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EXAMPLES OF RP-MEASURES

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ABSTRACT. We call a measure on the torus T^2 an RP-measure if its Poisson integral is the real part of a holomorphic function. Let RP_1 denote the set of RP-measures which are non-negative and have total mass one. We construct an extreme element μ of RP_1 such that the closed support of μ is all of T^2. We also construct an RP-measure which is not an extreme point, but which belongs to a proper weak* closed face of RP_1, is absolutely continuous with respect to Haar measure, and satisfies a certain necessary condition on extreme elements of RP_1.

The well known theorem of Herglotz asserts that, if u is a positive harmonic function on the open unit disk D, then there is a unique positive Borel measure μ on the unit circle T such that

$$u(z) = \int_T P_z(x)d\mu(x), \tag{1}$$

where $P_z(x)$ denotes the Poisson kernel $\mathrm{Re}(x + z)/(x - z)$. An equivalent way to state Herglotz's theorem is the following. Let f be holomorphic in D, have positive real part and satisfy $f(0) = 1$. Then there is a unique probability measure μ_f on T such that

$$\mathrm{Re}\, f(z) = \int_T P_z(x)d\mu_f(x). \tag{2}$$

It is clear from (2) that the correspondence $M_1(f) = \mu_f$ is a bijection between the convex sets $P(T)$ = Borel probability measures on T and

$$\mathscr{P}_1 = \{f | f \text{ holomorphic in } D,\ \mathrm{Re}\, f > 0, f(0) = 1\}.$$

Also, if $\mathscr{P}_1$ and $P(T)$ are equipped with the topology of uniform convergence on compacta and the weak* topology respectively, then M_1 is continuous. Moreover, M_1 is affine, i.e., it preserves convex combinations. It follows that

$$M_1(\mathrm{ex}\ \mathscr{P}_1) = \mathrm{ex}\ P(T),$$

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where ex $\mathscr{P}_1$ denotes the set of extreme points of $\mathscr{P}_1$. Since the extreme points of $P(T)$ are exactly the measures supported by singletons, it follows that $f \in$ ex $\mathscr{P}_1$ if and only if

$$\text{Re} f(z) = \int_T P_z(x) d\delta_y(x), \tag{3}$$

for some $y \in T$, where δ_y denotes the unit point mass measure concentrated at y.

When the disk D is replaced by the bi-disk $D \times D$, there is an analogue of (1). Namely, if $u(z, w)$ is positive on $D \times D$ and harmonic in each variable, then there is a unique positive measure μ on the torus T^2 such that

$$u(z, w) = \int_{T^2} P_z(x) P_z(y) d\mu(x, y).$$

Thus, as in the one-dimensional case, we have a mapping M_2: $\mathscr{P}_2 \to P(T^2)$, where

$$\mathscr{P}_2 = \{f \mid f \text{ holomphic on } D \times D, \text{ Re} f > 0 \text{ and } f(0, 0) = 1\}$$

and $M_2(f)$ is the unique probability measure μ_f which satisfies

$$\text{Re} f(z, w) = \int_{T^2} P_z(x) P_w(y) d\mu_f(x, y).$$

Like M_1, the mapping M_2 is affine, continuous, and one-to-one, but, in contrast to the one dimensional case, M_2 is not onto. In fact, it is easy to show that $\mu \in M_2(\mathscr{P}_2)$ if and only if $\int_{T^2} x^p y^q d\mu(x, y) = 0$ for all pairs of integers (p, q) with $pq < 0$. The set $M_2(\mathscr{P}_2)$ will, from now on, be denoted by RP_1. Since RP_1 is weak* compact and convex, it must have extreme points, but the problem of describing the extreme points of RP_1 posed by Rudin in [6], does not have such an easy solution as the problem of describing the extreme points of $P(T)$. Note, in particular, that PR_1 cannot contain point masses.

