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LOCALLY INJECTIVE TORSION MODULES

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ABSTRACT. Let R be a commutative ring and $\mathscr{F}$ a Gabriel topology of R. We discuss the R's satisfying the condition that for all $\mathscr{F}$-torsion R-modules T, T is $\mathscr{F}$-injective if and only if T is locally $\mathscr{F}$-injective. With one interpretation of locally $\mathscr{F}$-injective, this characterizes the $\mathscr{F}$-local R's. With another interpretation of locally $\mathscr{F}$-injective, every $\mathscr{F}$-local R has this property, but not conversely.

All rings considered will be commutative rings and R will always denote a ring. Concerning torsion theories, we follow mainly the notation from the B. Stenström text [7]. Our point of view will be mostly in terms of Gabriel topologies. Use spec$R$ for the set of all prime ideals of R and mspec$R$ for the set of all maximal ideals of R. If I is an ideal of R, then define $\text{mspec}(I) = \{M \in \text{mspec}R\colon I \subset M\}$. If T is an R-module and $M \in \text{mspec}R$, then define $T(M) = \{x \in T\colon \text{mspec}\ (\text{Ann}_R(x)) \subset \{M\}\} = \{0\} \bigcup \{x \in T\colon \text{mspec}\ (Ann_R(x)) = \{M\}\}$. Clearly $T(M)$ is then an R-submodule of T. For $\mathscr{F}$ a Gabriel topology of R, then R is $\mathscr{F}$-local if (1.) $|\text{mspec}(I)| < \infty$ for all $I \in \mathscr{F}$, and (2.) $|\text{mspec}(P)| = 1$ for all $P \in \mathscr{F} \bigcap \text{spec}R$. Then for $\mathscr{F}$ a Gabriel topology of R, the following three conditions are equivalent: (1.) R is $\mathscr{F}$-local, (2.) $T = \bigoplus_{M \in \text{mspec}R} T(M)$ for all $\mathscr{F}$-torsion R-modules T, and (3.) $T \cong \bigoplus_{M \in \text{mspec}R} T_M$ for all $\mathscr{F}$-torsion R-modules T [**2**, Theorem 1.2]. See [**2**] for a general discussion and the history of the $\mathscr{F}$-local concept.

We introduce the local Gabriel topologies $\mathscr{F}\{M\}$ along with a few observations. If $\mathscr{F}$ is a Gabriel topology of R and $M \in \text{mspec}R$, then $\mathscr{F}\{M\} = \{I \in \mathscr{F}\colon \text{mspec}(I) \subset \{M\}\}$. For $P \in \text{spec}R$, let $\mathscr{F}(P) = \{I\colon I$ is an ideal of R and $I \not\subset P\}$. Then $\mathscr{F}(P)$ is a Gabriel topology of R. Since $\mathscr{F}\{M\} = \mathscr{F} \bigcap (\bigcap \{\mathscr{F}(P)\colon P \in \text{mspec}R - \{M\}\})$, and the intersection of Gabriel topologies is a Gabriel topology, one infers that $\mathscr{F}\{M\}$ is a Gabriel topology of R. Note that if T is an R-module and $M \in \text{mspec}R$,

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then $T(M)$ is just the $\bigcap \{\mathscr{F}(P)\colon P \in \operatorname{mspec}R - \{M\}\}$-torsion submodule of T.

Suppose $\mathscr{F}$ is a Gabriel topology of R. Use t for the torsion radical corresponding to $\mathscr{F}$. If also $M \in \operatorname{mspec}R$, then use t_M for the torsion radical corresponding to $\mathscr{F}\{M\}$. It follows from the definitions that if $\mathscr{F}$ is a Gabriel topology of R, $M \in \operatorname{mspec}R$, and T is an $\mathscr{F}$-torsion R-module, then $T(M) = t_M(T)$. Another point of view, following the papers **[4]** and **[6]**, is that R is $\mathscr{F}$-local if and only if $t = \bigoplus_{M \in \operatorname{mspec}R} t_M$.

