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TRIPLE PRODUCTS IN THE GATEGORY OF SPECTRA OVER A SPACE

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ABSTRACT. We construct the category of spectra over a space and define triple products in this category. We prove that the semitensor product $H^*(B;Z_p) \circledast \mathscr{A}(p)$ may be interpreted as a certain set of morphisms in this category; as a consequence we define and discuss triple products in $H^*(B;Z_p) \circledast \mathscr{A}(p)$.

1. Introduction. In this paper we shall study triple products in the category of spectra over a space B. This category is a joint generalization of the stable category of J.F. Adams [**2**] and the category of spaces over a space introduced intependently by J. James [**5**], J. McClendon [**8**], and others. The definition of triple products in this category is a straightforward generalization of our earlier definition of triple products in the ordinary stable category [**4**], which in turn was a generalization of some work of E. Spanier [**10**].

We shall also prove a theorem that interprets the semitensor product $H^*(B; Z_p) \odot \mathscr{A}(p)$ [**6**, 7] as a set of morphisms in the category of spectra over B. This enables us to define triple products in $H^*(B; Z_p) \odot \mathscr{A}(p)$, and to show that if X is a stable two-stage Postnikov system over B, then the action of $H^*(B; Z_p) \odot \mathscr{A}(p)$ on $H^*_B(X; Z_p)$ is related to the triple-product structure of $H^*(B; Z_p) \odot \mathscr{A}(p)$. We shall also give a relationship between triple products in $H^*(B; Z_p) \odot \mathscr{A}(p)$ and those in $\mathscr{A}(p)$, as defined in [**4**].

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2. The category of spectra over B. In this section we shall construct the category of spectra over a space B, and we shall define triple products in this category.

Let B be a space. Then a space over B is a space X together with a map $p:X \to B$. A pointed space over B is a space X together with two maps, $p:X \to B$ and $s:B \to X$, such that $ps = 1_B$. Let (X, p, s) and (Y, q, r) be

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pointed spaces over B. Then a B-map f: $(X, p, s) \to (Y, q, r)$ is a map f: $X \to Y$ such that the following diagrams commute.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "s"'] & & Y \arrow[dl, "r"] \\ & B & \end{tikzcd}$$

Let (X, p, s) be a B-space, and let $Y \subset X$ be such that $s(B) \subset Y$. Then $(Y, p \mid Y, s)$ is a subspace of the B-space X. Note that the inclusion map i: $Y \to X$ is a B-map.

In this way the collection of pointed spaces over B forms a category. This category has been considered by several people, including I. James [**5**] and J. McClendon [**8**]. (See also [**9**].)

If X is a pointed space over B, let $\mathrm{Cone}_B X$ be the (reduced) cone of X over B and let $\sum_B X$ be the (reduced) suspension of X over B; these are both pointed spaces over B([**9**, p.13]). Furthermore, if f: $X \to Y$ is a B-map, then there are natural B-maps $\mathrm{Cone}_B(f)$: $\mathrm{Cone}_B X \to \mathrm{Cone}_B Y$ and $\sum_B f$: $\sum_B X \to \sum_B Y$.

If f: $X \to Y$ is a map of pointed spaces over B, let C_f be the (reduced) mapping cone of f over B; it is a pointed space over B, and there are B-maps i: $Y \to C_f$ and j: $C_f \to \sum_B X$ defined in the usual way [**9**, p.12].

A spectrum over B, or B-spectrum, is a sequence of pointed CW-complexes over B, $\{X_n\}$, together with cellular B-maps

$$\xi_n\colon \textstyle\sum_B X_n \to X_{n+1},$$

such that each ξ_n maps $\sum_B X_n$ homomorphically onto a subcomplex of X_{n+1}. These objects can be made into a category by mimicking the procedure by which J.F. Adams constructed the ordinary stable category in [**2**]. Furthermore, most of the constructions made in [**2**] for the ordinary category carry over to this setting as well. In particular, let $[X, Y]_r^B$ denote the set of homotopy classes of B-maps of degree r from X to Y. Then, as in the ordinary case, $[X, Y]_r^B$ has the structure of an abelian group, and $[X, X]_B^*$ has the structure of a graded ring, where the product of two B-maps is their composition.

Let f: $X \to Y$ be a B-map of degree r, and let Z be a third B-spectrum. Then we get, in the usual way, homomorphisms

$$f_*\colon [Z, X]_n \to [Z, Y]_{n+r}$$

and

$$f^*\colon [Y, Z]_n \to [X, Z]_{n+r}.$$

If $f_1 \simeq f_2$, then $f_{1*} = f_{2*}$ and $f_1^* = f_2^*$. If $f\colon X \to Y$ is homotopic over B to the constant map, then we shall write $f \simeq *$.

There are some other constructions we shall need. Let X be a B-spectrum. Then we define the cone spectrum of X over B, $\mathrm{Cone}_B X$, by $(\mathrm{Cone}_B X)_n = \mathrm{Cone}_B(X_n)$, with maps

$$\textstyle\sum_B \mathrm{Cone}_B(X_n) \approx \mathrm{Cone}_B \sum_B X_n \xrightarrow{\mathrm{Cone}_B(\xi_n)} \mathrm{Cone}_B(X_{n+1}).$$

There is a natural B-map of degree 0, $i\colon X \to \mathrm{Cone}_B X$. If $f\colon \mathrm{Cone}_B X \to Y$ is a B-map, then we shall let $f|X$ denote the composition fi. (See [**2** p.151.])

Also, given any B-map $f\colon X \to Y$ there is a B-map $\mathrm{Cone}_B f\colon \mathrm{Cone}_B X \to \mathrm{Cone}_B Y$ defined in the obvious way. Note that $f \simeq *$ if and only if there is a B-map $F\colon \mathrm{Cone}_B X \to Y$ such that $F|X = f$.

Again, let X be a B-spectrum. Then we can define a spectrum $\mathrm{Susp}_B X$ by $(\mathrm{Susp}_B X)_n = \sum_B(X_n)$, with the obvious structure maps. There is a B-map $\sigma\colon \mathrm{Susp}_B X \to X$ of degree -1, defined by

$$\sigma_n\colon (\mathrm{Susp}_B X)_n = \textstyle\sum_B X_n \xrightarrow{(-1)^n \xi_n} X_{n+1}.$$

(See [**4**].) Note that the image of σ is a cofinal subspectrum of X. Note also that if X' is a cofinal subspectrum of X, then $\mathrm{Cone}_B X'$ and $\mathrm{Susp}_B X'$ are cofinal subspectra of $\mathrm{Cone}_B X$ and $\mathrm{Susp}_B X$, respectively.