In this paper we present two examples. In our first example we show that if g is an appropriately chosen inner function, then

$$G(z, w) = \frac{(1 - ig(w))(1 + iz)}{1 - zg(w)} + ig(0)$$

is an extreme element of $\mathscr{P}_2$ having the property that $\mu_G = M_2(G)$ is an extreme RP_1 measure whose closed support is all of T^2. (Note the contrast with (3).) Our second example is a member of $\mathscr{P}_2$ of the form

$$F_0(z, w) = \frac{1 + z^n f_0 w/z)}{1 - z^n f_0 w/z)},$$

where f_0 is a certain polynomial of degree $n \geqq 2$. We show that F_0 satisfies a necessary condition on the extreme points of $\mathscr{P}_2$, given by Forelli in [1], that $\mu_{F_0} = M_2(F_0)$ is absolutely continuous with respect to the usual Lebesgue measure on T^2, and that μ_{F_0} belongs to a proper weak* closed face of RP_1. Unfortunately, the example F_0 happens not to be an extreme element of $\mathscr{P}_2$. Nevertheless, it suggests a conjecture which relates to another question raised by Rudin in [6], namely, does there exist an extreme element of RP_1 which is absolutely continuous with respect to Lebesgue measure on T^2? In constructing our examples we will develop some results which are, perhaps, of some independent interest.

EXAMPLE A. This example is derived from the following

THEOREM 1. *Let g be an inner function on D such that $g(0)$ is real. Then*

$$G(z, w) = \frac{(1 - ig(w))\,(1 + iz)}{1 - zg(w)} + ig(0) \tag{4}$$

is an extreme element of $\mathscr{P}_2$. (See [2] for a discussion of inner functions.)

Before giving a proof of Theorem 1 we will show how it leads to our example. We choose g such that it has no analytic continuation across any sub-arc of T. (For example, we could take g to be the Blaschke product

$$g(w) = \prod_{n=1}^{\infty} \prod_{k=0}^{n-1} \lambda_n^k \left(\frac{1 - 2^{-n})\,\lambda_n^k - w}{1-(1 - 2^{-n})\lambda_n^k w} \right),$$

where $\lambda_n = \exp\,(2\pi i/n)$. Of course, the crucial property possessed by this g is that its zeros accumulate at every point of T.) It is easy to show that if W is an open subset of T^2, then G cannot have an analytic continuation across W. It follows from a result due to Rudin [5, p. 23], that $\mu_G(W) > 0$. Thus, the closed support of μ_G is all of T^2.

PROOF OF THEOREM 1. Define a measure ν_0 on T^2 via

$$\int_{T^2} h(x, y) d\nu_0(x, y) = \int_T h(\overline{g(y)}, y)\,|dy|,$$

for $h \in C(T^2)$. Here, $C(T^2)$ denotes the space of continuous complex valued functions on T^2 and $|dy|$ denotes the element of arc-length on T normalized so that $\int_T |dy| = 1$. We remark that the Poisson integral of ν_0 is

$$\operatorname{Re}\left(\frac{1 + zg(w)}{1 - zg(w)} \right).$$

Note that $\nu_0 \in RP_1$, for, if $n, m > 0$, then

$$\int_{T^2} x^{-n} y^m d\nu_0(x, y) = \int_T (g(y))^n y^m |dy| = 0.$$

Consider

$$\mathscr{F}(\nu_0) = \{\nu \in RP_1 | \nu \text{ is absolutely continuous with respect to } \nu_0\}.$$

It is evident that ex $\mathscr{F}(\nu_0) \subseteq$ ex RP_1. The previous statement tells us nothing, however, unless we know that ex $\mathscr{F}(\nu_0) \neq \emptyset$. Since $\mathscr{F}(\nu_0)$ is not necessarily weak* closed, we cannot assert the existence of extreme elements of $\mathscr{F}(\nu_0)$ via the Krein-Milman theorem. Nevertheless, we will show that μ_G, where G is given by (4), is an extreme element of $\mathscr{F}(\nu_0)$.

