(x) & (x \in \mathbf{R}). \end{cases}$$

CLAIM 1. There is a strongly continuous semigroup P_t on the Banach space C of all $f = (f_1, f_2)$; $\mathbf{R} \to \mathbf{R}^2$ tuch that f_1 has bounded continuous derivatives up to order 3 and f_2 is continuous and bounded, the norm being

$$\|f\| = \sup |f_2(x)| + \sum_{i=0}^{3} \sup_x |f_1^{(i)}(x)|,$$

such that $P_t f$ solves (1) for $f \in C$.

PROOF. Put

$$S_t f_1(x) = \int_{-\infty}^{+\infty} \left(\frac{1}{4\pi t}\right)^{1/2} e^{-|x-y|^2/4t} f_1(y)\, dy = v(x, t).$$

since

$$(S_t f_1)^{(r)}(x) = \int_{-\infty}^{+\infty} \left(\frac{1}{4\pi t}\right)^{1/2} e^{-|y|^2/4t} f_1^{(r)}(x - y)\, dy \qquad (r = 0, 1, 2, 3)$$

$S_t f_1$ has continuous bounded derivatives up to order 3 and

$$|(S_t f_1)^{(2)}(x_1) - (S_t f_1)^{(2)}(x_2)| \leqq \sup_x |S_t f_1^{(3)}|\, |x_1 - x_2|.$$

Hence $(S_t f_1)^{(2)}$ is Hölder continuous on $\mathbf{R}$ uniformly in $t \geqq 0$. Now for any bounded Hölder continuous function φ there is a solution s to the Cauchy problem

$$(2)\qquad \begin{array}{ll} s_{xx} + s_\varphi = s_t & \text{on } \mathbf{R} \times \,]0, \infty[\\ s(x, 0) = f_2(x) & (x \in \mathbf{R}) \end{array}$$

(Friedmann [5] theorem 12 page 25). In this way we find a solution $P_t f = ((P_t f)_1, (P_t f)_2)$ of (1) with $(P_t f)_1 = S_t f_1$. (2) is equivalent to:

$$(se^{-Mt})_{xx} + (se^{-Mt})(\varphi - M) = (e^{-tM} s)_t.$$

If $M = \|\varphi\| = \sup_{x \in \mathbf{R}} |\varphi(x)|$ then by the maximum principle

$$|e^{-Mt}\, s(x, t)| \leqq |S_t f_2(x)| \leqq \|f_2\| \text{ and} \qquad (3)$$
$$|s(x, t)| \leqq e^{t\|\varphi\|}\, \|f_2\|.$$

Since by the maximum principle there is, for any $T > 0$, at most one bounded solution of (1) on $\mathbf{R} \times [0, T[$, and since $P_t f(x)$ is bounded for $(x, t) \in \mathbf{R} \times [0, T[$, the family of operators P_t is a semigroup

$$P_{t+s} f = P_t P_s f \quad (t, s \geqq 0).$$

Since by the maximum principle there is, for any $T > 0$, at most one bounded solution of (1) on $\mathbf{R} \times [0, T[$, and since $P_t f(x)$ is bounded for $(x, t) \in \mathbf{R} \times [0, T[$, the family of operators P_t is a semigroup: $P_{t+s} f = P_t P_s f$, $(t, s \geqq 0)$. Let us show that

$$\lim_{t \downarrow 0} \|P_t f - f\| = 0.$$

$$\|P_t f - f\| = \|w(\cdot, t) - f_2\| + \sum_{r=0}^{3} \|S_t f_1)^{(r)} - f_1^{(r)}\|;\ w(\cdot, t) = (P_t f)_2.$$