Finally, let X and Y be B-spectra, and let $f\colon X \to Y$ be a B-map of degree r. Then we can construct the mapping cone spectrum of f, E_f, as follows. The map f is represented by a function $f'\colon X' \to Y$ from some cofinal subspectrum X' of X. Define $(E_f)_n$ to be $E_{f'n}$, the mapping cone over B of the map $f'_n\colon X'_n \to Y_{n-r}$. The structure maps $\sum(E_f)_n \to (E_f)_{n+1}$ are the obvious ones. Following Adams, we can write $E_f = Y \cup_f \mathrm{Cone}_B X$. Then, as in the ordinary case, there are B-maps $i\colon Y \to E_f$ and $j\colon E_f \to X$, where $\deg i = 0$ and $\deg j = -1$, such that, for any B-spectrum Z, the following sequences are exact.

$$\cdots \to [Z, X]_n \xrightarrow{f_*} [Z, Y]_{n-r} \xrightarrow{i_*} [Z, E_f]_{n-r} \xrightarrow{j_*} [Z, X]_{n-r-1} \to \cdots$$

$$\cdots \leftarrow [X, Z]_n \xleftarrow{f^*} [Y, Z]_{n-r} \xleftarrow{i^*} [E_f, Z]_{n-r} \xleftarrow{j^*} [X, Z]_{n-r-1} \leftarrow \cdots$$

The triangle

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \arrow[dl, dashed, "i"] \\ & E_f \arrow[ul, "j"] & \end{tikzcd}$$

is called an exact triangle over B. More generally, a triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, dashed, "g"] \\ & Z & \end{tikzcd}$$

of B-spectra and B-maps is called exact (over B) if there is a homotopy equivalence $k\colon Z \to E_f$ such that the following diagram is homotopy commutative.

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[dr, "h"] \arrow[dd, "k"] & \\ Y & & X \\ & E_f \arrow[ul, "i"] \arrow[ur, "j"'] & \end{tikzcd}$$

We shall now show how to define triple products in the category of B-spectra. First, let X and Y be two spectra over B, and let f, g: $\mathrm{Cone}_B X \to Y$ be two B-maps of degree r such that $f|X = g|X$. We define a B-map $\delta(f, g)\colon X \to Y$ of degree $r + 1$ in the following way. Let X' be a cofinal subspectrum of X, and let f', g': $\mathrm{Cone}_B X' \to Y$ be two functions representing f and g, respectively. Then, for each n, f'_n: $\mathrm{Cone}_B X'_n \to Y_{n-r}$ and g'_n: $\mathrm{Cone}_B X'_n \to Y_{n-r}$ are such that $f'_n(x, 1) = g'_n(x, 1)$ for all $x \in X'_n$. We can then define a function

$$d(f'_n, g'_n)\colon \textstyle\sum_B X'_n \to Y_{n-r}$$

by the rule

$$d(f'_n, g'_n)\,(x, t) = \begin{cases} f'_n(x, 2t), & \text{if } \ 0 \leqq t \leqq 1/2 \\ g'_n(x, 2 - 2t), & \text{if } \ 1/2 \leqq t \leqq 1. \end{cases}$$

It is easy to check that $d(f'_n, g'_n)$ is a map of B-spaces, and that the following diagram commutes.

$$\begin{tikzcd} \sum_B(\sum_B X'_n) \arrow[r, "\sum_B d(f'_n\, g'_n)"] \arrow[d, "\sum_B \xi_n"'] & \sum_B Y_{n-r} \arrow[d, "\zeta_{n-r}"] \\ {}_B X'_{n+1} \arrow[r, "d(f'_{n+1},\, g'_{n+1})"'] & Y_{n-r+1} \end{tikzcd}$$

Thus we get a function $d(f', g')\colon \mathrm{Susp}\ X' \to Y$, of degree r. Since Susp

X' is cofinal in Susp X, we therefore get a B-map $\delta_1(f, g)$: Susp $E \to F$ of degree r. Finally, because of the B-map σ: $\text{Susp}_B\, X \to X$ of degree -1 whose image is cofinal in X, there is a B-map $\delta(f, g)$: $X \to Y$ of degree $r + 1$ such that the following diagram commutes.

$$\begin{tikzcd}
\text{Susp}_B X \arrow[r, "\delta_1"] \arrow[d, "\alpha"'] & Y \\
X \arrow[ur, dashed, "\delta"'] &
\end{tikzcd}$$

Let X, Y, Z, and W be B-spectra, and let f: $X \to Y$, g: $Y \to Z$, and h: $Z \to W$ be B-maps of degrees, n, m, and r respectively. Suppose $gf \simeq *$ and $hg \simeq *$. Since $gf \simeq *$, there is a B-map F: $\text{Cone}_B X \to Z$, of degree $n + m$ such that $F|X = gf$. Similarly, since $hg \simeq *$, there is a B-map G: $\text{Cone}_B Y \to W$ of degree $m + r$ such that $G|Y = hg$. Consider the B-maps hF: $\text{Cone}_B X \to W$ and $G(\text{Cone}_B f)$: $\text{Cone}_B X \to W$. These are B-maps of degree $n + m + r$. Furthermore, $hF|X = hgf$ and $G(\text{Cone}_B f)|hgf$. Thus there is a B-map

$$\delta(hF,\ G(\text{Cone}_B f)) \colon X \to Y$$

of degree $n + m + r + 1$. Define $\langle h, g, f\rangle$ to be the homotopy class of this map. Thus $\langle h, g, f\rangle \in [X, W]^B_{n+m+r+1}$.

