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FUNCTIONS DEFINED BY CONTINUED FRACTIONS MEROMORPHIC CONTINUATION

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1. Introduction. A continued fraction

$$K\left(\frac{a_n}{b_n}\right) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots\cdots = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{b_4 + \ddots}}}}, \tag{1.1}$$

where a_n, $b_n \in \mathbf{C}$, $a_n \neq 0$ for all n, is an infinite process resembling a series in many ways. Corresponding to the partial sums of a series, we have the approximants of $K(a_n/b_n)$,

$$f_n = \mathop{K}_{m=1}^{n}\left(\frac{a_m}{b_m}\right) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}, \qquad \text{for } n \geqq 0. \tag{1.2}$$

(Here $K_{m=1}^{0}(a_m/b_m) = 0$.) Further, we say that $K(a_n/b_n)$ converges to a value f, or that $K(a_n/b_n) = f$, if $\lim_{n\to\infty} f_n$ exists and is equal to f. (We permit convergence to ∞.)

Still in analogy with series, the elements a_n and b_n may be functions of a complex variable z. $K(a_n(z)/b_n(z))$ then defines a function of z in the subset $E \subseteq \mathbf{C}$ where $K(a_n(z)/b_n(z))$ converges. (Another way of defining functions by continued fractions, $K(a_n(z)/b_n(z))$, is by correspondence [3, §5.1]. In this paper, though, we shall use $f(z) = \lim_{n\to\infty} f_n(z)$ pointwise, for all z such that this limit exists.)

Finally, we also have modified approximants f_n^* of $K(a_n/b_n)$. They arise if we replace the n^{th} tail

$$\mathop{K}_{m=n+1}^{\infty}\left(\frac{a_m}{b_m}\right) = \frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \cdots \tag{1.3}$$

of $K(a_n/b_n)$, not by 0 as in the ordinary approximants (1.2), but by a modifying factor w_n. That is, $f_0^* = w_0$ and

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(1.4) $$f_n^* = \frac{a_1}{b_1} + \cdots + \frac{a_{n-1}}{b_{n-1}} + \frac{a_n}{b_n + w_n}, \qquad \text{for } n = 1, 2, 3, \ldots.$$

As in [2], we introduce the linear fractional transformations

(1.5) $$S_n^{(m)}(w) = \frac{a_{m+1}}{b_{m+1}} + \cdots + \frac{a_{m+n-1}}{b_{m+n-1}} + \frac{a_{m+n}}{b_{m+n} + w}, \qquad \text{for } n = 1, 2, 3, \ldots$$

and $S_0^{(m)}(w) = w$, for $m = 0, 1, 2, \ldots$. We also use the shorter notation $S_n^{(0)} = S_n$ if $m = 0$. Then we have $f_n = S_n(0)$ and $f_n^* = S_n(w_n)$ for all n. If $K(a_n/b_n)$ converges, then its tails, (1.3), also converge for all n. Their values are denoted by $f^{(n)} = K_{m=n+1}^{\infty}(a_m/b_m)$. Hence, in that case we have $f^{(n)} = \lim_{m\to\infty} S_m^{(n)}(0) = S_m^{(n)}(f^{(m+n)})$. If a_n and b_n are functions of z, $S_n^{(m)}$ will, in general, also depend on z. We shall emphasize this by the notation

(1.6) $$S_n^{(m)}(w, z) = \frac{a_{m+1}(z)}{b_{m+1}(z)} + \frac{a_{m+2}(z)}{b_{m+2}(z)} + \cdots\cdots + \frac{a_{m+n}(z)}{b_{m+n}(z) + w(z)},$$

for $n = 1, 2, 3, \ldots$, $S_0^{(m)}(w, z) = w(z)$, for all $m \geqq 0$. Furthermore, $S_n^{(0)}(w, z)$ is also written $S_n(w, z)$, for all n.

In this paper we shall use modified approximants to obtain analytic continuation of a function defined by a continued fraction. The idea of this application originated with Waadeland [6, 7]. He observed that the 1-periodic T-fraction $K(z/(1 - z))$ converges to z for $|z| < 1$ and to -1 for $|z| > 1$, but that its modified approximants $\{S_n(z)\}$ converge to z in the whole complex plane (they are all identically z), and $\{S_n(-1)\}$ converges to -1 in $\mathbf{C}$. Hence, in this simple example, the right modifying factor provides us with an analytic continuation of $K(z/(1 - z))$ from $|z| < 1$ to $\mathbf{C}$ or from $|z| > 1$ to $\mathbf{C}$.

As Waadeland himself pointed out, this example is not so interesting in itself, but it can be extended to limit-periodic T-fractions $K(F_n z/(1 + G_n z))$, where $F_n \to F \neq 0$ and $G_n \to G$ fast enough. Then, using the "wrong tail value" of the continued fraction $K(Fz/(1 + Gz))$ as a modifying factor, gives us analytic continuations of $K(F_n z/(1 + G_n z))$.

In 1980–1981, Thron and Waadeland [4, 5] published sufficient conditions for a limit 1-periodic continued fraction $K(a_n(z)/b_n(z))$ to be continued analytically by this method. [4] is based on continued fractions corresponding to Laurent series or meromorphic functions; [5] is based on continued fractions converging to meromorphic functions in some domains.

In this paper we shall use an extension of their idea [8]. Let $K(a_n(z)/b_n(z))$ be a continued fraction which converges to a holomorphic function $f(z)$ in a domain $\mathscr{D}_0$. Let, furthermore, $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ be an auxiliary continued fraction with holomorphic tails $\tilde{f}^{(n)}(z)$ in $\mathscr{D}_0$, which can be continued analytically to a larger domain $\mathscr{D}$. Loosely described, it follows that if a_n and b_n can be continued analytically to $\mathscr{D}$, $S_n(\tilde{f}^{(n)}, z)$ is holomorphic on compact Subsets of $\mathscr{D}$ from a certain n on, and $\{S_n(\tilde{f}^{(n)}, z)\}$ converges uniformly on compact subsets of $\mathscr{D}$, then $F(z) = \lim S_n(\tilde{f}^{(n)}, z)$ is holomorphic in $\mathscr{D}$. Since $\lim S_n(\tilde{f}^{(n)}, z) = f(z) = K(a_n(z)/b_n(z))$ in $\mathscr{D}_0$ under mild conditions, $F(z)$ is the analytic continuation of $f(z)$ to $\mathscr{D}$.

This method does not depend on the modifying factors $\tilde{f}^{(n)}(z)$ being the tails of an auxiliary continued fraction. We could use any $w_n(z)$ as long as $S_n(w_n, z)$ is holomorphic from some n on, converges uniformly on compact subsets of $\mathscr{D}$, and converges to $f(z)$ in a subset of $\mathscr{D}_0$ with at least one point of accumulation. However, to find useful and sufficient conditions for this to happen is considerably more difficult in that more general situation.

We shall generalize the approach of Thron and Waadeland [5]. To emphasize the similarities and differences, we shall use the same form of presentation as in [5].

If we had based the definition of the function $K(a_n(z)/b_n(z))$ on correspondence instead of convergence, we could have used the approach of [4] to the same problem. This is, however, beyond the scope of this paper, and an extension of [4] will be presented separately.