Since S_t is strongly continuous with respect to $\|\ \|$ and $(S_t f)^{(r)} = S_t(f^{(r)})$ the second term tends to zero as t tends to zero. By Friedmann ([5] theorem 12 page 25)

$$w(x, t) = S_t f_2(x) + \alpha \int_0^t S_{t-\tau}\, [(S_\tau f_1)^{(2)} (P_\tau f)_2](x) d\tau \text{ and}$$
$$|w(x, t) - f_2(x)| \leqq \|S_t f_2 - f_2\| + \text{const. } t. \text{ Hence}$$
$$\lim_{t \to 0} \|P_t f - f\| = 0.$$

Let us show that $f \to P_t f$ is continuous on C. This is obvious for $(P_t f)_1$. For $(P_t f)_2$ we easily see that

$$\|(P_t f)_2 - (P_t \tilde{f})_2\| \leqq c_1\, |f - \tilde{f}| + c^2 \int_0^t \|(P_\tau f)_2 - (P_\tau \tilde{f})_2\|\, d\tau.$$

Hence by Gronwall's inequality

$$\|(P_t f)_2 - (P_t \tilde{f})_2\| \leqq c_1\, |f - \tilde{f}|\, e^{c_2 t}.$$

CLAIM 2. P_t satisfies A_1, A_2 (as in §4), A_3 and A_6.

PROOF. A_1 and A_3 are obvious. A_6 is a consequence of the boundedness of P_t and the maximum principle. Obviously any constant $u = (u_1, u_2)$ is an equilibrium. Let us show that P_t is differentiable at u.

$$\frac{1}{h}(P_t(u+hf)-u) = (S_t f_1, \frac{1}{h}[(P_t(u+hf))_2 - u_2])$$

$$\frac{1}{h}[(P_t(u+hf))_2-u_2]=S_tf_2+\frac{\alpha}{h}\int_0^t S_{t-\tau}[(S_\tau(u_1+hf_1))^{(2)}(P_\tau(u+hf))_2]\,d\tau$$

$$=S_tf_2+\alpha\int_0^t S_{t-\tau}[(S_\tau f_1)^{(2)}(P_\tau(u+hf))_2]\,d\tau.$$

Now by (3)

(4) $\|(P_\tau(u+hf))_2\| \leqq e^{t|u_1|+\|f\|}\|u_2+hf_2\|\ (0 \leqq h \leqq 1, 0 \leqq \tau \leqq t).$

Multiplying the last equality by h and letting h tend to zero we find

$$\lim_{h\downarrow 0}\|(P_t(u+hf))_2 - u_2\| = 0.$$

Using this and (4) in the same equality yields

$$\lim_{h\downarrow 0}\frac{1}{h}[P_t(u+hf))_2 - u_2] = S_tf_2 + \alpha u_2\int_0^t S_{t-\tau}[(S_\tau f_1)^{(2)}]\,d\tau.$$

Hence P_t is differentiable at u and

$$Q_tf = D_uP_tf = (S_tf_1, S_tf_2 + \alpha u_2\int_0^t S_{t-\tau}(S_\tau f_1)^{(2)}\,d\tau).$$

Furthermore $D_P = C$, $D_Q = C$ and

$$\left.\frac{d}{dt}\right|_{t=0} Q_tf = ((f_1)_{xx}, (f_2)_{xx} + \alpha u_2(f_1)_{xx}), \left.\frac{d}{dt}\right|_{t=0}[P_t(u+hf) - u]$$

$$= (h(f_1)_{xx}, h(f_2)_{xx} + \alpha h_2(f_1)_{xx}(u_2 + hf_2)),$$

$$\lim_{h\downarrow 0}\frac{1}{h}\left(\left.\frac{d}{dt}\right|_{t=0}[P_t(u+hf) - u]\right) = \left.\frac{d}{dt}\right|_{t=0} Q_tf.$$

$Q_tf(x) = z(x, t)$ solves the Cauchy problem

$$\text{(5)}\qquad \begin{cases} (z_1)_{xx} = (z_1)_t \\ (z_2)_{xx} + \alpha u_2(z_1)_{xx} = (z_2)_t \\ z_1(x,0) = f_1(x) \qquad z_2(x,0) = f_2(x). \end{cases}$$

CLAIM 3. P_t satisfies A_5 only in case $\alpha = 0$. P_t satisfies A_7 for all $\alpha \geqq 0$.