The definition of $\langle h, g, f\rangle$ depends on the choice of the B-maps F and G. However, if F_1: $\text{Cone}_B X \to Z$ and G_1: $\text{Cone}_B Y \to W$ are other B-maps such that $F_1|X = gf$ and $G_1|Y = hg$, then there exist $\alpha \in [X, Z]^B_{n+m+1}$ and $\beta \in [Y, W]^B_{m+r+1}$ such that

$$(1) \qquad [\delta(hF_1, G_1(\text{Cone}_B f))] = h_*(\alpha) + [\delta(hF, G(\text{Cone}_B f))] + f^*(\beta).$$

Conversely, for any $\alpha \in [X, Z]^B_{n+m+1}$ and $\beta \in [Y, W]^B_{m+r+1}$, there are B-maps F_1: $\text{Cone}_B X \to Z$ and G_1: $\text{Cone}_B Y \to W$ such that $F_1|X = gf$ and $G_1|Y = gh$, and such that (1) holds. (The proof is identical to that given in **[4]** for the ordinary case.) Therefore $\langle h, g, f\rangle$ is a well-defined element of

$$[X,\ W]^B_{n+m+r+1}/\ (h_*[X,\ Z]^B_{n+m+1} + f^*[Y,\ W]^B_{m+r+1}).$$

This triple product has the following properties.

THEOREM 2.1. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ *be maps of B-spectra such that* $gf \simeq *$ *and* $hg \simeq *$. *Let* E_f *be the mapping cone over of* f, *and consider the following diagram.*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "g"] \arrow[dr, "i"'] & Z \arrow[r, "h"] & W \\ & & E_f \arrow[u, dotted, "b"] \arrow[r] & X \arrow[u, dotted, "a"'] \end{tikzcd}$$

(1) *There are B-maps* b: $E_f \to Z$ *and* $X \to W$ *such that this diagram is a homotopy commutative.*

(2) *The B-map* a *in* (1) *is not unique; however, the homotopy class* $[a] \in [X, W]^B_*$ *is a well-defined element of the group* $[X, W]^B_*/(h_*[X, Z]^B_* + f^*[Y, W]^B_*)$.

(3) *As an element of this group,* $[a] = \langle h, g, f\rangle$.

THEOREM 2.2. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ *be as in* Theorem 2.1. *Let* E_h *be the mapping cone over B of h, and consider the following diagram.*

$$\begin{tikzcd} W \arrow[r, "i"] & E_h \arrow[dr, "j"] & & \\ X \arrow[u, dotted, "a"] \arrow[r, "f"'] & Y \arrow[u, dotted, "b"] \arrow[r, "g"'] & Z \arrow[r, "h"'] & W \end{tikzcd}$$

(1) *There are B-maps* b: $Y \to E_h$ *and* a: $X \to W$ *such that this diagram is homotopy commutative.*

(2) *The homotopy class* $[a] \in [X, W]^B_*$ *is a well-defined element of the group* $[X, W]^B_* / (h_*[X, Z]^B_* + f^*[Y, W]^B_*)$.

(3) *As an element of this group,* $[a] = -\langle h, g, f\rangle$.

COROLLARY 2.3. *Let*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

be an exact triangle of B-spectra. Then the triple product $\langle h, g, f\rangle$ *is defined and equals the homotopy class of the identity B-map* $X \to X$ *as an element of the group* $[X, X]^B_* / (h_*[X, Z]^B_* + f^*[Y, X]^B_*)$.

THEOREM 2.4: *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ *be as in Theorem* 2.1, *and let* $X \xrightarrow{f_1} Y \xrightarrow{g_1} Z \xrightarrow{h_1} W$ *be such that* $f_1 \simeq f$, $g_1 \simeq g$, *and* $h_1 \simeq h$. *Then* $g_1 f_1 \simeq *$, $h_1 g_1 \simeq *$, and $\langle h_1, g_1, f_1\rangle = \langle h, g, f\rangle$.

THEOREM 2.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be as in Theorem 2.1. Suppose one of f, g, or h is the constant B-map. Then $\langle h, g, f\rangle$ is represented by the constant B-map $X \to W$.*

COROLLARY 2.6. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ be as above. Suppose $f \simeq *$, $g \simeq *$, or $h \simeq *$. Then $\langle h, g, f\rangle$ is represented by the constant B-map $X \to W$.*

THEOREM 2.7. *Suppose we have the following homotopy commutative diagram of B-spectra and B-maps*

$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "a"'] & Y_1 \arrow[r, "g_1"] \arrow[d, "b"'] & Z_1 \arrow[r, "h_1"] \arrow[d, "c"'] & W_1 \arrow[d, "d"'] \\
X_2 \arrow[r, "f_2"'] & Y_2 \arrow[r, "g_2"'] & Z_2 \arrow[r, "h_2"'] & W_2
\end{tikzcd}$$

such that $g_1f_1 \simeq *$, $h_1g_1 \simeq *$, $g_2f_2 \simeq *$, and $h_2g_2 \simeq *$. Then $a^*\langle h_2, g_2, f_2\rangle = d_*\langle h_1, g_1 f_1\rangle \in [X_1, W_2]^B_* / (h_{2*}[X_1, Z_2]^B_* + f_1^*[Y_1, W_2]^B_*$.

The proofs of these results are identical to those given in **[4]** for the ordinary case.

We can also define matrix triple products. For example, let X, Y, $Z_1, \ldots, Z_n$ be B-spectra, and let W be a ring B-spectrum with multiplication μ. Let $f\colon X \to Y$, $g_i\colon Y \to Z_i$, and $h_i\colon Z_i \to W$ be B-maps such that deg g_i + deg h_i is the same for all i. Suppose $g_if \simeq *$ for all i and $h_1 g_1 + \cdots + h_ng_n \simeq *$. We can interpret this as saying that the two matrix products

$$\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}(f) \quad \text{and} \quad (h_1 \cdots h_n)\begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$

are homotopically trivial. Then the matrix triple product

$$(*) \qquad \left\langle (h_1 \cdots h_n), \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}, (f) \right\rangle$$

is defined as follows. The maps $g_i\colon Y \to Z_i$ determine a map $g\colon Y \to Z_1 \wedge \cdots \wedge Z_n$, and the fact that $g_if \simeq *$ for all i implies that $gf \simeq *$. Also, let $h\colon Z_1 \wedge \cdots \wedge Z_n \to W$ be the composition

$$Z_1 \wedge \cdots \wedge Z_n \xrightarrow{h_1 \cdots h_n} W \wedge \cdots \wedge W \to W,$$

where the second map is induced by μ. Then the fact that $h_1g_1 + \cdots +$

$h_n g_n \simeq *$ implies that $hg \simeq *$. We then define the matrix triple product (*) to be the triple product $\langle h, g, f \rangle$.