2. Modified approximants. In this section we are, for convenience, looking at continued fractions $K(a_n/b_n)$ with constant elements. To construct modified approximants we use an auxiliary continued fraction $K(\tilde{a}_n/\tilde{b}_n)$, where

$$\delta_n = a_n - \tilde{a}_n \text{ and } \eta_n = b_n - \tilde{b}_n \tag{2.1}$$

converge to 0 "fast enough". (We assume that such an auxiliary continued fraction can be found. "Fast enough" will be defined later.) The modifying factors shall be a sequence $\{w_n\}$ of complex numbers such that $w_n + \tilde{b}_n \neq 0$, for all $n \geqq 1$, and

$$w_{n-1}(\tilde{b}_n + w_n) = \tilde{a}_n \qquad \text{for } n = 1, 2, 3, \ldots \tag{2.2}$$

(w_n is not necessarily the n^{th} tail $\tilde{f}^{(n)}$ of $K(\tilde{a}_n/\tilde{b}_n)$, although, if $K(\tilde{a}_n/\tilde{b}_n)$ converges, $w_n = \tilde{f}^{(n)}$ for all $n \geqq 0$ is an example of a sequence satisfying (2.2).) Since $\tilde{a}_n$ and $\tilde{b}_n \in \mathbf{C}$, and since we have added the condition $\tilde{b}_n + w_n \neq 0$ for all n, we see by (2.2) that $w_n \neq \infty$ for all n.

The definition of our continued fractions contains the condition $a_n \neq 0$ for all $n \geqq 1$. However, if we turn to the case where $\tilde{a}_n$, a_n are functions of z, it is often unnatural to avoid values of z where $a_n(z) = 0$ or $\tilde{a}_n(z) = 0$ for one or more values of n. Therefore we shall allow $a_n = 0$ and $\tilde{a}_n = 0$ in this and the following sections. If $\tilde{a}_N = 0$, then $w_{N-1} = 0$. This means that all sequences $\{w_n\}$, satisfying our conditions, will coincide for $n \leqq N - 1$.

We want to look at $S_n(w_n)$. For any continued fraction $K(a_n/b_n)$ we have

$$S_n(x) = \frac{A_n + A_{n-1}x}{B_n + B_{n-1}x} \qquad \text{for all } n \geqq 0, \tag{2.3}$$

where A_n, B_n are given by the following recursion relations:

$$A_{-1} = 1,\ A_0 = 0,\ B_{-1} = 0,\ B_0 = 1, \tag{2.4a}$$

$$A_n = b_n A_{n-1} + a_n A_{n-2},\ B_n = b_n B_{n-1} + a_n B_{n-2} \qquad \text{for } n = 1, 2, 3, \ldots. \tag{2.4b}$$

(This notation is in accordance with [3]. We shall also use $A_n^{(N)}$, $B_n^{(N)}$ to denote the corresponding expressions related to the N^{th} tail $K_{n=N+1}^{\infty}(a_n/b_n)$. Statements (2.3) and (2.4) are also valid if $a_n = 0$ for one or more values of n if $S_n(x)$ is still defined.) Therefore, under our conditions, we get,

$$\begin{aligned} A_n + A_{n-1}w_n &= (\tilde{b}_n + w_n)A_{n-1} + \tilde{a}_n A_{n-2} + \eta_n A_{n-1} + \delta_n A_{n-2} \\ &= (\tilde{b}_n + w_n)[A_{n-1} + w_{n-1}A_{n-2}] + \eta_n A_{n-1} + \delta_n A_{n-2} \\ &= [A_0 + w_0 A_{-1}] \cdot \prod_{j=1}^{n} (\tilde{b}_j + w_j) \\ &\quad + \sum_{m=1}^{n} \left[\eta_m A_{m-1} \prod_{j=m+1}^{n} (\tilde{b}_j + w_j) \right] \\ &\quad + \sum_{m=1}^{n} \left[\delta_m A_{m-2} \prod_{j=m+1}^{n} (\tilde{b}_j + w_j) \right] \\ &\qquad \text{for } n = 0, 1, 2, \ldots. \end{aligned} \tag{2.5}$$

By exactly the same method, we get similar expressions for $B_n + B_{n-1}w_n$, where all the A's on the right hand side of (2.5) are replaced by B's. Since $A_0 + w_0 A_{-1} = w_0$ and $B_0 + w_0 B_{-1} = 1$ by (2.4), $S_n(w_n)$ can be written

$$S_n(w_n) = \frac{w_0 + \sum_{m=1}^{n} \eta_m \dfrac{A_{m-1}}{\prod_{j=1}^{m} (\tilde{b}_j + w_j)} + \sum_{m=1}^{n} \delta_m \dfrac{A_{m-2}}{\prod_{j=1}^{m} (\tilde{b}_j + w_j)}}{1 + \sum_{m=1}^{n} \eta_m \dfrac{B_{m-1}}{\prod_{j=1}^{m} (\tilde{b}_j + w_j)} + \sum_{m=1}^{n} \delta_m \dfrac{B_{m-2}}{\prod_{j=1}^{m} (\tilde{b}_j + w_j)}} \tag{2.6}$$

for all $n \geqq 0$ if $\tilde{b}_j + w_j \neq 0$, for all $j \geqq 1$.

3. Convergence of $\{S_n(w_n)\}$. As in [5] we now find sufficient conditions for each of the four series in (2.6) to converge absolutely. Furthermore, we find upper bounds for the sums of these series, and thus for $S_n(w_n)$, if $|\eta_m|$ and $|\delta_m|$ are small enough.

We are still looking at continued fractions $K(a_n/b_n)$ with constant elements. This corresponds to looking at $K(a_n(z)/b_n(z))$ for fixed values of z. (The upper bounds for $S_n(w_n)$ will then, in the next section, be used to prove uniform convergence of $S_n(w_n, z)$ in appropriate domains.)

We assume that the modifying factors w_n related to the auxiliary continued fraction $K(\tilde{a}_n/\tilde{b}_n)$, by (2.2), are $\neq -\tilde{b}_n$. We permit $\tilde{a}_n = 0$ or $a_n = 0$ for one or more values of n, even though $K(a_n/b_n)$ and/or $K(\tilde{a}_n/\tilde{b}_n)$, then, are not continued fractions by the usual definition as long as $\{S_n\}$ and $\{w_n\}$ are still well defined.

LEMMA 3.1. *Let $K(a_n/b_n)$ and $K(\tilde{a}_n/\tilde{b}_n)$ be continued fractions (possibly with a_n or $\tilde{a}_n = 0$ for some $n \in \mathbf{N}$), and let $\{w_n\}$ be a sequence of complex numbers such that $\tilde{b}_n + w_n \neq 0$, for all $n \geqq 1$, and*

$$w_{n-1}(\tilde{b}_n + w_n) = \tilde{a}_n \qquad \text{for } n = 1, 2, 3, \ldots. \tag{3.1}$$

If there exist positive numbers $C \geqq 0$ and $M > 1$ such that

$$\prod_{j=m}^{n} \left| \frac{w_j}{\tilde{b}_j + w_j} \right| \leqq CM^{n-m+1} \qquad \text{for all } n \geqq m \geqq 1, \tag{3.2}$$

and

$$\frac{|a_n - \tilde{a}_n|}{|\tilde{b}_n + w_n|\,|\tilde{b}_{n-1} + w_{n-1}|} \leqq \frac{\alpha}{M^{n-1}} \qquad \text{for } n = 2, 3, 4, \ldots, \tag{3.3a}$$

$$\frac{|b_n - \tilde{b}_n|}{|\tilde{b}_n + w_n|} \leqq \frac{\beta}{M^n} \qquad \text{for } n = 1, 2, 3, \ldots, \tag{3.3b}$$

where α and β are positive constants such that

$$\alpha + \beta + (n-2)\alpha C + (n-1)\beta C \leqq M^n(M-1) \qquad \text{for } n = 2, 3, 4, \ldots, \tag{3.4}$$

then

$$\left| \frac{A_n}{\prod_{j=1}^{n} (\tilde{b}_j + w_j)} \right| \leqq \left| \frac{a_1}{\tilde{b}_1 + w_1} \right| (1 + (n-1)C)M^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots, \tag{3.5}$$

and

$$\left| \frac{B_n}{\prod_{j=1}^{n} (\tilde{b}_j + w_j)} \right| \leqq (1 + nC)M^{n-1} \max\left\{M, 1 + \frac{\beta}{M}\right\} \qquad \text{for } n = 0, 1, 2, \ldots. \tag{3.6}$$