Let $\nu \in \mathscr{F}(\nu_0)$. Define a measure ρ_ν on T via

$$\int_T f(y)d\rho_\nu(y) = \int_{T^2} f(y)d\nu(x, y),$$

where $f \in C(T)$. It is claimed that ρ_ν is absolutely continuous with respect to Lebesgue measure on T. Let $K \subseteq T$ be closed and have Lebesgue measure 0. Let $f \in C(T)$ be chosen such that $f_0(y) = 1$ for $y \in K$ and $|f_0(y)| < 1$ for $y \in T\backslash K$. Let $\varepsilon > 0$ be given. Choose $H_0 \in C(T^2)$ such that

$$\int_{T^2}\left|\frac{d\nu}{d\nu_0} - H_0\right|d\nu_0 < \varepsilon. \tag{5}$$

Then, for $n = 1, 2, \ldots,$ we have

$$\begin{aligned}\left|\int_T (f_0(y))^n d\rho_\nu(y)\right| &= \left|\int_{T^2} (f_0(y))^n d\nu(x, y)\right| \\ &\leqq \left|\int_{T^2} (f_0(y))^n H_0(x, y)d\nu_0(x, y)\right| + \varepsilon \\ &\leqq \left|\int_T (f_0(y))^n H_0(\overline{g(y)}, y)\,|dy|\right| + \varepsilon.\end{aligned}$$

Letting $n \to \infty$, we obtain $\rho_\nu(K) \leqq \varepsilon$. Since ε was chosen arbitrarily from $(0, \infty)$, it follows that $\rho_\nu(K) = 0$. Thus, the measure ρ_ν is absolutely continuous with respect to arc-length measure on T.

Now define a measure $\tilde{\nu}$ via

$$\int_{T^2} h(x, y)d\tilde{\nu}(x, y) = \int_T h(\overline{g(y)}, y)d\rho_\nu(y), \qquad h \in C(T^2).$$

We will show that $\tilde{\nu} = \nu$ by showing that

$$\int_{T^2} k(x, y)d\tilde{\nu}(x, y) = \int_{T^2} k(x, y)d\nu(x, y).$$

for every $k \in C(D \times D)$ such that $|k| \leqq 1$. Let $\varepsilon > 0$ be given. Choose H_0 as in the previous paragraph. Choose $r \in (0, 1)$ such that

$$\left|\int_T k(\overline{g(ry)}, y)H_0(\overline{g(y)}, y)|dy| - \int_T k(\overline{g(y)}, y)H_0(\overline{g(y)}, y)\,|dy|\right| < \varepsilon \tag{6}$$

and

$$\left|\int_T k(\overline{g(ry)}, y)d\rho_\nu(y) - \int_T k(\overline{g(y)}, y)d\rho_\nu(y)\right| < \varepsilon. \tag{7}$$

From (5) and the definition of ν_0, we have

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_T k(\overline{g(y)}, y)H_0(\overline{g(y)}, y)\,|dy|\right| < \varepsilon. \tag{8}$$

From (6) and (8), we have

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_T k(\overline{g(ry)}, y)H_0(\overline{g(y)}, y)\,|dy|\right| < 2\varepsilon.$$

From the definition of ν_0 and from the continuity of $y \to g(ry)$, it follows that

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_T k(\overline{g(ry)}, y)H_0(x, y)d\nu_0(x, y)\right| < 2\varepsilon.$$

Again, using (5), we have

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_{T^2} k(\overline{g(ry)}, y)d\nu(x, y)\right| < 3\varepsilon.$$

From the definition of ρ_ν, it follows that

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_T k(\overline{g(ry)}, y)\, d\rho_\nu(y)\right| < 3\varepsilon.$$

Using (7), we have

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_T k(\overline{g(y)}, y)\, d\rho_\nu(y)\right| < 4\varepsilon.$$

Finally, by the definition of $\tilde{\nu}$, we have

$$\left|\int_{T^2} k(x, y)d\nu(x, y) - \int_{T^2} k(x, y)d\tilde{\nu}(x, y)\right| < 4\varepsilon.$$

Since ε is an arbitrary positive number, it follows that $\int kd\nu = \int kd\tilde{\nu}$. Thus, $\nu = \tilde{\nu}$.