3. Triple Products in $\mathbf{H^*(B; Z_p) \odot \mathscr{A}(p)}$. In this section we shall show how the semi-tensor product $H^*(B; Z_p) \odot \mathscr{A}(p)$ can be interpreted as a set of morphisms in the category of spectra over B. We shall then define triple products in $H^*(B; Z_p) \odot \mathscr{A}(p)$ and derive some easy consequences of the results of the last section.

To begin, let Y be a pointed space. Then we can make $B \times Y$ into a B-space as follows. Define $\pi: B \times Y \to B$ and $\sigma: B \to B \times Y$ by $\pi(b, y) = b$ and $\sigma(b) = (b, *)$. Then, clearly, $(B \times Y, \pi, \sigma)$ is a pointed space over B. Now, it is easy to see that $\sum_B(B \times Y) = B \times \sum Y$ and that the maps $\pi_1: \sum_B(B \times Y) \to B$ and $\sigma_1: B \to \sum_B(B \times Y)$ are the same as the maps $\pi: B \times \sum Y \to B$ and $\sigma: B \to B \times \sum Y$. If we have an ordinary spectrum Y, we can form a B-spectrum $B \times Y$ by letting $(B \times Y)_n = B \times Y_n$, with maps $\sum_B(B \times Y_n) = B \times \sum Y_n \xrightarrow{1\times\xi_n} B \times Y_{n+1}$. These structure maps are B-maps because the following diagrams commute.

$$\begin{tikzcd}
\sum_B(B \times Y_n) \arrow[dr, dotted, "\pi_1"'] & = B \times \sum Y_n \arrow[r, "1\times\xi_n"] \arrow[d, "\pi"] & B \times Y_{n+1}B \arrow[dl, "\pi"] \\
& B &
\end{tikzcd}$$

$$\begin{tikzcd}
\sum_B(B \times Y_n) \arrow[dr, "\sigma_1"'] & = B \times \sum Y_n \arrow[r, "1\times\xi_n"] & B \times Y_{n+1} \arrow[dl, "\sigma"] \\
& B \arrow[u, "\sigma"] &
\end{tikzcd}$$

Therefore $B \times Y$ is a spectrum over B.

We now state the main theorem of this section.

THEOREM 3.1. *For any prime number p, we have*

$$H^*(B; Z_p) \odot \mathscr{A}(p) \approx [B \times HZ_p, B \times HZ_p]_*^B$$

as algebras over Z_p.

For the proof of Theorem 3.1 we need some more constructions. For the remainder of this section, we shall write $\mathscr{A}$ for $\mathscr{A}(p)$, $H^*(X)$ for $H^*(X; Z_p)$ and H for HZ_p.

Let X be a pointed space over B. Then X has an associated pointed space $\bar{X}$, defined by $\bar{X} = X/s(B)$ with the image of $s(B)$ as the base-point. It is easy to see that

$$\overline{\sum_B X} = \sum \bar{X}.$$

Also, if f: $X \to Y$ is a map of B-spaces, then there is an induced map $\bar{f}: \bar{X} \to \bar{Y}$ of pointed spaces.

Let X be a spectrum over B. Then X has an associated ordinary spectrum X, defined as $(\bar{X})_n = \bar{X}_n$ with structure maps

$$\sum \overline{X_n} = \overline{\sum_B X_n} \xrightarrow{\bar{\xi}_{n+1}} \overline{X_{n+1}}.$$

Let (X, p, s) be a B-spectrum, and let $(B \times Y, \pi, \sigma)$ be a trivial B-spectrum. Let f: $X \to B \times Y$ be a B-map of degree k. Then, for each integer n, the following diagrams are commutative.

$$\begin{tikzcd} X_n \arrow[rr, "f_n"] \arrow[dr, "p"'] & & B \times Y_{n-k} \arrow[dl, "\pi"] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[rr, "f_n"] & & B \times Y_{n-k} \\ & B \arrow[ul, "s"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Now $f_n(x) = (f_n'(x), f_n''(x))$, where f_n': $X_n \to B$ and f_n'': $X_n \to Y_{n-k}$. Thus

$$p(x) = \pi(f_n(x)) = \pi(f_n'(x), f_n''(x)) = f_n'(x).$$

Thus $f_n' = p$. So $f_n(x) = (p(x), f_n''(x))$. Also, $f_n(s(b)) = \sigma(b) = (b, *)$; but $f_n(s(b)) = (p(s(b)), f_n''(s(b))) = (b, f_n''(s(b)))$. Thus $f_n''(s(b)) = *$ for all $b \in B$. Therefore, f_n'' induces a map $\overline{f_n''}$: $X_n \to Y_{n-k}$.

Since f: $X \to B \times Y$ is a map of B-spectra, the following diagram is commutative.

$$\begin{tikzcd} \sum_B X_n \arrow[rr, "\xi_n"] \arrow[d, "\Sigma_B f_n"] & & X_{n+1} \arrow[d, "f_{n+1}"] \\ \sum_B (B \times Y_{n-k}) = B \times \sum Y_{n-k} \arrow[rr, "1 \times \xi_n"] & & B \times Y_{n-k+1} \end{tikzcd}$$

It follows that the following diagram is commutative.

$$\begin{tikzcd} \sum \overline{X_n} = \overline{\sum_B X_n} \arrow[r] \arrow[d] & \overline{X_{n+1}} \arrow[d] \\ \sum Y_n \arrow[r] & Y_{n+1} \end{tikzcd}$$

Therefore, the maps $\bar{f}_n''$ form a map of spectra $\bar{f}''$: $\bar{X} \to Y$.

Thus a map $f\colon X \to B \times Y$ of B-spectra determines a map $\bar{f}''\colon \bar{X} \to Y$ of spectra. Conversely, let $\bar{f}\colon \bar{X} \to Y$ be a map of spectra. Then define $f_n\colon X_n \to B \times Y_{n-k}$ by

$$f_n(x) = \begin{cases} (p(x), \bar{f}_u(x)) & \text{if } x \notin s(B) \\ (p(x), *) & \text{if } x \in s(B). \end{cases}$$

Then the following diagrams are commutative.