The proof is obvious from (1).

REMARK. It is not difficult to work out the preceding example with $\Omega = \mathbf{R}$ replaced by $\Omega =]0, 1[$. This is because a nice semigroup solution of

$$V_{xx} = V_t \text{ on }]0,1[\times]0,\infty[$$

$$V(x,0) = f_1(x) \quad x \in]0,1[$$

is given by $u(x, t) = S_t\tilde{f}_1(x)$, $x \in]0, 1[$, $t \geqq 0$ where $\tilde{f}_1$ is a real valued function on $\mathbf{R}$ such that

$$\tilde{f}_1(x) = f_1(x) \quad \text{for} \quad 0 < x < 1$$
$$\tilde{f}_1(1 + x) = -\tilde{f}_1(1 - x) \quad 0 \leqq x$$
$$\tilde{f}_1(-x) = -\tilde{f}_1(x) \quad 0 \leqq x.$$

See also [9] for a system with cross population pressure.

EXAMPLE 2. Consider the system

$$(6) \qquad \begin{cases} W = V_t \\ V_{xx} = W_t \qquad \text{on } \mathbf{R} \times]0, \infty[\\ v(x, 0) = f_1(x),\ w(x, 0) = f_2(x) \quad (x \in \mathbf{R}). \end{cases}$$

This is the simplest wave equation. The Cauchy problem for (6) admits the following semigroup solution:

$$P_tf(x) = ((P_tf(x))_1, P_tf(x))_2) \text{ with}$$
$$(P_tf(x))_1 = \frac{1}{2}[f_1(x + t) + f_1(x - t)] + \frac{1}{2}\int_{x-t}^{x+t} f_2(y)dy$$
$$(P_tf(x))_2 = \frac{1}{2}[f_1^{(1)}(x + t) - f_1^{(1)}(x - t)] + \frac{1}{2}[f_2(x + t) + f_2(x - t)].$$

P_t is defined, for example, on the space C of all (f_1, f_2) such that f_1, f_2 and $f_1^{(1)}$ are continuous with compact support. It is easy to see that P_t satisfies A_1, A_2, A_3, A_4, $A_7(0)$. P_t does not satisfy A_6, however. This can be seen taking $f = (f_1, 0)$ such that $f_1 \geqq 0$ and $f_1(x) = 1 - x^2$ for $|x| \leqq 1/2$. Then $(P_tf(x))_2 = -(x + t) + (x - t) = -2t$ for small $t > 0$.

5. Weak parabolicity. In this section we ask the following question: Suppose

$$Lf(x) = \sum_{i,k} a^{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} f(x) + \sum_i b^i(x) \frac{\partial}{\partial x_i} f(x) + c(x) f(x)$$

and L is the infinitesimal generator of some semigroup P_t. Does this impose any algebraic structure on the set of coefficients $a^{ik}_{j,\ell}(x)$, $b^i_{j,\ell}(x)$, $c_{j,\ell}(x)$? The problem is particularly interesting in the case where P_t is the linearisation at a nonzero equilibrium of a process of "reaction and diffusion" (see §4).

DEFINITION. We say that L is weakly parabolic if there is a real number $\varepsilon \geqq 0$ such that for all $x \in \Omega$, all $y \in \mathbf{R}^n$ and all eigenvalues λ of the matrix

$$P(x, y) = \sum_{i,k} a^{ik}(x) y_i y_k$$

the real part of λ is greater or equal ε.

REMARK. If $\varepsilon > 0$, L is called parabolic in the sense of Petrovski (see [5] Chapt. 9).