The argument above shows that

$$\int_{T^2} h(x, y)d\nu(x, y) = \int_T h(\overline{g(y)}, y)D_\nu(y)|dy|,$$

for every $h \in C(T^2)$, where $D_\nu = (d\rho_\nu)/|dy|$. Taking $h(x, y) = x^{-1}y^n$, where n is a positive integer, and using the fact that $\nu \in RP_1$, we have

$$\int_T y^n g(y)D_\nu(y)|dy| = 0.$$

It follows that $F_\nu = fD_\nu$ belongs to the Hardy space H_1. (See [2, Chapt. 4] for a discussion of H_1, the F. and M. Riesz Theorem, and related topics.) It is now clear that the mapping $\nu \to F_\nu$ is an affine bijection between $\mathscr{F}(\nu_0)$ and a convex subset of

$$R_g = \left\{F \in H_1 \,|\, F\bar{g} \geqq 0 \text{ and } \int_T F(y)\overline{g(y)}|dy| = 1\right\}.$$

In fact, it is easy to show that each $F \in R_g$ is of the form F_ν for some $\nu \in \mathscr{F}(\nu_0)$. It follows that ν is an extreme element of $\mathscr{F}(\nu_0)$ if and only if F_ν is an extreme element of R_g. In [3, example 3] we showed that the extreme elements of R_g are exactly the outer functions which lie in R_g. (See [2] for a discussion of outer functions.) An example of an outer function in R_g is $F_1 = (2i)^{-1}(g + i)^2$. (Note the fact that $F_1 \in R_g$ depends on $g(0)$ being real.) The measure ν_1 such that $F_1 = F_{\nu_1}$ is given by

$$\int_{T^2} h(x, y)d\nu_1(x, y) = \int_T h(\overline{g(y)}, y)\,(1 + \operatorname{Im} g(y))\,|dy|.$$

We will now calculate the Poisson integral of ν_1. Let $z, w \in D$. Then

$$\int_{T^2} P_z(x)P_w(y)d\nu_1(x, y) = \sum_{n=1}^{\infty} z^{-n} \int_{T^2} x^n P_w(y)d\nu_1(x, y) + \sum_{n=0}^{\infty} z^n \int_{T^2} x^{-n}P_w(y)d\nu_1(x, y).$$

For $n > 0$, we have

$$\int_{T^2} x^n P_w(y)d\nu_1(x, y) = \int_T \overline{g(y)}^n P_w(y)(1 + \operatorname{Im}(g(y))|dy| = \overline{g(w)}^n + (2i)^{-1}(\overline{g(w)}^{n-1} - \overline{g(w)}^{n+1})$$

and

$$\int_{T^2} x^{-n}P_w(y)d\nu_1(x, y) = \int_T g(y)^n P_w(y)\,(1 + \operatorname{Im}(g(y))|dy| = g(w)^n + (2i)^{-1}(g(w)^{n+1} - g(w)^{n-1}).$$

Also,

$$\int_{T^2} P_w(y)d\nu_1(x, y) = 1 + (2i)^{-1}(g(w) - \overline{g(w)}).$$

A straightforward calculation now shows that

$$\int_{T^2} P_z(x)P_w(y)d\nu_1(x, y) = \operatorname{Re}\left(\frac{1 - ig(w)\,(1 + iz)}{1 - zg(w)}\right).$$

Since ν_1 is an extreme element of RP_1, it follows that

$$G(z, w) = \frac{(1 - ig(w))(1 + iz)}{1 - zg(w)} + ig(0)$$

is an extreme element of $\mathscr{P}_2$.

EXAMPLE B. Consider a function $F \in \mathscr{P}_2$. For λ, $z \in D$, we have the expansion

$$F(z, \lambda z) = 1 + 2\sum_{n=1}^{\infty} F_n(\lambda)z^n, \tag{9}$$

where $F_n(\lambda)$ is a polynomial in λ of degree $\leqq n$, which satisfies $\sup_{\lambda \in D}|F_n(\lambda)| \leqq 1$. Let $\mathscr{U}_n$ denote the set of polynomials of degree $\leqq n$ which are bounded in absolute value by 1 on D. Of course $\mathscr{U}_n$ is a compact convex set of (complex) dimension $n + 1$. Let $f \in \text{ex}\ \mathscr{U}_n$, where $n > 1$. Consider the set

$$\mathscr{F}(f, n) = \{F \in \mathscr{P}_2 | F_n(\lambda) = f(\lambda)\}.$$

It is easy to see that $\mathscr{F}(f, n)$ is a proper face of $\mathscr{P}_2$ which is closed in the topology of uniform convergence on compact subsets of $D \times D$. Furthermore, we have the following