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[dr, "p"'] & B \times Y_{n-k} \arrow[d, "\pi"] \\ & B \end{tikzcd} \qquad \begin{tikzcd} X_n \arrow[r, "f_n"] & B \times Y_{n-k} \\ & B \arrow[ul, "s"] \arrow[u, "\sigma"'] \end{tikzcd}$$

Thus f_n is a B-map. Also, the following diagram is commutative, for each n.

$$\begin{tikzcd} \sum_B X_n \arrow[rr, "\xi_n"] \arrow[d, "\Sigma_B f"] & & X_{n+1} \arrow[d, "f_{n+1}"] \\ \sum_B(B \times Y_{n-k}) = B \times \sum Y_{n-1} \arrow[rr, "1\times\xi_{n-k}"] & & B \times Y_{n-k+1} \end{tikzcd}$$

Therefore the maps f_n form a map of B-spectra $f\colon X \to B \times Y$. It is easy to see that the correspondences $f \leftrightarrow \bar{f}''$ are inverses of each other. Thus there is an isomorphism

$$\phi\colon [X, B \times Y]_*^B \approx [\bar{X}, Y]_*$$

defined by $\phi(f) = \bar{f}''$.

Let $Y = H$. Then we have

$$\phi\colon [X, B \times H]_*^B \approx [\bar{X}, H]_* = H^*(\bar{X}).$$

Let E be an ordinary spectrum, and let $X = B \times E$. Then we have

$$\phi\colon [B \times E, B \times H]_*^B \approx [\overline{B \times E}, H]_* = H^*(\overline{B \times E}).$$

Let $\mathrm{Sp}(B)$ be the spectrum

$$(\mathrm{Sp}(B))_n = \begin{cases} \{pt\}, & \text{if } n < 0 \\ \sum^n B, & \text{if } n \geqq 0. \end{cases}$$

Recall that $H^*(B) \approx H^*(\mathrm{Sp}(B)) = [\mathrm{Sp}(B), H]_*$.

We now have the following lemma.

LEMMA. $\overline{B \times E}$ *is cofinal in* $\mathrm{Sp}(B) \wedge E$.

PROOF.

$$\begin{aligned}
(\overline{B \times E})_n &= \overline{B \times E_n} \\
&= (B \times E_n)/\sigma(B) \\
&= (B \times E_n)/(b, *) \sim * \\
&= (B^+ \times E_n)/(b, *) \sim * \sim (*, x) \\
&= (B^+ \times E_n)/B^+ \vee E_n \\
&= B^+ \wedge E_n,
\end{aligned}$$

where B^+ denotes the union of B with a disjoint base point. The lemma follows.

It follows from this lemma that the inclusion map $i\colon \overline{B \times E} \to \mathrm{Sp}(B) \wedge E$ induces an isomorphism

$$i^*\colon H^*(\mathrm{Sp}(B) \wedge E) \to H^*(\overline{B \times E}).$$

Recall that if X and Y are ordinary spectra, then we have cross products $\chi\colon H^*(X) \otimes H^*(Y) \to H^*(X \wedge Y)$ defined as follows. Let $u \in H^*(X)$ and $v \in H^*(Y)$; then $u\colon X \to H$ and $v\colon Y \to H$. Then $\chi(u \otimes v)$ is the composition

$$X \wedge Y \xrightarrow{u \wedge v} H \wedge H \xrightarrow{\mu} H.$$

Note that $\chi(u \otimes v) = \mu_*(u \wedge v)$, where $\mu_*\colon [X \wedge Y, H \wedge\ ] \to [X \wedge Y, H]_*$. By the Kunneth Theorem [**1**], χ is an isomorphism.

If X is a CW-complex, then cup products in $H^*(X)$ can be defined as follows. The diagonal map $\Delta\colon X \to X \times X$ induces a map $\mathrm{Sp}(\Delta)\colon \mathrm{Sp}(X) \to \mathrm{Sp}(X \times X) \subset \mathrm{Sp}/X) \wedge \mathrm{Sp}(X)$. This map induces a homomorphism

$$\mathrm{Sp}(\Delta)^*\colon H^*(\mathrm{Sp}(X) \wedge \mathrm{Sp}(X)) \to H^*(\mathrm{Sp}(X)) = H^*(X).$$

Now let $x, y \in H^*(X)$. Then the cup product $x \cup y$ is $\mathrm{Sp}(\Delta)^*\chi(x \otimes y)$.

Let E be any spectrum. Then we have an isomorphism of Z_p-vector spaces

$$\begin{aligned}
&H^*(B) \otimes H^*(E) \\
&= H^*(\mathrm{Sp}(B)) \otimes H^*(E) \overset{\chi}{\approx} H^*(\mathrm{Sp}(B) \wedge E) \overset{i^*}{\approx} H^*(\overline{B \times E}) \\
&= [\overline{B \times E}, H]_* \xrightarrow{\phi^{-1}} [B \times E, B \times H]^B_*.
\end{aligned}$$

Let $\Phi = \phi^{-1} i^* \chi$. Then we have proved

THEOREM 3.2. *Φ is an isomorphism of vector spaces*

$$\Phi\colon H^*(B) \otimes H^*(E) \to [B \times E, B \times H]_*^B.$$

Let $E = H$. Then $H^*(H) = \mathscr{A}$, and so we have

$$\Phi\colon H^*(B) \otimes \mathscr{A} \to [B \times H, B \times H]_*^B.$$

THEOREM 3.3. *Φ is a homomorphism of algebras*

$$\Phi\colon H^*(B) \odot \mathscr{A} \to [B \times H, B \times H]_*^B.$$

Once we prove Theorem 3.3, then, from it and from Theorem 3.2, it will follow that Φ is an isomorphism of algebras. This will prove Theorem 3.1.

PROOF (of Theorem 3.3). Let $f\colon X \to Y$ be a map of B-spectra. Then there is an induced map of spectra $\bar{f}\colon \bar{X} \to \bar{Y}$. Furthermore, if Y is an ordinary spectrum, then there is a "projection" $\pi_2\colon \overline{B \times Y} \to Y$ that is a map of spectra. If $f\colon X \to B \times Y$ is a B-map, then $\phi(f) = \pi_2 \bar{f}$, where ϕ: $[X, B \times Y]_*^B \to [\bar{X}, Y]_*^B$ is the above isomorphism.