In order to prove that L is weakly parabolic we assume that

(A) P_t is a strongly continuous semigroup on the Hilbert space $L^2(\Omega, \mathbf{C}^m, \lambda)$ of all $\mathbf{C}^m$ valued Lebesgue-square integrable functions $f = (f_1, \ldots, f_m)$ on Ω, the scalar product being

$$\langle f, g\rangle = \sum_{i=1}^{m} \int_\Omega f_i \, \overline{g_i} \, d\lambda.$$

The domain of definition D_A of the infinitesimal generator A of P_t contains the space C_0^∞ of all infinitely differentiable, $\mathbf{C}^m$-valued functions on Ω with compact support and

$$Af = Lf \text{ for all } f \in C_0^\infty.$$

Finally $c_{j,\ell}$, $b^i_{j,\ell}$ are locally bounded, measurable functions and $a^{ik}_{j,\ell}$ are continuous on Ω:

THEOREM 4. *If $\Omega = \mathbf{R}^n$ then L is weakly parabolic.*

PROOF. Suppose $x_0 \in \mathbf{R}^n$, $y \in \mathbf{R}^n$, $v \in \mathbf{C}^m$, $|v| = 1$, $\lambda \in \mathbf{C}$ such that

$$P(x_0, y) = \Big(\sum_{i,k} a^{ik}(x_0) y_i y_k\Big) v = \lambda v \text{ and}$$
$$Re\, \lambda = -\varepsilon < 0.$$

Without loss of generality we may assume $x_0 = 0$. Let $\varphi_n = \cos(x\ n\ y)v$. By an easy calculation we can see that there exists a strictly positive constant C with:

$$\left(\int_{B(0,1)} |\varphi_n(x)|^2 \, dx\right)^{1/2} > C \text{ for all } n \in N,$$

where $B(0, 1)$ is the unit ball with radius 1 and center 0. Let $g \in C_0^\infty$ with $g = 1$ on $B(0, 1)$. Hence $\psi_n = \cos(x \cdot ny) g(x) \cdot v$ is in C_0^∞ and

$$\|\psi_n\|_{L^2} > C > 0.$$

$$\text{Let } A^\infty f(x) := \sum_{i,k} a^{ik}(0) \frac{\partial^2}{\partial x_i \partial x_k} f(x) \text{ for all } f \in C_0^\infty.$$

Then

$$A^\infty \psi_n = -n^2 \lambda \psi_n - n\left(\sum_{i,k} y_k \sin(x \cdot ny) \frac{\partial g}{\partial x_i} + y_i \sin(x \cdot ny) \frac{\partial g}{\partial x^i}\right) a^{ik}(0)$$

$$+ v \cos(x \cdot ny) \sum_{i,k} \frac{\partial^2 g}{\partial x_i \partial x_k} a^{ik}(0).$$

Hence

$$(1) \qquad \|A^\infty \psi_n + n^2 \lambda\psi_n\|_{L^2} \leqq nA + B,$$

where A and B are constants independent of n. For any $\rho > 0, f \in L^2$ set

$$f_\rho(x) = f(\rho x).$$
$$P_t^\rho f(x) = P_{\rho^{-2}t} f_\rho(\rho^{-1}x).$$

It is easy to verify that $f_\rho \in L^2$, if $f \in L^2$, and that P_t^ρ is a strongly continuous semigroup on L^2. By the Hille-Yoshida theorem ([7] for example) there are constants M, $w > 0$ such that

$$|P_t| \leqq M\, e^{wt} \quad (t > 0).$$

It is easily verified that, as a consequence, we have

$$|P_t^\rho| \leqq M\, e^{w\rho^{-2}t}.$$

Again, by the Hille-Yoshida theorem,

$$(2) \qquad \|\mu f - A^\rho f\|_{L^2} \geqq \frac{Re\mu - w\rho^{-2}}{M} \|f\|_{L^2} \qquad (f \in D_{A^\rho}, \ \mathrm{Re}\mu > \rho^{-2}w),$$

where A^ρ is the infinitesimal generator of P_t^ρ and D_{A^ρ} its domain of definition. The relation

$$\left\|\frac{1}{t}(P_t^\rho f - f) - g\right\|_2^2 = \rho^n \left\|\frac{1}{t}(P_{\rho^{-2}t} f_\rho - f_\rho) - g_\rho\right\|_2^2$$
$$= \rho^{n-2}\left\|\frac{1}{\rho^{-2}t}(P_{\rho^{-2}t} f_\rho - f_\rho) - \rho^2 g_\rho\right\|_2^2$$

shows that

$$(f \in D_{A^\rho} \text{ and } A^\rho f = g) \Leftrightarrow (f_\rho \in D_A \text{ and } Af_\rho = \rho^2 g_\rho).$$