THEOREM 2. *For $n > 1$ and $f \in \text{ex}\ \mathscr{U}_n$, $\mathscr{F}(f, n)$ contains an extreme element of $\mathscr{P}_2$.*

PROOF. Since $\mathscr{F}(f, n)$ is a compact convex set, by the Krein-Milman theorem it suffices to show that $\mathscr{F}(f, n)$ is non-empty. Thus, the observation that

$$F(z, w) = \frac{1 + z^n f(w/z)}{1 - z^n f(w/z)}$$

belongs to $\mathscr{F}(f, n)$ completes the proof.

The interested reader can check that Theorem 2 is simply a re-phrasing of Example 4 of [3]. Also, Examples 1 and 2 of [3] correspond to the cases in which $f(\lambda) = \lambda$ and $f(\lambda) = \lambda^m$ respectively. A natural idea, then, is to study the case in which f is an extreme element of $\mathscr{U}_n$ which is not of the form $c\lambda^k$. It requires a little work, however, to show that such extreme elements of $\mathscr{U}_n$ exist. We will show below that, for $n \geqq 2$, there is a polynomial $f_0 \in \mathscr{U}_n$ having the following properties:

$$f_0^{-1}(T) \backslash D = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \tag{10}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct points of T; and

$$f_0 \in \text{ex}\ \mathscr{U}_n. \tag{11}$$

Let $x_1, x_2, \ldots, x_n$ be distinct points of T. And let $K^{-1} = \sup \prod_{j=1}^{n}|x - x_j|^2$. Since $1 - K\prod_{j=1}^{n}|e^{it} - x_j|^2$ is a non-negative trigonometric polynomial, it follows from a well known result of Fejer and Riesz [4, p. 259] that there is a polynomial $h \in \mathscr{U}$ of degree n such that

$$|h(e^{it})|^2 = 1 - K \prod_{j=1}^{n} |e^{it} - x_j|^2.$$

Note that $|h(x_j)| = 1$, for $j = 1, 2, \ldots, n$, and that h is not of the form cz^n. Consider

$$E = \{g \in U_n | g(x_i) = h(x_i),\ i = 1, 2, \ldots, n\}.$$

It is easy to see that E is a proper face of $\mathscr{U}_n$ and can contain cz^n, for at most one value of c. Note also that $h \in E$. It follows from the Krein-Milman theorem that E has an extreme point f_0 which is not of the form cz^n. It is claimed that $f_0(x) \notin T$ for $x \neq x_i$, $i = 1, 2, \ldots, n$. The set $T \cap f_0^{-1}(T)$ contains $x_1, x_2, \ldots x_n$. Suppose $x_0 \in T \cap f_0^{-1}(T)$ where $x_0 \neq x_i$, $i = 1, 2, \ldots, n$. Then the trigonometric polynomial

$$g_0(e^{it}) = 1 - |f_0(e^{it})|^2$$

has degree $\leqq n$ and vanishes at more than n points. It follows from the result of Fejer and Riesz mentioned above that $g(e^{it}) = 0$, for all $t \in [0, 2\pi)$. Thus, $|f_0(x)| = 1$ for every $x \in T$. By Schwarz reflection it must follow that f_0 is of the form cz^n. This contradiction shows that f_0 satisfies (10). That f_0 also satisfies (11) follows from the fact that f_0 is an extreme element of the face E.

(The author is indebted to the referee for this proof of the existence of an $f_0 \in U_n$ satisfying (10) and (11). The original proof submitted was much longer.)