Let $x \otimes \alpha$ and $y \otimes \beta \in H^*(B) \odot \mathscr{A}$. We must show that $\Phi((x \otimes \alpha)(y \otimes \beta)) = \Phi(x \otimes \alpha)\Phi(y \otimes \beta)$. Let $\Phi(x \otimes \alpha) = f$ and $\Phi(y \otimes \beta) = g$. We must show that $\Phi((x \otimes \alpha)(y \otimes \beta)) = fg$. From $\Phi(x \otimes \alpha) = f$ we have $\phi^{-1} i^* \chi(x \otimes \alpha) = f$, or $i^*\chi(x \otimes \alpha) = \phi(f)$, and similarly $i^*\chi(y \otimes \beta) = \phi(g)$. We must show that $i^*\chi((x \otimes \alpha)(y \times \beta)) = \phi(fg)$. Now, $i^*\chi(x \otimes \alpha)$ is the composition

$$B \times H \xrightarrow{i} \mathrm{Sp}(B) \wedge H \xrightarrow{x \wedge \alpha} H \wedge H \xrightarrow{\mu} H,$$

and $\phi(f)$ is the composition

$$\overline{B \times H} \xrightarrow{\bar{f}} \overline{B \times H} \xrightarrow{\pi_2} H.$$

Thus the following diagram commutes.

(1)
$$\begin{tikzcd}
 & \mathrm{Sp}(B) \wedge H \arrow[r, "x \wedge \alpha"] & H \wedge H \arrow[dr, "\mu"] & \\
\overline{B \times H} \arrow[ur, "i"] \arrow[drr, "\bar{f}"'] & & & H. \\
 & & \overline{B \times H} \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Similarly, the following diagram commutes.

(2)
$$\begin{tikzcd}
 & \mathrm{Sp}(B) \wedge H \arrow[r, "y \wedge \beta"] & H \wedge H \arrow[dr, "\mu"] & \\
\overline{B \times H} \arrow[ur, "i"] \arrow[drr, "\bar{g}"'] & & & H. \\
 & & \overline{B \times H} \arrow[ur, "\pi_2"'] &
\end{tikzcd}$$

Now let $\psi\colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ be the coproduct, and let $\psi(\alpha) = \sum_i \alpha_i' \otimes \alpha_i''$. Then $(x \otimes \alpha)(y \otimes \beta) = \sum_i (x \cup \alpha_i'(y)) \otimes a_i''\beta$. So

$$\begin{aligned}
\chi((x \otimes \alpha)(y \otimes \beta)) &= \chi(\textstyle\sum (x \cup \alpha_i'(y)) \otimes \alpha_i''\beta) \\
&= \textstyle\sum \chi((x \cup a_i'(y)) \otimes \alpha_i''\beta) \\
&= \textstyle\sum \mu_*((x \cup \alpha_i'(y)) \wedge \alpha_i''\beta) \\
&= \mu_*((x \cup (\textstyle\sum\alpha')(y)) \wedge (\textstyle\sum\alpha_i'')\beta) \\
&= \chi((x \cup (\textstyle\sum\alpha_i')(y)) \otimes (\textstyle\sum\alpha_i'')\beta).
\end{aligned}$$

Thus $i^*\chi((x \otimes \alpha)(y \otimes \beta)) = i^*\chi((x \cup (\sum\alpha_i')(y)) \otimes (\alpha_i'')\beta)$. This last is

(3) $$\overline{B \times H} \xrightarrow{i} \mathrm{Sp}(B) \wedge H \xrightarrow{(x\cup(\Sigma\alpha')(y))\wedge(\Sigma\alpha''_i)\beta} H \wedge H \xrightarrow{\mu} H.$$

Now $x \cup (\sum\alpha_i')(y)$ is given by

(4) $$\mathrm{Sp}(B) \xrightarrow{\mathrm{Sp}(\Delta)} \mathrm{Sp}(B) \wedge \mathrm{Sp}(B) \xrightarrow{x\wedge(\Sigma\alpha')(y)} H \wedge H \xrightarrow{\mu} H.$$

Furthermore, $(\sum\alpha_i')(y)$ is given by

(5) $$\mathrm{Sp}(B) \xrightarrow{y} y\, H \xrightarrow{\Sigma\alpha'} H.$$

Substituting (5) into (4), we have

(6) $$\mathrm{Sp}(B) \xrightarrow{\mathrm{Sp}(\Delta)} \mathrm{Sp}(B) \wedge \mathrm{Sp}(B) \xrightarrow{x\wedge y} H \wedge H \xrightarrow{1\wedge\Sigma\alpha_i'} H \wedge H \xrightarrow{\mu} H.$$

Now $(\sum\alpha_i'')\beta$ is given by

(7) $$H \xrightarrow{\beta} H \xrightarrow{\Sigma\alpha''} H.$$

Substituting (6) and (7) into (3), we have

(8) $$\begin{aligned}
&\overline{B \times H} \xrightarrow{i} \mathrm{Sp}(B) \wedge H \xrightarrow{\mathrm{Sp}(\Delta)\wedge 1} \mathrm{Sp}(B) \wedge \mathrm{Sp}(B) \wedge H \xrightarrow{x\wedge y\wedge\beta} H \wedge H \wedge H \\
&\xrightarrow{1\wedge\Sigma\alpha'\wedge\Sigma\alpha''} H \wedge H \wedge H \xrightarrow{\mu\wedge 1} H \wedge H \xrightarrow{\mu} H.
\end{aligned}$$

On the other hand, $\phi(fg)$ is given by

(9) $$\overline{B \times H} \xrightarrow{\overline{fg}} \overline{B \times H} \xrightarrow{\pi_2} H.$$

Now $\overline{fg}$ is just the composition

(10) $$\overline{B \times H} \xrightarrow{\bar{g}} \overline{B \times H} \xrightarrow{\bar{f}} \overline{B \times H}.$$

Substituting (10) into (9), we have

(11) $$\overline{B \times H} \xrightarrow{\bar{g}} \overline{B \times H} \xrightarrow{\bar{f}} \overline{B \times H} \xrightarrow{\pi_2} H.$$

We must show that (8) and (11) are equal.