We shall need the following preliminary result involving the comparison of $T(M)$ and T_M. For $M \in \operatorname{mspec}R$ and T an R-module, define $f_{T,M}\colon T(M) \to T_M$ by $f_{T,M}(x) = x/1$ for all $x \in T(M)$. It is straightforward to verify that $f_{T,M}$ is an R_M-monomorphism.

THEOREM 1. *Let $\mathscr{F}$ be a Gabriel topology of R. The following statements are equivalent.*

(1.) *R is $\mathscr{F}$-local.*

(2.) *$f_{T,M}\colon T(M) \to T_M$ is an R_M-isomorphism for all $\mathscr{F}$-torsion R-modules T and for all $M \in \operatorname{mspec}R$.*

PROOF. (1.) $\to$ (2.) **[2**, Corollary 1.4]. (2.) $\to$ (1.) We prove the contrapositive. Suppose R is not $\mathscr{F}$-local. There exists an $\mathscr{F}$-torsion R-module T such that $T \neq \bigoplus_{M \in \operatorname{mspec}R} T(M)$. Define $T' = \bigoplus_{M \in \operatorname{mspec}R} T(M)$, and so $T \supsetneqq T'$. Since T/T' is a nonzero R-module, there exists an $M \in \operatorname{mspec}R$ such that $(T/T')_M \supsetneqq \{0\}$; i.e., $T_M \supsetneqq T'_M$. It follows that $f_{T,M}$ is not surjective and so statement (2.) is false. This completes the proof of the theorem.

We discuss the first form of locally injective. The interested reader may wish to refer to [3] for some general background. Recall that if $\mathscr{F}$ is a Gabriel topology of R and T is an R-module, then T is $\mathscr{F}$-injective if the canonical homomorphism $\operatorname{Hom}_R(R, T) \to \operatorname{Hom}_R(I, T)$ is surjective for all $I \in \mathscr{F}$. In other words, if $I \in \mathscr{F}$ and $f\colon I \to T$ is an R-homomorphism, then f extends to an R-homomorphism $R \to T$. We shall first be concerned with describing the rings with the property that an $\mathscr{F}$-torsion R-module T is $\mathscr{F}$-injective if and only if $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in \operatorname{mspec}R$.

LEMMA 2. *Let I be an ideal of R. Suppose $|\operatorname{mspec}(I)| = \infty$ and there does not exist a $P \in \operatorname{spec}R$ with $I \subset P$ and $|\operatorname{mspec}(P)| > 1$. Then there exists $\{M_n\}_{n=1}^{\infty} \subset \operatorname{mspec}(I)$ such that $\bigcap_{i=k}^{\infty} M_i \subsetneqq \bigcap_{i=k+1}^{\infty} M_i$ for all $k = 1, 2, \ldots$.*

PROOF. Recall that an ideal of R is a j-ideal if it is the intersection of some maximal ideals of R, and j-spec$R$ is the set of all prime ideals of R that are j-ideals. The hypotheses imply that the set of minimal j-primes over I is $\operatorname{mspec}(I)$, which is an infinite set. Thus j-specR/I is not Noetherian. Let $I_1 \subsetneqq I_2 \subsetneqq \cdots$ be j-ideals of R containing I, and for each

n, choose a maximal ideal M_n containing I_n but not containing I_{n+1}. These M_n's have the required properties.

LEMMA 3. *Let $\mathscr{F}$ be a Gabriel topology of R and let T be an R-module. If T is an $\mathscr{F}$-injective R-module, then $T(M)$ is an $\mathscr{F}\{M\}$-injective R-module for all $M \in$ mspec$R$.*

PROOF. Suppose T is $\mathscr{F}$-injective and consider $M \in \text{mspec}R$. Let $I \in \mathscr{F}\{M\}$ and let $f\colon I \to T(M)$ be an R-homomorphism. Since $I \in \mathscr{F}\{M\} \subset \mathscr{F}$ and T is $\mathscr{F}$-injective, there exists an R-homomorphism $g\colon R \to T$ such that $g \mid I = f$.