PROOF. We shall first prove (3.6). Let δ_n and η_n be defined by (2.1), and let

$$D_n = \frac{B_n}{\prod_{j=1}^{n} (\tilde{b}_j + w_j)} \qquad \text{for } n = -1, 0, 1, \ldots. \tag{3.7}$$

Then $D_{-1} = 0$, $D_0 = 1$ and

$$(3.8)\qquad D_n = \frac{\tilde{b}_n D_{n-1}}{\tilde{b}_n + w_n} + \frac{w_{n-1} D_{n-2}}{\tilde{b}_{n-1} + w_{n-1}} + \frac{\eta_n D_{n-1}}{\tilde{b}_n + w_n} + \frac{\delta_n D_{n-2}}{(\tilde{b}_n + w_n)(\tilde{b}_{n-1} + w_{n-1})}$$

for $n = 1, 2, 3, \ldots$, by (2.4b) and (3.1). With

$$(3.9)\qquad P_n = D_n + \frac{w_n D_{n-1}}{\tilde{b}_n + w_n} \qquad \text{for } n = 0, 1, 2, \ldots,$$

we get $P_0 = 1$, and, by (3.8),

$$(3.10)\qquad P_n = P_{n-1} + \frac{\eta_n D_{n-1}}{\tilde{b}_n + w_n} + \frac{\delta_n D_{n-2}}{(\tilde{b}_n + w_n)(\tilde{b}_{n-1} + w_{n-1})} \qquad \text{for } n = 1, 2, 3, \ldots$$

Since, by (3.9),

$$(3.11)\qquad \begin{aligned} D_n &= P_n - \frac{w_n D_{n-1}}{\tilde{b}_n + w_n} = P_n - \frac{w_n}{\tilde{b}_n + w_n} P_{n-1} + \frac{w_n}{\tilde{b}_n + w_n} \cdot \frac{w_{n-1}}{\tilde{b}_{n-1} + w_{n-1}} D_{n-2} \\ &= \sum_{\nu=0}^{n} \left[(-1)^{n+\nu} P_\nu \prod_{j=\nu+1}^{n} \frac{w_j}{\tilde{b}_j + w_j} \right] \qquad \text{for } n = -1, 0, 1, 2, \ldots, \end{aligned}$$

we get

$$(3.12)\qquad \begin{aligned} P_n = P_{n-1} &+ \frac{\eta_n}{\tilde{b}_n + w_n} \sum_{\nu=0}^{n-1} \left[(-1)^{n-1+\nu} P_\nu \prod_{j=\nu+1}^{n-1} \frac{w_j}{\tilde{b}_j + w_j} \right] \\ &+ \frac{\delta_n}{(\tilde{b}_n + w_n)(\tilde{b}_{n-1} + w_{n-1})} \sum_{\nu=0}^{n-2} \left[(-1)^{n+\nu} P_\nu \prod_{j=\nu+1}^{n-2} \frac{w_j}{\tilde{b}_j + w_j} \right] \\ &\qquad \text{for } n = 1, 2, 3, \ldots. \end{aligned}$$

By induction we shall see that $|P_n| \leqq M^{n-1} \max\{M, 1, + \beta/M\}$ for $n = 1, 2, 3, \ldots.$

We note first that $|P_0| = 1 = M^0$ and $|P_1| = |1 + \eta_1/(\tilde{b}_1 + w_1)| \leqq 1 + \beta/M$. If $|P_m| \leqq M^{m-1} \max\{M, 1 + \beta/M\}$, for all $m < n$, where n is an integer $\geqq 2$, then, by (3.12),

$$(3.13)\qquad \begin{aligned} |P_n| \leqq |P_{n-1}| &+ \frac{\beta}{M^n} \left[|P_{n-1}| + \sum_{\nu=0}^{n-2} \left[M^{\nu-1} \max\left\{M, 1 + \frac{\beta}{M}\right\} C M^{n-1-\nu} \right] \right] \\ &+ \frac{\alpha}{M^{n-1}} \left[|P_{n-2}| + \sum_{\nu=0}^{n-3} \left[M^{\nu-1} \max\left\{M, 1 + \frac{\beta}{M}\right\} C M^{n-2-\nu} \right] \right] \\ \leqq M^{n-1} &\max\left\{M, 1 + \frac{\beta}{M}\right\} \cdot \left[\frac{1}{M} + \frac{\beta}{M^{n+1}} + (n-1) \frac{\beta C}{M^{n+1}} \right. \\ &\left. + \frac{\alpha}{M^{n+1}} + (n-2) \frac{\alpha C}{M^{n+1}} \right] \leqq M^{n-1} \max\left\{M, 1 + \frac{\beta}{M}\right\}, \end{aligned}$$

by use of (3.2), (3.3) and (3.4). Therefore, by (3.11),

$$(3.14)\qquad |D_n| \leqq |P_n| + \sum_{\nu=0}^{n-1} |P_\nu| CM^{n-\nu} \leqq (1+nC)M^{n-1}\max\left\{M,\ 1+\frac{\beta}{M}\right\},$$

for $n = 0, 1, 2, \ldots$, which gives (3.6).

To prove (3.5), we follow the same pattern, this time with

$$(3.15)\qquad D_n = \frac{A_n}{\prod_{j=1}^{n} (\tilde{b}_j + w_j)} \qquad \text{for } n = -1, 0, 1, \ldots.$$

Then we get, in particular, that $D_{-1} = 1$, $D_0 = 0$ and $D_1 = a_1/(\tilde{b}_1 + w_1)$. With P_n defined by (3.9), we now have $P_1 = D_1$ and

$$(3.16)\qquad \begin{aligned} P_n = P_{n-1} &+ \frac{\eta_n}{\tilde{b}_n + w_n} \sum_{\nu=1}^{n-1}\left[(-1)^{n-1+\nu} P_\nu \prod_{j=\nu+1}^{n-1} \frac{w_j}{\tilde{b}_j + w_j}\right] \\ &+ \frac{\delta_n}{(\tilde{b}_n + w_n)(\tilde{b}_{n-1} + w_{n-1})} \sum_{\nu=1}^{n-2}\left[(-1)^{n+\nu} P_\nu \prod_{j=\nu+1}^{n-2} \frac{w_j}{\tilde{b}_j + w_j}\right], \end{aligned}$$

for $n = 2, 3, 4, \ldots$. By induction we get that

$$(3.17)\qquad |P_n| \leqq \left|\frac{a_1}{\tilde{b}_1 + w_1}\right| M^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots.$$

To see this, first note that $|P_1| = |a_1|/|\tilde{b}_1 + w_1|$ and $|P_2| \leqq |P_1| + |P_1|\,\beta/M^2 \leqq M|P_1|$ since $\beta \leqq \alpha + \beta + (2-2)\alpha C + (2-1)\beta C \leqq M^2(M-1)$, by (3.4) (with $n = 2$). If (3.17) is valid for all $n \leqq m - 1$, where $m \geqq 3$, then, by (3.16) and (3.4),

$$(3.18)\qquad \begin{aligned} |P_m| \leqq P|_{m-1}| &+ \frac{\beta}{M^m}\left[|P_{m-1}| + \sum_{\nu=1}^{m-2}\left[\left|\frac{a_1}{\tilde{b}_1 + w_1}\right| M^{\nu-1} CM^{m-1-\nu}\right]\right] \\ &+ \frac{\alpha}{M^{m-1}}\left[|P_{m-2}| + \sum_{\nu=1}^{m-3}\left[\left|\frac{a_1}{\tilde{b}_1 + w_1}\right| M^{\nu-1} CM^{m-2-\nu}\right]\right] \\ &\leqq \left|\frac{a_1}{\tilde{b}_1 + w_1}\right| M^{m-1}\left(\frac{1}{M} + \frac{\beta}{M^{m+1}} + (m-2)\frac{\beta C}{M^{m+1}} + \frac{\alpha}{M^{m+1}}\right. \\ &\left.+ (m-3)\frac{\alpha C}{M^{m+1}}\right) \leqq \left|\frac{a_1}{\tilde{b}_1 + w_1}\right| M^{m-1}. \end{aligned}$$