$$
\begin{tikzcd}[column sep=small]
\mathrm{Sp}(B)\wedge H \arrow[rr, "\mathrm{Sp}(\Delta)\wedge 1"] & & \mathrm{Sp}(B)\wedge\mathrm{Sp}(B)\wedge H \arrow[rr, "2\wedge y\wedge\beta"] \arrow[dl, "1\wedge\bar{g}"'] \arrow[dr, "1\wedge y\wedge\beta"] \arrow[d, phantom, "(\mathrm{F})"] \arrow[drr, phantom, "(\mathrm{C})"] & & H\wedge H\wedge H \arrow[r, "1\wedge\sum\alpha_i'\wedge\sum\alpha_i''"] \arrow[d, "1\wedge\mu"] \arrow[dr, phantom, "(\mathrm{B})"] & H\wedge H\wedge H \arrow[r, "\mu\wedge 1"] \arrow[d, "1\wedge\mu"'] \arrow[dr, phantom, "(\mathrm{A})"] & H\wedge H \arrow[d, "\mu"] \\
B\times H \arrow[u, "i"] \arrow[d, "\bar{g}"'] \arrow[dr, phantom, "(\mathrm{G})"] & \mathrm{Sp}(B)\wedge\mathrm{Sp}(B)\wedge H \arrow[d, "1\wedge\pi_2"] & & \mathrm{Sp}(B)\wedge H\wedge H \arrow[r, "x\wedge\mu"] \arrow[dll, "1\wedge\mu"'] \arrow[dr, phantom, "(\mathrm{D})"] & H\wedge H \arrow[r, "1\wedge\alpha"] \arrow[d, phantom, "(\mathrm{E})"] & H\wedge H \arrow[r, "\mu"] & H \\
B\times H \arrow[r, "i"] \arrow[rrrrr, bend right, "\bar{f}"'] & \mathrm{Sp}(B)\wedge H \arrow[urrrr, "x\wedge\alpha"] & & & & B\times H \arrow[ur, "\pi_2"'] &
\end{tikzcd}
$$

Figure 3.1

Consider the diagram in Figure 3.1. The proof will be complete if we show that the various parts of this diagram commute.

(A). This commutes by the associativity of μ.

(B). The map μ: $H \wedge H \to H$ induces

$$\begin{array}{ccc} H^*(H) \xrightarrow{\mu^*} H^*(H \wedge H) & \underset{\approx}{\xrightarrow{\chi^{-1}}} & H^*(H) \otimes H^*(H) \\ \| & & \| \\ \mathscr{A} & \xrightarrow{\psi} & \mathscr{A} \otimes \mathscr{A} \end{array}$$

Since $\psi(\alpha) = \sum \alpha_i' \otimes \alpha_i''$, we have $\chi^{-1}\mu^*(\alpha) = \sum \alpha_i' \otimes \alpha_i''$. Thus we have

$$\begin{aligned} \alpha \circ \mu &= \mu^*(\alpha) \\ &= \chi(\textstyle\sum \alpha_i' \otimes \alpha_1'') \\ &= \textstyle\sum \chi(\alpha_i' \otimes \alpha_i'') \\ &= \textstyle\sum \mu^*(\alpha_i' \wedge \alpha_i'') \\ &= \mu \circ (\textstyle\sum \alpha_i' \wedge \sum \alpha_i''). \end{aligned}$$

Thus the following diagram commutes.

$$\begin{array}{ccc} H \wedge H & \xrightarrow{\mu} & H \\ \downarrow{\scriptstyle \Sigma\alpha' \wedge \Sigma\alpha''\alpha} & & \downarrow{\scriptstyle \mu} \\ H \wedge H & \xrightarrow[d]{} & H \end{array}$$

Thus (B) commutes.

(C) and (D). These commute by basic properties of the smash product.

(E). This commutes because diagram (1) commutes.

(F). This commutes because diagram (2) commutes.

(G). To show that (G) commutes, it suffices to show that the following diagram commutes, for each n.

$$\begin{array}{ccc} B^+ \wedge H_n & \xrightarrow{\Delta \wedge 1} & B^+ \wedge B^+ \wedge H_n \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle 1 \wedge g_n} \\ B^+ \wedge H_{n-k} & \xrightarrow[1 \wedge \pi_2]{} & B^+ \wedge B^+ \wedge H_{n-k} \end{array}$$

The commutativity of this diagram follows from the fact that $g_n(b, x) = (b, \pi_2 g_n(b, x))$. This completes the proof of the theorem.

Let X be a spectrum over B. Earlier we showed that $[X, B \times H]_*^B \approx H^*(\bar{X})$. This suggests defining the B-cohomology of X, $H_B^*(X) = [X, B \times H]_*^B$. Then $H_B^*(X)$ can be given the structure of a module over $H^*(B) \odot \mathscr{A}$ in the same way that $H^*(X)$ is a module over $\mathscr{A}$ for ordinary spectra X; that is, let $u \in H_B^*(X)$ and let $a \in H^*(B) \odot \mathscr{A}$. Then u is represented by a B-map $X \to B \times H$, and by Theorem 3.1, a is represented by a B-map $B \times H \to B \times H$. Then $a \cdot u$ is represented by the composition $X \xrightarrow{u} B \times H \xrightarrow{a} B \times H$. It is easy to see that this makes $H_B^*(X)$ into a module over $H^*(B) \odot \mathscr{A}$.

Let a, b, and $c \in H^*(B) \odot \mathscr{A}$ be such that $ba = 0$ and $cb = 0$. By Theorem 3.1 we may regard a, b, and c as elements of $[B \times H, B \times H]_*^B$, and, therefore, by the material in §2, the triple product $\langle c, b, a\rangle$ is defined and is an element of $[B \times H, B \times H]_*^B = H^*(B) \odot \mathscr{A}$. More precisely, $\langle c, b, a\rangle = \Phi^{-1} \langle \Phi(c), \Phi(b), \Phi(a)\rangle$, where Φ: $H^*(B) \odot \mathscr{A} \to [B \times H, B \times H]_*^B$ is the isomorphism of Theorem 3.2. The indeterminacy of $\langle c, b, a\rangle$ is $[H^*(B) \odot \mathscr{A}]\, a + c[H^*(B) \odot \mathscr{A}]$. Thus $\langle c, b, a\rangle$ is a well-defined element of

$$H^*(B) \odot \mathscr{A} \,/\, ([H^*(B) \odot \mathscr{A}]a + c[H^*(B) \odot \mathscr{A}]).$$

Let $a \in H^*(B) \odot \mathscr{A}$. Regard a as a B-map a: $B \times H \to B \times H$, and let E_a be the mapping cone over B of a. We can think of E_a as a stable two-stage Postnikov system over B, with k-invariant a. Then we can get information about the structure of $H_B^*(E_a)$, as a module over $H^*(B) \odot \mathscr{A}$, from the triple product structure of $H^*(B) \odot \mathscr{A}$ as follows.