$$\begin{array}{ccc} 0 \to I & \xrightarrow{\text{incl.}} & R \\ f \downarrow & & \downarrow g \\ T(M) & \xrightarrow{\text{incl.}} & T \end{array}$$

Note that $I((T(M) + Rg(1))/T(M)) = \{0\}$. Hence $T(M)$ and $(T(M) + Rg(1))/T(M)$ are both $\bigcap\{\mathscr{F}(P)\colon P \in \text{mspec}R\text{-}\{M\}\}$-torsion R-modules. Since torsions are closed under extensions, $T(M) + Rg(1)$ is also an $\bigcap\{\mathscr{F}(P)\colon P \in \text{mspec}R\text{-}\{M\}\}$-torsion R-module. Therefore $Rg(1) \subset T(M)$, and so one may view $g\colon R \to T(M)$ with $g \mid I = f$. This verifies that $T(M)$ is an $\mathscr{F}\{M\}$-injective R-module.

The proof given above is similar to the proof of the following fact. If $\mathscr{F}$ is a Gabriel topology of R and T is an $\mathscr{F}$-injective R-module, then the $\mathscr{F}$-torsion submodule of T is also an $\mathscr{F}$-injective R-module. Of course, this generalizes the well-known fact: If R is an integral domain and T is an injective R-module, then the torsion submodule of T is also an injective R-module.

THEOREM 4. *Let $\mathscr{F}$ be a Gabriel topology of R. The following statements are equivalent.*

(1.) R is $\mathscr{F}$-local.

(2.) For all $\mathscr{F}$-torsion R-modules T, T is an $\mathscr{F}$-injective R-module if and only if $T(M)$ is an $\mathscr{F}\{M\}$-injective R-module for all $M \in$ mspec$R$.

PROOF. (1.) → (2.) Suppose R is $\mathscr{F}$-local. Let T be an $\mathscr{F}$-torsion R-module. If T is $\mathscr{F}$-injective, then $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in$ mspec$R$ by Lemma 3. Conversely, suppose $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in \text{mspec}R$. We shall use the $\mathscr{F}$-injective envelope of T, denoted $E_{\mathscr{F}}(T)$. For example, $E_{\mathscr{F}}(T) = \{x \in E(T)\colon (T\colon x) \in \mathscr{F}\}$ by [7, Ch. 9, Proposition 2.2]. Consider an $M \in \text{mspec}R$. Since R is $\mathscr{F}$-local, we can write $T = T(M) \oplus C$ where $C = \bigoplus_{N \in \text{mspec}R\text{-}\{M\}} T(N)$. Let $D = E_{\mathscr{F}\{M\}}(C)$. Since $C(M) = \{0\}$ and D is an essential extension of C, we have $D(M) = \{0\}$. By [7, Ch. 6, Proposition 3.2], D is $\mathscr{F}$-torsion. Hence $D = \bigoplus_{N \in \text{mspec}R-\{M\}}$

$D(N)$. Since $D = E_{\mathscr{F}\{M\}}(C) = \{x \in E(C)\colon (C\colon x) \in \mathscr{F}\{M\}\}$, it follows that $D = C$. Therefore, C is $\mathscr{F}\{M\}$-injective and so T is $\mathscr{F}\{M\}$-injective. Similarly, $E_{\mathscr{F}}(T)$ is $\mathscr{F}$-torsion and $E_{\mathscr{F}}(T) = \oplus_{M \in \text{mspec}R} E_{\mathscr{F}}(T)(M)$. Since T is $\mathscr{F}\{M\}$-injective for all $M \in \text{mspec}R$, $E_{\mathscr{F}}(T)(M) = T(M)$. It follows that $E_{\mathscr{F}}(T) = T$ and so T is $\mathscr{F}$-injective.