Therefore,

$$(3.19)\qquad \begin{aligned} |D_n| &= \left|\sum_{\nu=1}^{n}\left[(-1)^{\nu+n} P_\nu \prod_{j=\nu+1}^{n} \frac{w_j}{\tilde{b}_j + w_j}\right]\right| \leqq |P_n| + \sum_{\nu=1}^{n-1} |P_\nu| CM^{n-\nu} \\ &\leqq \left|\frac{a_1}{\tilde{b}_1 + w_1}\right| (1 + (n-1)C) M^{n-1} \qquad \text{for } n = 1, 2, 3, \ldots, \end{aligned}$$

which gives (3.5).

COMMENT 1. We can always find $\alpha > 0$ and $\beta > 0$ satisfying (3.4). In fact, if $C > 0$, then

$$\begin{aligned}[\alpha + \beta + (n-2)\alpha C + (n-1)C\beta]M^{-n} \\ \leqq [\alpha + \beta + (n-1)\alpha C + n\beta C]M^{-(n+1)}\end{aligned} \tag{3.20}$$

if and only if

$$n \leqq 2 + \frac{1}{M-1} - \frac{1}{C} - \frac{\beta}{\alpha + \beta}. \tag{3.21}$$

Therefore, (3.4) is satisfied for all $n \geqq 2$ if

$$\alpha + \beta + \mu\alpha C + (\mu + 1)\beta C \leqq M^{2+\mu}(M-1), \tag{3.22}$$

$$\text{where } \mu = \max\left\{0, \frac{1}{M-1} - \frac{1}{C} - \frac{\beta}{\alpha+\beta}\right\}. \tag{3.23}$$

In particular, this is so if

$$\alpha + \beta \leqq \frac{M^{2+\mu}(M-1)}{1 + (\mu+1)C}. \tag{3.24}$$

COMMENT 2. If, in particular, we require that $C \geqq M - 1$, then we know, by (3.4), that

$$\alpha + \beta + \beta C \leqq M^2(M-1), \tag{3.25}$$

and therefore $\beta \leqq (M^2(M-1))/(C+1) \leqq M(M-1)$. Hence, max $\{M, 1 + \beta/M\} = M$, which simplifies (3.6).

COMMENT 3. If we restrict ourselves to continued fractions $K(a_n/1)$ and $K(\tilde{a}_n/1)$, we can let $\beta = 0$. Then (3.4) reduces to

$$\alpha \leqq \frac{M^n(M-1)}{1 + (n-2)C}, \qquad \text{for } n = 2, 3, 4, \ldots. \tag{3.26}$$

Further, max $\{M, 1 + \beta/M\} = \max\{M, 1\} = M$, which simplifies (3.6). (By the same reasoning as in Comment 1, (3.26) is surely satisfied for all integers $n \geqq 2$ if it is satisfied for $n = \max\{2, 2 + 1/(M-1) - 1/C\}$.)

Similarly, if we restrict ourselves to continued fractions $K(1/b_n)$ and $K(1/\tilde{b}_n)$, we can use $\alpha = 0$. Then (3.4) reduces to

$$\beta \leqq \frac{M^n(M-1)}{1 + (n-1)C} \qquad \text{for } n = 2, 3, 4, \ldots \tag{3.27}$$

(which is surely satisfied if it is valid for $n = \max\{2, M/(M-1) - 1/C\}$.).

The condition (3.2) in Lemma 3.1 imposes a restriction on the auxiliary continued fraction $K(\tilde{a}_n/\tilde{b}_n)$ and the sequence $\{w_n\}$ we choose to use. If $K(\tilde{a}_n/\tilde{b}_n)$ is limit k-periodic, we know by [**2**, Theorem 4.1] that $\{|w_n|/|\tilde{b}_n + w_n|\}$ is limit k-periodic, except in special situations. Therefore, the existence of appropriate constants M and C can be easily proved in these cases.

In more general cases we know, by [**1**, Proposition 4.2], that $\lim_{n\to\infty} \prod_{j=m}^{n} |f^{(j)}/(b_j + f^{(j)})| = 0$ if $K(a_n/b_n)$ converges and $\{f^{(n)}\}_{n=0}^{\infty}$ is bounded. Therefore, any $M > 1$ can be used, if we choose $w_n = \tilde{f}^{(n)}$ in Lemma 3.1, and $\{\tilde{f}^{(n)}\}$ is bounded (if $K(\tilde{a}_n/\tilde{b}_n)$ converges).

The next step in the process of finding conditions for the convergence of (2.6), is to use Lemma 3.1 to ensure absolute convergence of the four series on the right hand side of (2.6).

LEMMA 3.2. *Under the conditions of Lemma* 3.1, *except that* (3.3) *is replaced by*

$$(3.28)\qquad \begin{aligned} \frac{|a_n - \tilde{a}_n|}{|\tilde{b}_n + w_n|\,|\tilde{b}_{n-1} + w_{n-1}|} &\leqq \frac{\alpha}{M^{*n-1}} \qquad \text{for } n \geqq 2,\\ \left|\frac{b_n - \tilde{b}_n}{\tilde{b}_n + w_n}\right| &\leqq \frac{\beta}{M^{*n}} \qquad \text{for } n \geqq 1, \end{aligned}$$

where $1 < M < M^*$, *we have, with* δ_n *and* η_n *defined by* (2.1),

$$(3.29)\qquad \sum_{m=1}^{\infty}\left|\eta_m \frac{A_{m-1}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right| \leqq \left|\frac{a_1}{\tilde{b}_1 + w_1}\right| \frac{\beta}{M^*}\,\frac{M^* - M + CM}{(M^* - M)^2},$$

$$(3.30)\qquad \sum_{m=1}^{\infty}\left|\delta_m \frac{A_{m-2}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right| \leqq \left|\frac{\delta_1}{\tilde{b}_1 + w_1}\right| + \left|\frac{a_1}{\tilde{b}_1 + w_1}\right| \frac{\alpha}{M^*}\,\frac{M^* - M + CM}{(M^* - M)^2},$$

$$(3.31)\qquad \sum_{m=1}^{\infty}\left|\eta_m \frac{B_{m-1}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right| \leqq \frac{\beta}{M}\,\frac{M^* - M + CM}{(M^* - M)^2}\max\left\{M,\ 1 + \frac{\beta}{M}\right\},$$

and

$$(3.32)\qquad \sum_{m=1}^{\infty}\left|\delta_m \frac{B_{m-2}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right| \leqq \frac{\alpha}{M}\,\frac{M^* - M + CM}{(M^* - M)^2}\max\left\{M,\ 1 + \frac{\beta}{M}\right\}.$$