Since, for any $\rho > 0$,

$$f_\rho \in C_0^\infty \subset D_A, \text{ if } f \in C_0^\infty,$$

we get, for $f \in C_0^\infty$

$$(A^\rho f)(x) = \rho^{-2}(Af_\rho)(\rho^{-1}x)$$
$$= \sum_{i,k} a^{ik}(\rho^{-1}x) \frac{\partial^2}{\partial x_i \partial x_k} f(x) + \rho^{-1} \sum_i b^i(\rho^{-1}x) \frac{\partial}{\partial x_i} f(x)$$
$$+ \rho^{-2} c(\rho x) f(x).$$

Hence

$$\|A^\rho f - A^\infty f\|_{L^2} \leqq \left(\int_{\mathbf{R}^n}\left(\sum a^{ik}(\rho^{-1}x) - a^{ik}(0)\right)\frac{\partial^2}{\varrho x_i \partial x_k} f\right)^2 \lambda(dx)\Big)^{1/2}$$
$$+ \frac{1}{\rho}\left(\left(\int_{\mathbf{R}^n}\sum_i b^i(\rho^{-1}x)\frac{\partial}{\partial x_i}f(x)\right)^2 \lambda(dx)\right)^{1/2}$$
$$+ \frac{1}{\rho^2}\left(\int (c(\int\rho^{-1}x)f(x))^2\, \lambda(dx)\right)^{1/2}.$$

The first term on the right hand side tends to zero as $\rho \to \infty$, since, by assumption (A) $\lim_{\rho\to\infty} |a^{ik}(\rho^{-1}x) - a^{ik}(0)| = 0$ uniformly in x on every compact subset of $\mathbf{R}^n$ and $\partial^2/\partial x_i \partial x_k\, f$ has compact support, if $f \in C_0^\infty$. The second and third term tend to zero because b^i and c are locally bounded, measurable functions on Ω. Now take the limit as ρ tends to ∞ in (2). This yields

$$\|\mu f - A^\infty f\|_2 \geqq \frac{\mathrm{Re}\mu}{M}\|f\|_2 \qquad (f \in C_0^\infty,\ \mathrm{Re}\mu > 0).$$

In particular,

$$\|n^2\lambda\psi_n + A^\infty\psi_n\|_n \geqq \frac{n^2\varepsilon}{9M}\|\psi_n\|_2 \geqq \frac{n^2\varepsilon}{M}C, \text{ for every } n \in \mathbf{N}.$$

This is a contradiction to (1).

For applications the case where Ω is a bounded open subset of $\mathbf{R}^n$ is more interesting than $\Omega = \mathbf{R}^n$. In the following let Ω be a bounded open set in $\mathbf{R}^n$.

THEOREM 5. *L is weakly parabolic.*

PROOF. *Let* $x_0 \in \Omega$, $y \in \mathbf{R}^n$, $v \in \mathbf{C}^m$ with $|v| = 1$, $\lambda \in \mathbf{C}$ such that

$$P(x_0, y)v = \left(\sum_{i,k} a^{ik}(x_0)y_i y_n\right)v = \lambda v.$$

Without loss of generality we can assume that $x_0 = 0$. For a measurable function f on Ω we define

$$f_\rho(x) = \begin{cases} f(\rho x) & \text{if } \rho x \in \Omega \\ 0 & \text{if } \rho x \notin \Omega. \end{cases}$$