(2.) → (1.) Suppose that statement (2.) is satisfied. We first claim that $|\text{mspec}(P)| = 1$ for all $P \in \mathscr{F} \cap \text{spec}R$. Suppose this is not so; i.e., suppose there exists $P \in \mathscr{F} \cap \text{spec}R$ with $|\text{mspec}(P)| > 1$. Let T be the $\mathscr{F}$-torsion R-module R/P. If $x \in T - \{0\}$, then $\text{Ann}_R(x) = P$ and so $|\text{mspec}(Ann_R(x))| > 1$. In particular $T(M) = \{0\}$ and so $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in \text{mspec}R$. Choose $M_1 \in \text{mspec}(P)$, choose $x \in M_1 - P$, and define $I = P + Rx$. Then $I \in \mathscr{F}$. Define $f\colon I \to T$ by $f(p + rx) = r + P$ for $p \in P$ and $r \in R$. One checks that f is a well-defined R-homomorphism, and there does not exist an R-homomorphism $R \to T$ that extends f. Hence T is not $\mathscr{F}$-injective. This contradicts statement (2.) and verifies the claim that $|\text{mspec}(P)| = 1$ for all $P \in \mathscr{F} \cap \text{spec}R$.

We next claim that $|\text{mspec}(I)| < \infty$ for all $I \in \mathscr{F}$. Suppose this is not so; i.e., suppose there exists $I \in \mathscr{F}$ with $|\text{mspec}(I)| = \infty$. The hypotheses of Lemma 2 are satisfied, so there exists $\{M_n\}_{n=1}^{\infty} \subset \text{mspec}(I)$ such that $\bigcap_{i=k}^{\infty} M_i \subsetneqq \bigcap_{i=k+1}^{\infty} M_i$ for all $k = 1, 2, \ldots$. Define $I_n = \bigcap_{i=n}^{\infty} M_i$ for $n = 1, 2, \ldots$. Again we use $E_{\mathscr{F}}$ for the $\mathscr{F}$-injective envelope functor. Define $T = \oplus_{n=1}^{\infty} E_{\mathscr{F}}(R/I_n)$. We shall prove that T is an $\mathscr{F}$-torsion R-module that is not $\mathscr{F}$-injective and $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in \text{mspec}R$. Identify R/I_n as a submodule of $E_{\mathscr{F}}(R/I_n)$ and identify $\oplus_{n=1}^{\infty} R/I_n$ as a submodule of T.

For a given n, one has $I_n \supset I \in \mathscr{F}$, so R/I_n is $\mathscr{F}$-torsion. Then $E_{\mathscr{F}}(R/I_n)$ and T are also $\mathscr{F}$-torsion R-modules.

We claim that T is not $\mathscr{F}$-injective. Define $\bar{I} = \bigcup_{n=1}^{\infty} I_n \in \mathscr{F}$, and define the R-homomorphism $f\colon \bar{I} \to T$ by $f(x) = \langle x + I_n \rangle \in \oplus_{n=1}^{\infty} R/I_n \subset \oplus_{n=1}^{\infty} E_{\mathscr{F}}(R/I_n) = T$ for $x \in \bar{I}$. Note that the infinite tuple $\langle x + I_n \rangle$ has only finitely many nonzero components. By the way the M_i's were chosen, $I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq \bar{I}$. Hence there does not exist an R-homomorphism $R \to T$ extending f. This verifies the claim that T is not $\mathscr{F}$-injective.