PROOF. Since the conditions of Lemma 3.1 are satisfied, the conclusion (3.5) – (3.6) is valid. Therefore we get

$$\sum_{m=1}^{\infty}\left|\eta_m \frac{A_{m-1}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right|$$

$$\begin{aligned}
&\leqq \sum_{m=2}^{\infty}\left(\frac{\beta}{M^{*m}}\left|\frac{a_1}{\tilde{b}_1 + w_1}\right|(1 + (m-2)C)M^{m-2}\right) \\
&= \left|\frac{a_1}{\tilde{b}_1 + w_1}\right|\frac{\beta}{M^{*2}}\left[\sum_{m=2}^{\infty}\left(\frac{M}{M^*}\right)^{m-2} + C\frac{M}{M^*}\sum_{m=3}^{\infty}(m-2)\left(\frac{M}{M^*}\right)^{m-3}\right] \qquad (3.33)\\
&= \left|\frac{a_1}{\tilde{b}_1 + w_1}\right|\frac{\beta}{M^{*2}}\left[\frac{1}{1-\frac{M}{M^*}} + C\frac{M}{M^*}\cdot\frac{1}{\left(1-\frac{M}{M^*}\right)^2}\right] \\
&= \left|\frac{a_1}{\tilde{b}_1 + w_1}\right|\frac{\beta}{M^*}\,\frac{M^* - M + CM}{(M^* - M)^2},
\end{aligned}$$

since $A_0 = 0$. Furthermore, since $A_{-1} = 1$, it follows that

$$\sum_{m=1}^{\infty}\left|\delta_m \frac{A_{m-2}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right|$$

$$\begin{aligned}
&\leqq \left|\frac{\delta_1}{\tilde{b}_1 + w_1}\right| + \sum_{m=3}^{\infty}\left(\frac{\alpha}{M^{*m-1}}\left|\frac{a_1}{\tilde{b}_1 + w_1}\right|(1 + (m-3)C)M^{m-3}\right) \qquad (3.34)\\
&= \left|\frac{\delta_1}{\tilde{b}_1 + w_1}\right| + \left|\frac{a_1}{\tilde{b}_1 + w_1}\right|\frac{\alpha}{M^{*2}}\left[\sum_{m=3}^{\infty}\left(\frac{M}{M^*}\right)^{m-3} + C\frac{M}{M^*}\sum_{m=3}^{\infty}(m-3)\left(\frac{M}{M^*}\right)^{m-4}\right] \\
&= \left|\frac{\delta_1}{\tilde{b}_1 + w_1}\right| + \left|\frac{a_1}{\tilde{b}_1 + w_1}\right|\frac{\alpha}{M^*}\,\frac{M^* - M + CM}{(M^* - M)^2}.
\end{aligned}$$

Similarly,

$$\sum_{m=1}^{\infty}\left|\eta_m \frac{B_{m-1}}{\prod_{j=1}^{m}(\tilde{b}_j + w_j)}\right|$$

$$\begin{aligned}
&\leqq \sum_{m=1}^{\infty}\left(\frac{\beta}{M^{*m}}(1 + (m-1)C)M^{m-2}\max\left\{M, 1 + \frac{\beta}{M}\right\}\right) \qquad (3.35)\\
&= \frac{\beta}{M^{*2}}\left[\frac{M^*}{M}\cdot\frac{1}{1-\frac{M}{M^*}} + C\cdot\frac{1}{\left(1-\frac{M}{M^*}\right)^2}\right]\max\left\{M, 1 + \frac{\beta}{M}\right\} \\
&= \frac{\beta}{M}\,\frac{M^* - M + CM}{(M^* - M)^2}\max\left\{M, 1 + \frac{\beta}{M}\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{m=1}^{\infty}\left|\delta_m \frac{B_{m-2}}{\prod_{j=1}^{m}(\tilde{b}_j - w_j)}\right| \\
(3.36)\qquad &\leqq \sum_{m=2}^{\infty}\left(\frac{\alpha}{M^{*m-1}}(1+(m-2)C)M^{m-3}\max\left\{M, 1+\frac{\beta}{M}\right\}\right) \\
&= \frac{\alpha}{M^{*2}}\left[\frac{M^*}{M}\cdot\frac{1}{1-\frac{M}{M^*}} + C\frac{1}{\left(1-\frac{M}{M^*}\right)^2}\right]\max\left\{M, 1+\frac{\beta}{M}\right\} \\
&= \frac{\alpha}{M}\frac{M^* - M + CM}{(M^* - M)^2}\max\left\{M, 1+\frac{\beta}{M}\right\}.
\end{aligned}$$

We can now use Lemma 3.2 to obtain sufficient conditions for the convergence of (2.6).

PROPOSITION 3.3 *Let $K(a_n/b_n)$ and $K(\tilde{a}_n/\tilde{b}_n)$ be continued fractions (possibly with a_n or $\tilde{a}_n = 0$ for some $n \in \mathbf{N}$), and $\{w_n\}$ a sequence of complex numbers such that, for an $N^* \in \mathbf{N} \cup \{0\}$,*

(i)

$$(3.37)\qquad w_{n-1}(\tilde{b}_n + w_n) = \tilde{a}_n,\ \tilde{b}_n + w_n \neq 0 \text{ for } n = N^* + 1, N^* + 2, \ldots,$$

and

(ii) there exist constants $C \geqq 0$ and $M > 1$ such that

$$(3.38)\qquad \prod_{j=m}^{n}\left|\frac{w_j}{\tilde{b}_j + w_j}\right| \leqq CM^{n-m+1} \qquad \text{for all } n \geqq m \geqq N^* + 1.$$

If

$$(3.39)\qquad |a_n - \tilde{a}_n| \leqq |\tilde{b}_n + w_n|\,|\tilde{b}_{n-1} + w_{n-1}|\frac{\alpha}{M^{*n-1}} \qquad \text{for } n = N^* + 2, N^* + 3, \ldots,$$

and

$$(3.40)\qquad |b_n - \tilde{b}_n| \leqq |\tilde{b}_n + w_n|\frac{\beta}{M^{*n}} \qquad \text{for } n = N^* + 1, N^* + 2, \ldots,$$

where $M < M^* \leqq M^2$ and $\alpha, \beta \geqq 0$, then the following three statements are true.

A. $\{S_n(w_n)\}$ converges, possibly to ∞.

B. For a fixed continued fraction $K(\tilde{a}_n/\tilde{b}_n)$, a fixed sequence $\{w_n\}$ and fixed values C, M, M^*, there is an extended real-valued function $H(\alpha, \beta)$, so that $H(\alpha, \beta) \to 0$ as $(\alpha, \beta) \to (0, 0)$ and so that

$$(3.41)\qquad \left|\lim_{n\to\infty} S_n^{(N^*)}(w_{n-N^*}) - w_{N^*}\right| \leqq H(\alpha, \beta),$$

provided (3.39) and (3.40) are satisfied and $|a_{N^*+1} - \tilde{a}_{N^*+1}| \leqq \alpha \cdot P$, where P is a fixed, positive constant.

C. Let $q \in (0, 1)$ be an arbitrary constant. Then

$$(3.42) \qquad |\lim_{m\to\infty} S_m^{(n)}(w_{n+m}) - w_n| \leqq \frac{K(n)}{1-q} \qquad \text{for } n = N, N+1, \ldots,$$

where N is the smallest integer $\geqq N^*$ such that

$$(3.43) \qquad \frac{\alpha + \beta}{M^{*N}} \leqq q \frac{(M^* - M)^2}{M^* - M + CM},$$

and where

$$(3.44) \quad K(n) = \left|\frac{a_{n+1} - \tilde{a}_{n+1}}{\tilde{b}_{n+1} + w_{n+1}}\right| + \left(\left|\frac{a_{n+1}}{\tilde{b}_{n+1} + w_{n+1}}\right| + |w_n| M^*\right) \frac{\alpha + \beta}{M^{*n+1}} \frac{M^* - M + CM}{(M^* - M)^2}.$$

PROOF. We shall first prove C, and thereafter use C to prove A and B.