Let $r > 0$ such that $B(0, r) \in \Omega$ and $\rho_0 > 0$ such that

$$\frac{1}{\rho_0}\Omega \subset B(0, r) \ (\rho_0 \text{ exists, since } \Omega \text{ is bounded}).$$

Then, we have, for $\rho > \rho_0$

$$f_\rho \in L^2(\Omega), \text{ if } f \in L^2(\Omega), \text{ and}$$
$$f_\rho \in C_0^\infty(\Omega), \text{ if } f \in C_0^\infty(\Omega).$$

We can then define P_t^ϱ and the same proof as that of Theorem 4 leads to the assertion.

REMARK. We can replace $L^2(\Omega)$ in Theorem 4 and 5 by a Banach subspace of the space of bounded continuous functions on Ω with the supremum norm.

EXAMPLE. Mimura **[9]** proposed the system

$$\begin{cases} u_t = \Delta\{(d_{11} + d_{12}v)u\} + (R_1 - a_1u - b_1v)u \\ v_t = \Delta\{(d_{22} + d_{21}u)v\} + (R_2 - a_2v - b_2u)v \end{cases}$$

as a model of two competing species with self and cross-population pressures. Here d_{ij}, R_i, a_i, b_i are positive constants or zero. The linearisation at a constant concentration (α, β) with $\alpha > 0$, $\beta > 0$ is

$$\begin{cases} u_t = (d_{11} + d_{12}\beta)\Delta u + d_{12}\alpha\Delta v + \cdots \\ v_t = (d_{22} + d_{21}\,\alpha)\Delta v + d_{21}\beta\Delta u + \cdots \end{cases}$$

This system is weakly parabolic if and only if all eigenvalues of

$$\begin{pmatrix} d_{11} + d_{12}\beta, & \alpha d_{12} \\ \beta d_{21}, & d_{22} + \alpha d_{21} \end{pmatrix} = B$$

have non-negative real part. It is easily seen that this is true, because trace $B \geqq 0$ and det $B \geqq 0$.

REMARKS. Theorems 4 and 5 look like special cases of a general theorem which works for any open subset Ω of $\mathbf{R}^n$ and a wide class of Banach spaces of $\mathbf{C}^m$ valued functions on Ω including $L^2(\Omega, \mathbf{C}^m)$. Weak parabolicity is certainly not a sufficient condition for L to be the restriction of a strongly continuous semigroup on L^2. If, for example, all $a^{ik} = 0$, then by the same method of proof we find that necessarily $Re\,\lambda \geqq 0$ for any eigenvalue λ of

$$\sum_k ib^k(x)y_k \; (y \in \mathbf{C}^m,\ x \in \Omega,\ i^2 = -1).$$

On the other hand, parabolicity in the sense of Petrovski is a sufficient condition provided the a^{ik}, b^i, c are sufficiently smooth (see **[5]** chap 9). Parabolicity in the sense of Petrovski however is not a necessary condition, the simplest counter example being $P_tf = f$ $(t \geqq 0)$. The question treated in this section is a classical one. When is a problem "correctly posed" in the sense of Hadamard? Further results and references may be found in a chapter called "inverse theorems" in **[6]**. These inverse theorems however only work for coeffidients depending on t but not on x.

6. Summary and a problem. We have shown that smooth, local, memoryless processes which obey a kind of maximum principle are governed by systems of weakly parabolic semilinear second order partial differential equations, at least near an equilibrium.

It is remarkable that the former properties are common not only to a wide class of processes of reaction and diffusion but also to processes governed by wave equations.

From the point of view of biological sciences, the axiom that the process should have no memory seems to be the most restrictive. From a mathematical point of view, however, it seems more promising to investigate the absence of locality (A_3). In fact very little is known about processes with memory whereas for non local processes we have as a natural generalization of a differential operator the notion of a pseudo differential operator. Hence the following question may lead to interesting further research.

What kind of assumptions may replace (A_3) in order that the resulting evolutionary law still is a system of pseudo differential equations?

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