We need to show that $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in \text{mspec}R$. This will require several steps. Let $k \geqq 1$ and let $M \in \text{mspec}R - \{M_n\}_{n=k}^{\infty}$. We claim that R/I_k is an $\mathscr{F}\{M\}$-torsion-free R-module. Suppose this is not so and $r + I_k$ is a nonzero $\mathscr{F}\{M\}$-torsion element of R/I_k with $r \in R - I_k$. Then $\text{Ann}_R(r + I_k) = (I_k\colon r)$ and $\text{mspec}((I_k\colon r)) = \{M\}$. For $j \geqq k$ there exist $x_j \in (I_k\colon r) - M_j$. Hence $rx_j \in I_k = \bigcap_{n=k}^{\infty} M_n$. But $x_j \notin M_j$ implies $r \in M_j$. This is the case for all $j \geqq k$, so $r \in \bigcap_{j=k}^{\infty} M_j = I_k$, contradicting $r \notin I_k$. This verifies the claim that R/I_k is an $\mathscr{F}\{M\}$-torsion-free R-module for all $k \geqq 1$ and $M \in \text{mspec}R\text{-}\{M_n\}_{n=k}^{\infty}$. Since $E_{\mathscr{F}}(R/I_k)$ is an essential extension of R/I_k, it follows that $E_{\mathscr{F}}(R/I_k)$ is an

$\mathscr{F}\{M\}$-torsion-free R-module for all $k \geqq 1$ and $M \in \text{mspec}R\text{-}\{M_n\}_{n=k}^{\infty}$.

Let $M \in \text{mspec}R$. We claim that $T(M)$ is an $\mathscr{F}\{M\}$-injective R-module. First suppose that $M \notin \{M_n\}_{n=1}^{\infty}$. Then $T(M) = (\oplus_{n=1}^{\infty} E_{\mathscr{F}}(R/I_n))(M) = \oplus_{n=1}^{\infty} E_{\mathscr{F}}(R/I_n)(M) = \{0\}$, which is clearly an $\mathscr{F}\{M\}$-injective R-module. On the other hand, suppose $M = M_k$ for some k. Then $T(M) = T(M_k) \cong \oplus_{n=1}^{k} E_{\mathscr{F}}(R/I_n)(M_k)$. By Lemma 3, $E_{\mathscr{F}}(R/I_n)(M_k)$ is an $\mathscr{F}\{M_k\}$-injective R-module for $1 \leqq n \leqq k$. Hence $T(M)$ is an $\mathscr{F}\{M\}$-injective R-module, as claimed.

We have shown that if there exists $I \in \mathscr{F}$ with $|\text{mspec}(I)| = \infty$, then statement (2.) is false. By definition, then R satisfies statement (1.). This completes the proof of the theorem.

We discuss the second form of locally injective. Following [1], if $\mathscr{F}$ is a Gabriel topology of R and $M \in \text{mspec}R$, then $\mathscr{F}_M = \{I_M\colon I \in \mathscr{F}\}$ is a Gabriel topology of R_M. We shall now be concerned with the rings R having the property that for all $\mathscr{F}$-torsion R-modules T, T is an $\mathscr{F}$-injective R-module if and only if T_M is an $\mathscr{F}_M$-injective R_M-module for all $M \in \text{mspec}R$. It is perhaps more natural to consider this second form of locally injective than the first form. Unfortunately, we are unable to characterize this property, but present only some partial results.

We begin with a result of E. Matlis [5, Theorem 3.3]. Let R be an integral domain and let $\mathscr{N}$ be the set of all nonzero ideals of R. Thus $\mathscr{N}$ is the Gabriel topology of R that corresponds to the classical torsion theory of R. If R is an $\mathscr{N}$-local integral domain (called "h-local" by E. Matlis), then an R-module T is an injective R-module if and only if T_M is an injective R_M-module for all $M \in \text{mspec}R$. Another way of saying this is: if R is an $\mathscr{N}$-local integral domain, then an R-module T is an $\mathscr{N}$-injective R-module if and only if T_M is an $\mathscr{N}_M$-injective R_M-module for all $M \in \text{mspec}R$. We shall generalize this, although restricted to torsion modules.

COROLLARY 5. *If $\mathscr{F}$ is a Gabriel topology of R and R is $\mathscr{F}$-local, then for all $\mathscr{F}$-torsion R-modules T, T is an $\mathscr{F}$-injective R-module if and only if T_M is an $\mathscr{F}_M$-injective R_M-module for all $M \in$* mspec$R$.