C. The conditions of Lemma 3.2 are satisfied for the N^{th} tails $K_{n=N+1}^{\infty}(a_n/b_n)$ and $K_{n=N+1}^{\infty}(\tilde{a}_n/\tilde{b}_n)$ (with α and β in Lemma 3.2 replaced by αM^{*-N} and βM^{*-N}), because (3.43) implies (3.4) (with the said change of notation). This follows from Comment 1 to Lemma 3.1 since by (3.43) we have, with $M^* = M(1 + \delta)$, where $\delta > 0$,

$$\frac{\alpha + \beta}{M^{*N}} \leqq q \frac{\delta^2 M}{\delta + C} < \frac{\delta^2 M}{\delta + C}$$

and

$$M^2(M - 1) > \frac{\alpha + \beta}{M^{*N}} \frac{\delta + C}{\delta^2} M(M - 1) \geqq \frac{\alpha + \beta}{M^{*N}} \left(M + \frac{CM}{M - 1}\right)$$
$$> \frac{\alpha + \beta}{M^{*N}} \left(1 + \left(1 + \frac{1}{M-1}\right) C\right) > \frac{\alpha + \beta}{M^{*N}} (1 + (1 + \mu)C),$$

since $M^* = M(1 + \delta) \leqq M^2$, i.e., $\delta \leqq M - 1$. Hence (3.43) implies (3.22) with μ given by (3.23), which again implies (3.4). Therefore we have, by Lemma 3.2, that

$$(3.45) \qquad \sum_{m=1}^{\infty} \left|(b_{N+m} - \tilde{b}_{N+m}) \frac{A_{m-1}^{(N)}}{\prod_{j=N+1}^{N+m} (\tilde{b}_j + w_j)}\right| \leqq \left|\frac{a_{N+1}}{\tilde{b}_{N+1} + w_{N+1}}\right| \frac{\beta}{M^{*N+1}} \frac{M^* - M + CM}{(M^* - M)^2},$$

$$(3.46) \qquad \sum_{m=1}^{\infty} \left|(a_{N+m} - \tilde{a}_{N+m}) \frac{A_{m-2}^{(N)}}{\prod_{j=N+1}^{N+m} (\tilde{b}_j + w_j)}\right| \leqq \left|\frac{a_{N+1} - \tilde{a}_{N+1}}{\tilde{b}_{N+1} + w_{N+1}}\right| + \left|\frac{a_{N+1}}{\tilde{b}_{N+1} + w_{N+1}}\right| \frac{\alpha}{M^{*N+1}} \frac{M^* - M + CM}{(M^* - M)^2},$$

$$(3.47)\quad \sum_{m=1}^{\infty}\left|(b_{N+m}-\tilde{b}_{N+m})\frac{B_{m-1}^{(N)}}{\prod_{j=N+1}^{N+m}(\tilde{b}_j+w_j)}\right| \leqq \frac{\beta}{M^{*N}}\frac{M^*-M+CM}{(M^*-M)^2},$$

and

$$(3.48)\quad \sum_{m=1}^{\infty}\left|(a_{N+m}-\tilde{a}_{N+m})\frac{B_{m-2}^{(N)}}{\prod_{j=N+1}^{N+m}(\tilde{b}_j+w_j)}\right| \leqq \frac{\alpha}{M^{*N}}\frac{M^*-M+CM}{(M^*-M)^2},$$

since (3.43) implies that $\max\{M, 1+\beta/(MM^{*N})\} = M$. In fact, the same is true for all $n \geqq N$. By (2.6), we therefore get, for all $n \geqq N$,

$$(3.49)\quad \begin{aligned} &\left|\lim_{m\to\infty} S_m^{(n)}(w_{m+n}) - w_n\right| \\ &\leqq \left\{\left|\frac{a_{n+1}-\tilde{a}_{n+1}}{\tilde{b}_{n+1}+w_{n+1}}\right| + \left[\left|\frac{a_{n+1}}{\tilde{b}_{n+1}+w_{n+1}}\right|\frac{\alpha+\beta}{M^{*n+1}} + |w_n|\cdot\frac{\alpha+\beta}{M^{*n}}\right]\right. \\ &\quad \left.\frac{M^*-M+CM}{(M^*-M)^2}\right\}\Big/\left\{1-\frac{\alpha+\beta}{M^{*n}}\frac{M^*-M+CM}{(M^*-M)^2}\right\} \\ &\leqq \frac{K(n)}{(1-q)}, \end{aligned}$$

because

$$\frac{\alpha+\beta}{M^{*n}}\frac{M^*-M+CM}{(M^*-M)^2} \leqq q,$$

by (3.43)

A. Since $\{S_m^{(N)}(w_{N+m})\}_{m=1}^{\infty}$ converges by part C, we get $\lim_{n\to\infty} S_n(w_n) = S_N(\lim_{m\to\infty} S_m^{(N)}(w_{N+m}))$.

B. By part C we we can for instance, use

$$(3.50)\quad H(\alpha,\beta) = \begin{cases} 2K(N^*) & \text{if } (\alpha,\beta) \text{ satisfies (3.43) with } N=N^* \text{ and } q=1/2 \\ \infty & \text{otherwise.} \end{cases}$$

COMMENT. 1. The bound given by (3.42) is obviously not particularly good for n much larger than N, since the denominator is replaced by $1-q$. But we are only interested in establishing a finite upper bound here, which approaches 0 as $n\to\infty$, and not necessarily a best one.

COMMENT 2. Proposition 3.3 is also valid if a_n or $\tilde{a}_n$ is equal to 0 for one or more values of n. It can even be made valid for cases where $\tilde{b}_n + w_n = 0$ for some values of n, if we define (3.38) in a natural way. Except in special cases, we then need $C=\infty$ or $M=\infty$, and hence, $a_n = \tilde{a}_n$ and $b_n = \tilde{b}_n$ for all n. Therefore $S_n(w_n) = w_0$ and $S_n^{(m)}(w_{m+n}) = w_m$ for all n and m, and the result of Proposition 3.3 holds trivially. This will be a point of interest when we proceed to the next section, where we look at functions defined by continued fractions.

4. Analytic or meromorphic functions represented by continued fractions. In this section we return to the situation described in §1. We let the elements a_n and b_n of the continued fractions be functions of a complex variable z, and, in particular, we restrict ourselves to the case where $a_n(z)$ and $b_n(z)$ are analytic in the domain where we choose to define our continued fraction. Questions of interest will then be:

1. Does the continued fraction define an analytic or meromorphic function in some domain?
2. Can it, by any means, be continued analytically or meromorphically beyond that domain?
3. Can anything be said about possible singularities of this function?

Some results along these lines are already known. Question 1 is answered positively for certain T-fractions, C-fractions, π-fractions, etc. (See, for instance, [**3**, Chapter 7].) Question 2 is answered positively for the case where $K(a_n(z)/b_n(z))$ is limit 1-periodic, and $a_n(z)$ and $b_n(z)$ approach their limits geometrically and fast enough (see [**4**, **5**]). In this case, Question 3 is also positively answered, with respect to branch points. Several results about possible poles are also known. (See [**3**, Chapter 7].)

We are going to find sufficient conditions for positive answers to these questions in some new cases, by using Proposition 3.3 and its proof. The method involved is the one mentioned in the introduction, namely to use auxiliary continued fractions $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ to construct modified approximants. We shall see that if the two continued fractions are "near enough", properties of $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ are, to a certain extent, inherited by $K(a_n(z)/b_n(z))$.

The following theorem is the main result of this paper. It provides the definition of "near enough". In doing so, the term "domain" is used. By that we shall mean a domain in the usual, strict sense, i.e., an open connected set.