Consider the following exact triangle of maps over B

$$\begin{tikzcd} B \times H \arrow[rr, "a"] & & B \times X \arrow[dl, dashed, "i"] \\ & E_a \arrow[ul, "j"] & \end{tikzcd}$$

Passing to cohomology, we have

$$\begin{tikzcd} H^*(B) \odot \mathscr{A} \arrow[rr, "a^*"] \arrow[dr, "j^*"'] & & H^*(B) \odot \mathscr{A} \\ & H_B^*(E_a) \arrow[ur, "i^*"'] & \end{tikzcd}$$

Let $b \in H^*(B) \odot \mathscr{A}$ be such that $ba = 0$, and let $c \in H^*(B) \cdot \mathscr{A}$ be such that $cb = 0$. Then $b = i^*(x)$, for some $x \in H_B^*(E_a)$, and $c \cdot x = j^*(y)$, for some $y \in H^*(B) \odot \mathscr{A}$.

THEOREM 3.4. *The indeterminacy in the definition of* y *is* $[H^*(B) \odot \mathscr{A}]a + c[H^*(B) \odot \mathscr{A}]$, *and* $y = \langle c, b, a \rangle$ *modulo this indeterminacy.*

PROOF. This follows from Theorem 2.1.

Finally, it should be noted that Theorem 3.1 can be used to give a triple product definition of twisted secondary cohomology operations; this is entirely analogous to the triple product definition of ordinary secondary operations as discussed by Spanier in **[10]**. Properties of these twisted operations can then be easily deduced from the corresponding properties of triple products we listed in §2. We shall leave the details to the reader. One application is a relatively painless proof of the Generating Class Theorem of E. Thomas (**[11**, Theorem 5.9]); the reader can consult **[3]** for the details.

4. A relationship between triple products in $\mathscr{A}$ and in $H^*(B) \odot \mathscr{A}$. In this section we shall continue the practice of writing $\mathscr{A}$ for $\mathscr{A}(p)$, $H^*(X)$ for $H^*(X; Z_p)$, and H for HZ_p. Let $\alpha, \beta \in \mathscr{A}$, and suppose $\psi(\alpha) = 1 \otimes \alpha + \sum \alpha_i' \otimes \alpha_i'' + \alpha \otimes 1$. Then, for $1 \in H^0(B)$, $\alpha_i'(1) = 0$ unless $\alpha_i' = 1 \in \mathscr{A}$. Thus, in $H^*(B) \odot \mathscr{A}$,

$$\begin{aligned}(1 \otimes \alpha)(1 \otimes \beta) &= \textstyle\sum 1 \cup \alpha_i'(1) \otimes \alpha_i''\beta \\ &= 1 \cup 1(1) \otimes \alpha\beta \\ &= 1 \otimes \alpha\beta.\end{aligned}$$

Thus $(1 \otimes \alpha)(1 \otimes \beta) = 1 \otimes \alpha\beta$.

Let α, β, and $\gamma \in \mathscr{A}$ be such that $\beta\alpha = 0$ and $\gamma\beta = 0$. Then $\langle \gamma, \beta, \alpha \rangle$ is defined and is an element of $\mathscr{A}/(\mathscr{A}\cdot\alpha + \gamma\cdot\mathscr{A})$. (See **[4]**.) But also,

$$\begin{aligned}(1 \otimes \beta)(1 \otimes \alpha) &= 1 \otimes \beta\alpha \\ &= 1 \otimes 0 \\ &= 0\end{aligned}$$

and similarly $(1 \otimes \gamma)(1 \otimes \beta) = 0$. Thus the triple product $\langle 1 \otimes \gamma, 1 \otimes \beta, 1 \otimes \alpha \rangle$ is defined and is an element of

$$H^*(B) \odot \mathscr{A} \,/\, ([H^*(B) \odot \mathscr{A}]\,(1 \otimes \alpha) + (1 \otimes \gamma)\,[H^*(B) \odot \mathscr{A}]).$$

Define $1 \otimes \langle \gamma, \beta, \alpha \rangle = \{1 \otimes \delta \in H^*(B) \odot \mathscr{A}: \delta \in \langle \gamma, \beta, \alpha \rangle\}$. Then $1 \times \langle \gamma, \beta, \alpha \rangle \subset H^*(B) \odot \mathscr{A}$. If $\delta \in \langle \gamma, \beta, \alpha \rangle$, then also $\delta + a\alpha + \gamma b \in \langle \gamma, \beta, \alpha \rangle$, for any $a, b \in \mathscr{A}$ of appropriate degrees. Then

$$1 \otimes (\delta + a\alpha + \gamma b) \in 1 \otimes \langle \gamma, \beta, \alpha \rangle.$$

But

$$\begin{aligned}1 \otimes (\delta + a\alpha + \gamma b) &= (1 \otimes \delta) + (1 \otimes a\alpha) + (1 \otimes \gamma b) \\ &= 1 \otimes \delta + (1 \otimes a)(1 \otimes \alpha) + (1 \otimes \gamma)(1 \otimes b).\end{aligned}$$

Thus $1 \otimes \langle \gamma, \beta, \alpha \rangle$ is a well defined element of

$$H^*(B) \odot \mathscr{A} \,/\, ([H^*(B) \odot \mathscr{A}]\,(1 \otimes \alpha) + (1 \otimes \gamma)\,[H^*(B) \odot \mathscr{A}]).$$

THEOREM 4.1. $\langle 1 \otimes \gamma, 1 \otimes \beta, 1 \otimes \alpha \rangle = 1 \otimes \langle \gamma, \beta, \alpha \rangle$ *as elements of*

$$H^*(B) \odot \mathscr{A} \,/\, ([H^*(B) \odot \mathscr{A}]\,(1 \otimes \alpha) + (1 \otimes \gamma)\,[H^*(B) \odot \mathscr{A}]).$$

The proof is straightforward and is omitted.

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