PROOF. Suppose R is $\mathscr{F}$-local and T is an $\mathscr{F}$-torsion R-module. First suppose T is $\mathscr{F}$-injective. Let $M \in \text{mspec}R$, $J \in \mathscr{F}_M$, and $f\colon J \to T_M$ an R_M-homomorphism. There exists $I \in \mathscr{F}$ with $J = I_M$. Since R is $\mathscr{F}$-local, $R/I = (R/I)(M) \oplus (\oplus_{N \in \text{mspec}R-\{M\}}(R/I)(N))$. Write $(R/I)(M) \cong R/I_1$ and $\oplus_{N \in \text{mspec}R-\{M\}}(R/I)(N) \cong R/I_2$ for some ideals I_1 and I_2 of R. Then $R/I \cong R/I_1 \oplus R/I_2$ with $\text{mspec}(I_1) \subset \{M\}$ and $M \notin \text{mspec}(I_2)$. Thus $R_M/J = R_M/I_M \cong (R/I)_M \cong (R/I_1)_M \oplus (R/I_2)_M \cong R_M/(I_1)_M$. Therefore $J = (I_1)_M$ with $I_1 \in \mathscr{F}\{M\}$. There is a canonical R-homomorphism $I_1 \to J$. Let $\bar{f}$ be the composition $I_1 \to J \xrightarrow{f} T_M$. By Theorems 1 and 4, $T_M \cong$

$T(M)$, and so T_M is $\mathscr{F}\{M\}$-injective. There exists an R-homomorphism $\bar{g}: R \to T_M$ with $\bar{g}|I_1 = \bar{f}$. One checks that this induces an R_M-homomorphism $g: R_M \to T_M$ with $g|J = f$. This verifies that T_M is an $\mathscr{F}_M$-injective R_M-module for all $M \in \text{mspec}R$.

Conversely, suppose T_M is $\mathscr{F}_M$-injective for all $M \in \text{mspec}R$. Let $M \in \text{mspec}R$. We will show that $T(M)$ is $\mathscr{F}\{M\}$-injective. Let $I \in \mathscr{F}\{M\}$ and let $f: I \to T(M)$ be an R-homomorphism. This induces an R_M-homomorphism $I_M \to T(M)_M$. By [2, Lemma 1.1(3.)], $T(M)_M \cong T(M)$ as R_M-modules. Let $\bar{f}$ be the R_M-homomorphism which is the composition $I_M \to T(M)_M \cong T(M)$. By Theorem 1, $T(M) \cong T_M$ as R_M-modules, and since T_M is $\mathscr{F}_M$-injective, there exists an R_M-homomorphism $\bar{g}: R_M \to T(M)$ with $\bar{g} \mid I_M = \bar{f}$. If g is the composition $R \to R_M \xrightarrow{\bar{g}} T(M)$, then g is an R-homomorphism with $g \mid I = f$. This verifies that $T(M)$ is $\mathscr{F}\{M\}$-injective for all $M \in \text{mspec}R$. By Theorem 4, T is $\mathscr{F}$-injective. This completes the proof of the corollary.

One could give an alternate proof of Corollary 5 similar to the argument of E. Matlis [5, proof of Theorem 3.3]. Unfortunately the converse of Corollary 5 is not true. Choose R to be the ring of integers and choose $\mathscr{F}$ to be the set of all ideals of R. Then R is not $\mathscr{F}$-local and an $\mathscr{F}$-torsion R-module T is $\mathscr{F}$-injective if and only if T_M is $\mathscr{F}_M$-injective for all $M \in \text{mspec}R$. (This is well known, or see [5, Theorem 3.3].)

A Gabriel topology $\mathscr{F}$ of R is said to be nonminimal if $\mathscr{F}$ does not have a minimal prime ideal of R as an element. In [2] there are several localization results that are proved only for nonminimal Gabriel topologies. The counterexample to the converse of Corollary 5 does not use a nonminimal Gabriel topology. Is it possible that the converse of Corollary 5 is true if one assumes that $\mathscr{F}$ is a nonminimal Gabriel topology?

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