THEOREM 4.1. *Let two continued fractions $K(a_n(z)/b_n(z))$ and $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ and a sequence $\{w_n(z)\}$ be given such that*

$$(4.1)\qquad w_{n-1}(z)\,[\tilde{b}_n(z) + w_n(z)] = \tilde{a}_n(z) \qquad \textit{for } n = 1, 2, 3, \ldots$$

in a domain $\mathscr{D}^*$. Let $\mathscr{D} \subseteq \mathscr{D}^*$ be a domain, and $N:\mathscr{D} \to [0, \infty)$ be a continuous function such that the following conditions are satisfied in $\mathscr{D}$:

(*i*) a_n, b_n, $\tilde{a}_n$ and $\tilde{b}_n$ are analytic in $\mathscr{D}$ for all $n \geqq 1$;
(*ii*) *$w_n(z) + \tilde{b}_n(z) \neq 0$ in $\mathscr{D}$ for all $n \geqq N(z) + 1$;*
(*iii*) *for each $z \in \mathscr{D}$, w_n is analytic at z for all $n > N(z)$;*
(*iv*) *there exist continuous functions $M:\mathscr{D} \to (1, \infty)$ and $C:\mathscr{D} \to [0, \infty)$, such that, in $\mathscr{D}$*

$$(4.2)\qquad \prod_{j=m}^{n}\left|\frac{w_j(z)}{\tilde{b}_j(z)+w_j(z)}\right| \leqq C(z)M(z)^{n-m+1} \textit{ for all } n \geqq m \geqq N(z)+1;$$

(*v*) *there exist continuous functions* α, β, δ: $\mathscr{D} \to [0, \infty)$ *such that* $0 < \delta(z) \leqq M(z) - 1$ in $\mathscr{D}$,

$$(4.3)\qquad |a_n(z) - \tilde{a}_n(z)| \leqq \frac{|\tilde{b}_n(z) + w_n(z)|\,|\tilde{b}_{n-1}(z) + w_{n-1}(z)|\alpha(z)}{[(1+\delta(z))M(z)]^{n-1}},$$

for all $n \geqq N(z) + 2$; *and*

$$(4.4)\qquad |b_n(z) - \tilde{b}_n(z)| \leqq \frac{|\tilde{b}_n(z)+w_n(z)|\beta(z)}{[(1+\delta(z))M(z)]^n} \qquad \textit{for all } n \geqq N(z)+1,$$

for all $z \in \mathscr{D}$.

If $\mathscr{D} \neq \emptyset$, *the following statements are true.*

A. $\{S_n(w_n, z)\}_{n=1}^{\infty}$ *converges to a meromorphic function* $F(z)$ *in* $\mathscr{D}$ *or to* $F(z) \equiv \infty$ *in* $\mathscr{D}$. *The convergence is uniform in compact subsets of* $\mathscr{D}$ *which contain no poles of* $F(z)$, *if* F *is meromorphic.*

B. *For every compact subset* $\mathscr{C}$ *of* $\mathscr{D}$ *there exist a natural number* $N_{\mathscr{C}}^* \geqq N_{\mathscr{C}} = \max_{z\in\mathscr{C}} N(z)$ *and a positive sequence* $\{\mathscr{B}_{\mathscr{C}}^{(n)}\}$ *such that* $\{S_m^{(n)}(w_{m+n}, z)\}_{m=1}^{\infty}$ *converges uniformly to an analytic function* $F^{(n)}(z)$ *in* $\mathscr{C}$, *for all* $n \geqq N_{\mathscr{C}}^*$, *and*

$$(4.5)\qquad |F^{(n)}(z) - w_n(z)| \leqq \frac{\mathscr{B}_{\mathscr{C}}^{(n)}}{[(1+\delta(z))M(z)]^n} \textit{ in } \mathscr{C} \qquad \textit{for } n \geqq N_{\mathscr{C}}^*.$$

C. *If* $N(z)$ *is bounded in* $\mathscr{D}$ *by some* $N \in \mathbf{N}$ *such that, for a given continuous function* $q: \mathscr{D} \to (0, 1)$,

$$(4.6)\qquad \alpha(z) + \beta(z) \leqq q(z)\frac{\delta(z)^2}{C(z)+\delta(z)} M(z)^{N+1}[1+\delta(z)]^N,$$

then $\{S_m^{(n)}(w_{n+m}, z)\}_{m=1}^{\infty}$ *converges uniformly to an analytic function* $F^{(n)}(z)$ *in* $\mathscr{D}$ *for all* $n \geqq N$.

D. $\lim_{n\to\infty}(F^{(n)}(z) - w_n(z)) = 0$, *for all* $z \in \mathscr{D}$ *for which*

$$(4.7)\qquad \lim_{n\to\infty}\{|\tilde{b}_n(z) + w_n(z)| + |w_n(z)|\}[M(z)(1+\delta(z))]^{-n} = 0.$$

PROOF. Suppose that $\mathscr{D} \neq \emptyset$. For each $z \in \mathscr{D}$, the hypothesis of Proposition 3.3A is satisfied for the continued fractions

$$\mathop{K}_{n=-[-N(z)]+1}^{\infty}\left(\frac{a_n(z)}{b_n(z)}\right) \quad\text{and}\quad \mathop{K}_{n=-[-N(z)]+1}^{\infty}\left(\frac{\tilde{a}_n(z)}{\tilde{b}_n(z)}\right)$$

(with $M^* = M(z)(1+\delta(z))$, $\alpha = \alpha(z)(M(z)(1+\delta(z)))^{[-N(z)]}$, and $\beta = \beta(z)(M(z)(1+\delta(z)))^{[-N(z)]}$ ($[x] = \max\{n \in Z;\ n \leqq z\}$)). Therefore, we know that $S_n(w_n, z)_{n=1}^{\infty}$ converges for each $z \in \mathscr{D}$.

From Proposition 3.3C it follows that, for each $z \in \mathscr{D}$, we have

$$|F^{(n)}(z) - w_n(z)| \leqq \frac{K(n, z)}{1 - q(z)} \quad \text{for} \tag{4.8}$$
$$n = N^*(z), N^*(z) + 1, N^*(z) + 2, \ldots,$$

where $N^*(z)$ is the least integer in $[N(z), \infty)$ such that (4.6) is satisfied with $N = N^*(z)$, and

$$\begin{aligned} K(n, z) = &\left| \frac{a_{n+1}(z) - \tilde{a}_{n+1}(z)}{\tilde{b}_{n+1}(z) + w_{n+1}(z)} \right| \\ &+ \left(\left| \frac{a_{n+1}(z)}{\tilde{b}_{n+1}(z) + w_{n+1}(z)} \right| + |w_n(z)| M(z)(1 + \delta(z)) \right) \\ &\cdot \frac{\alpha(z) + \beta(z)}{(M(z)(1 + \delta(z)))^{n+1}} \frac{\delta(z) + C(z)}{M(z)\delta(z)^2} . \end{aligned} \tag{4.9}$$

We shall use this to prove B. Thereafter, we shall apply B. to prove A., C. and D.

B. Let $\mathscr{C}$ be an arbitrary compact subset of $\mathscr{D}$. Then the four series (3.45) – (3.48), where N is any integer $\geqq N_{\mathscr{C}}$, have analytic terms and converge absolutely and uniformly in $\mathscr{C}$. Therefore each of these four series converges to an analytic function in $\mathscr{C}$.