Given f_0, we define an element F_0 of $\mathscr{F}(f_0, n)$ by

$$F_0(z, w) = \frac{1 + z^n f_0(w/z)}{1 - z^n f_0(w/z)}.$$

We will show below that F_0 is not an extreme element of $\mathscr{P}_2$. However, F_0 is in some sense close to being extreme, for, besides belonging to a proper closed face of $\mathscr{P}_2$, namely, $\mathscr{F}(f_0, n)$, it possesses another property in common with the extreme elements of $\mathscr{P}_2$. We are referring to a necessary condition on members of ex $\mathscr{P}_2$ given by Forelli in [1]. Forelli's condition may be described in our context as follows. Each $F \in \mathscr{P}_2$ may be written uniquely in the form

$$F(z, w) = \frac{1 + f(z, w)}{1 - f(z, w)},$$

where f is analytic on $D \times D$, $f(0, 0) = 0$, and

$$\sup_{z, w \in D} |f(z, w)| \leqq 1. \tag{12}$$

The function of f is said to be irreducible if, whenever $f = f_1 f_2$, where f_1 and f_2 are analytic on $D \times D$ and satisfy (12), then either f_1 or f_2 is a constant of modulus 1. Forelli proved that, if $F \in \text{ex}\, \mathscr{P}_2$, then f is irreducible. We will show that

$$\tilde{f}_0(z, w) = z^n f_0(w/z)$$

is irreducible. Suppose that $\tilde{f}_0(z, w) = f_1(z, w) f_2(z, w)$, where f_1 and f_2 are analytic in $D \times D$ and satisfy (12). Letting $w = \lambda z$, we have

$$z^n f_0(\lambda) = f_1(z, \lambda z) f_2(z, \lambda z) = \Big(\sum_{k=0}^{\infty} f_{1,k}(\lambda) z^k\Big)\Big(\sum_{k=0}^{\infty} f_{2,k}(\lambda) z^k\Big),$$

where $f_{i,k}(\lambda)$ is a polynomial of degree $\leqq k$ satisfying $\sup_{\lambda \in D} |f_{i,k}(\lambda)| \leqq 1$ for $i = 1, 2$ and $k = 0, 1, 2 \ldots.$ Let p and q be, respectively, the first integers such that $f_{1,p}(\lambda) \neq 0$ and $f_{2,q}(\lambda) \neq 0$. Then we may write

$$f_0(\lambda) = f_{1,p}(\lambda) f_{2,q}(\lambda)$$

and assert that $p + q = n$. It is claimed that either $p = 0$ or $q = 0$. Suppose $p > 0$ and $q > 0$. Then $f_{1,p}$ and $f_{2,q}$ both have degree $< n$. Also, since $|f_{1,p}(\lambda_i)|, |f_{2,q}(\lambda_i)| \leqq 1$ and since

$$|f_{1,p}(\lambda_i)|\, |f_{2,q}(\lambda_i)| = |f_0(\lambda_i)| = 1,$$

it follows that

$$|f_{1,p}(\lambda_i)| = |f_{2,q}(\lambda_i)| = 1,$$

for $i = 1, 2, \ldots n$. Thus, $1 - |f_p(e^{it})|^2$ is a non-negative trigonometric polynomial of degree $p < n$ having n zeros. It follows from the result of Fejer and Riesz that $|f_{1,p}(e^{it})| = 1$. Similarly, $|f_{2,q}(e^{it}|) = 1$. It follows that $|f_0(\lambda)| = 1$ for $\lambda \in T$. But f_0 has the property that $|f_0(\lambda)| < 1$, for $\lambda \in T \backslash \{\lambda_1, \ldots, \lambda_n\}$. Thus, one of p, q, say p, must be 0. It has now been shown that

$$f_0(\lambda) = f_{1,0} f_{2,n}(\lambda).$$

Since $|f_0(\lambda_1)| = 1$, it follows that $|f_{1,0}| = 1$. From

$$|f_1(z, \lambda z)| \leqq 1, \qquad z, \lambda \in D,$$

$f(0, 0) = f_{1,0}$, and the maximum modulus principle, it follows that $f_1(z, \lambda z) \equiv f_{1,0}$. Hence, $f_1(z, w) \equiv f_{1,0}$.

Next, we show that $M_2(F_0)$ is absolutely continuous with respect to Lebesgue measure on T^2. Define a measure σ on T^2 via

$$d\sigma = \text{Re}\left(\frac{1 + x^n f_0(y\bar{x})}{1 - x^n f_0(y\bar{x})}\right) |dy|\, |dx|. \tag{13}$$

Note that

$$\int_T \operatorname{Re}\left(\frac{1 + x^n f_0(y)}{1 - x^n f_0(y)}\right)|dx| = 1,$$

for $y \in T \backslash \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Thus,

$$\begin{aligned}\sigma(T^2) &= \int_T\int_T \operatorname{Re}\left(\frac{1 + x^n f_0(y\bar{x})}{1 - x^n f_0(y\bar{x})}\right)|dy|\,|dx| \\ &= \int_T\int_T \operatorname{Re}\left(\frac{1 + x^n f_0(y)}{1 - x^n f_0(y)}\right)|dy|\,|dx| \\ &= \int_T 1|dy| \\ &= 1.\end{aligned}$$

It follows that $\sigma(T^2)$ is a finite measure. We will show that the Poisson integral of σ is Re F_0 and, hence, that $\sigma = M_2(F_0)$. We have

$$\begin{aligned}\int_{T^2} P_z(x)P_w(y)d\sigma(x, y) &= \int_T\int_T P_z(x)P_w(y) \operatorname{Re}\left(\frac{1 + x^n f_0(y\bar{x})}{1 - x^n f_0(y\bar{x})}\right)|dy|\,|dx| \\ &= \int_T\int_T P_z x)P_{w\bar{x}}(y) \operatorname{Re}\left(\frac{1 + x^n f_0(y)}{1 - x^n f_0(y)}\right)|dy|\,|dx|.\end{aligned}$$

For all but finitely many values of $x \in T$, we have $x^{-n} \notin f_0(T)$. Consequently, we have

$$\int_T P_{w\bar{x}}(y)\, Re\left(\frac{1 + x^n f_0(y)}{1 - x^n f_0(y)}\right)|dy| = \operatorname{Re}\left(\frac{1 + x^n f_0(w\bar{x})}{1 - x^n f_0(w\bar{x})}\right).$$

It follows that

$$\begin{aligned}\int_{T^2} P_z(x)P_w(y)d\sigma(x, y) &= \int_T P_z(x) \operatorname{Re}\left(\frac{1 + x^n f_0(w\bar{x})}{1 - x^n f_0(w\bar{x})}\right)|dx| \\ &= \operatorname{Re}\left(\frac{1 + z^n f_0(w/z)}{1 - z^n f_0(w/z)}\right) \\ &= \operatorname{Re} F_0(z. w).\end{aligned}$$

Unfortunately, F_0 is not an extreme point of $\mathscr{P}_2$, as the following argument shows. By the result of Fejer and Riesz, there is an analytic polynomial g of degree $\leqq n$ such that

$$1 - |f_0(e^{it})|^2 = |g(e^{it})|^2.$$

We will show that

$$\operatorname{Re}\,(F_0(z, w) \pm \frac{1}{2}\frac{(z^n g(w/z))^2}{1 - z^n f_0(w/z))} > 0, \tag{14}$$

for $|z|, |w| < 1$. To prove (14) we observe first that the left hand side of that inequality can be re-written in the form

$$(15)\qquad \frac{1 - |z^n f_0(w/z)|^2 \pm 2^{-1}\mathrm{Re}((z^n g(w/z))^2 + z^n|z|^{2n} f_0(\overline{w/z})(g(w/z))^2)}{|1 - z^n f_0(w/z)|^2}.$$

Thus, (14) will follow, if we can show that the numerator of (15) is non-negative when $|z| = |w| = 1$. Clearly, it suffices to show that the expression

$$(16)\qquad 1 - |f_0(e^{it})|^2 \pm 2^{-1}\,\mathrm{Re}(z^n((g(e^{it}))^2 + f_0(e^{it})\,(g(e^{it}))^2))$$

is non-negative, for $|z| = 1$ and $t \in [0, 2\pi)$. But (16) is non-negative because the $\pm$ term is dominated by the expression $1 - |f_0(e^{it})|^2$. It follows that (14) holds and, hence that $F_0 \in \mathrm{ex}\,\mathscr{P}_2$.

While F_0 is not an extreme element of $\mathscr{F}(f_0, n)$, it suggests the following

QUESTION. Does $\mathscr{F}(f_0; n)$ have an extreme element F_1 such that $M_2(F_1)$ is absolutely continuous with respect to Lebesgue measure on T^2?

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