If, in addition, we choose $n = N \geqq \sup\{N^*(z); z \in \mathscr{C}\} = N^*_{\mathscr{C}}$, we know, by (4.8), that (4.5) is valid with

$$\mathscr{B}^{(n)}_{\mathscr{C}} = \max_{z \in \mathscr{C}} \left\{ \frac{K(n, z)}{1 - q(z)} [M(z)(1 + \delta(z))]^n \right\} \qquad \text{for } n \leqq N^*_{\mathscr{C}}. \tag{4.10}$$

Therefore, $F^{(n)}$ is analytic in $\mathscr{C}$ for all $n \geqq N^*_{\mathscr{C}}$. Furthermore, by (2.6) (the uniform convergence of the four series and the boundedness away from 0 of the denominator), we see that the convergence of $\{S^{(n)}_m(w_{n+m}, z)\}_{m=1}^{\infty}$ is uniform with respect to $\mathscr{C}$ for all $n \geqq N^*_{\mathscr{C}}$.

A. Since $F(z) = \lim_{n\to\infty} S_n(w_n, z) = S_{N^*_{\mathscr{C}}}(F^{(N^*_{\mathscr{C}})}, z)$ for all $z \in \mathscr{C}$, where $F^{(N^*_{\mathscr{C}})}$ and the coefficients of $S_{N^*_{\mathscr{C}}}$ are analytic in $\mathscr{C}$ by condition (i) and part B, we know that $F(z)$ is meromorphic or identically ∞ in $\mathscr{C}$. Furthermore, the convergence is uniform if $F(z) \neq \infty$ in $\mathscr{C}$. Since $\mathscr{C}$ was arbitrarily chosen, A follows.

C. This follows directly, since (4.6) implies (4.8) with $n = N$ for all $z \in \mathscr{D}$.

D. This follows directly from (4.5) and (4.10).

COMMENT 1. The constant $N^*_{\mathscr{C}}$ and the sequence $\{\mathscr{B}^{(n)}_{\mathscr{C}}\}_{n=N^*_{\mathscr{C}}}^{\infty}$ in part B, are (by the proof) for a fixed $q:\mathscr{D} \to (0, 1)$, given by (4.10) and by

$N^*_{\mathscr{C}} = \max\{N^*(z); z \in \mathscr{C}\}$, where $N^*(z)$ is the smallest integer in $[N(z), \infty)$ such that (4.6) is satisfied (with $N = N^*(z)$).

COMMENT 2. Part C states sufficient conditions for $F^{(n)}$ to be analytic in $\mathscr{D}$. If, in particular, these conditions are satisfied for $N = 0$, then F and $F^{(n)}$ are analytic in $\mathscr{D}$, for all n.

So, also, in part B, if $N^*_{\mathscr{C}} = 0$ is possible in some compact subset $\mathscr{C}$, then F and $F^{(n)}$ are analytic in $\mathscr{C}$ for all n.

COMMENT 3. As mentioned in Comment 2 to Proposition 3.3, we can permit $a_n(z) = 0$ and/or $\tilde{a}_n(z) = 0$ in $\mathscr{D}$. This is convenient, since a_n and $\tilde{a}_n$ are functions. For the same reason, we also want to include some of the cases where $\tilde{b}_n(z) + w_n(z) = 0$ or $w_n(z) = \infty$, for some n.

Case 1. If z_0 is an isolated point in $\mathscr{D}^*\backslash\mathscr{D}$, such that

(a) a_n, b_n, $\tilde{a}_n$ and $\tilde{b}_n$ are analytic and w_n is meromorphic at z_0 for all n;

(b) $\overline{\lim}_{z\to z_0} \alpha(z)$, $\overline{\lim}_{z\to z_0} \beta(z)$ and $\overline{\lim}_{z\to z_0} N(z)$ are finite; and

(c) $\tilde{b}_n(z_0) + w_n(z_0) = 0$ for at most a finite number of indices n, then we can include z_0 in $\mathscr{D}$ by adjusting α, β and N in a neighborhood of z_0.

If, however, $\tilde{b}_n(z_0) + w_n(z_0) = 0$ and/or $w_n(z_0) = \infty$ for infinitely many indices n, we need another approach as in Case 2.

Case 2. Suppose that z_0 is an isolated point in $\mathscr{D}^*\backslash\mathscr{D}$, such that the conditions (a) and (b) above hold, and, in addition, that

(c)′ $\tilde{b}_n(z_0) + w_n(z_0) = 0$, for some or all n,

(d)′ w_n is analytic at z_0, for sufficiently large indices n, and

(e)′ $\tilde{b}_n(z_0) \neq 0$, for some sufficiently large indices n for which $\tilde{b}_n(z_0) + w_n(z_0) = 0$.

Then we can define $N(z)$ such that N is also continuous at z_0. (We achieve this by defining $N(z_0) \geqq \overline{\lim}_{z\to z_0} N(z)$, and redefining $N(z)$ in a neighborhood of z_0, such that N is continuous in $\mathscr{D} \cup \{z_0\}$.) From (4.2), we see that (e)′ implies that we need $M(z_0) = \infty$ (in fact, $\lim_{z\to z_0} M(z) = \infty$) or $C(z_0) = \infty$ ($\lim_{z\to z_0} C(z) = \infty$), which, by (4.6), implies that $\alpha(z_0) = \beta(z_0) = 0$ ($\lim_{z\to z_0}\alpha(z) = \lim_{z\to z_0}\beta(z) = 0$). In both cases we therefore need $a_{n+1}(z_0) = \tilde{a}_{n+1}(z_0)$ and $b_n(z_0) = \tilde{b}_n(z_0)$, for all $n \geqq N(z_0) + 1$.

If $\lim_{z\to z_0} M(z) = \infty$, we see, by (4.5), that $\lim_{z\to z_0} F^{(n)}(z) = w_n(z_0) = S_m^{(n)}(w_{n+m}, z_0) = \lim_{m\to\infty} S_m^{(n)}(w_{n+m}, z_0) = F^{(n)}(z_0)$, for all $n \geqq N(z_0) + 1$. Hence, $F^{(n)}(z)$ is analytic at z_0, for all $n > N(z_0)$.

If $\lim_{z\to z_0} C(z) = \infty$, the question is more open. Again, we have $F^{(n)}(z_0) = w_n(z_0)$. But we need $\lim_{z\to z_0} (\alpha(z) + \beta(z))C(z) = 0$ to conclude from Theorem 4.1 that $F^{(n)}$ is analytic at z_0.

Case 3. If we have that $\tilde{b}_n(z_0) + w_n(z_0) = 0$ if and only if $\tilde{b}_n(z_0) = 0$, or if we have that w_n has a pole at z_0 for some or all indices $n \geqq N(z_0) + 1$, the picture gets more complicated. These cases are best handled separately in each specific application of Theorem 4.1.

COMMENT 4. The generality of this theorem has made the checking of the conditions in specific situations rather troublesome. In particular, this is so for the restrictions on the auxiliary continued fraction $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ and its tails. Therefore, this theorem should mainly be used to develop a library of usable (that is, already checked) auxiliary continued fractions along with instructions for which continued fractions $K(a_n(z)/b_n(z))$ they can be used. A beginning of such a library will be presented in a separate paper in the near future.

COMMENT 5. If we have, in addition to the conditions of Theorem 4.1, that both $K(a_n(z)/b_n(z))$ and $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ converge in $\mathscr{D}$, that $\lim_{n\to\infty} S_n(\tilde{f}^{(n)}, z) = f(z)$ in $\mathscr{D}$, and furthermore that $w_n = \tilde{f}^{(n)}$, then $f(z)$ is meromorphic or identically ∞ in $\mathscr{D}$, by Theorem 4.1.

The condition $f(z) = \lim_{n\to\infty} S_n(\tilde{f}^{(n)}, z)$ is not as restrictive as it may seem. For instance, if $K(a_n(z)/b_n(z))$ is limit k-periodic, then we know by [**2**, Theorem 3.1] that it is satisfied, except, possibly, in very special situations.

If, in particular, $N(z) \equiv 0$ and (4.6) is satisfied for $N = 0$, then $f(z)$ is analytic in $\mathscr{D}$, and so are all the tails $f^{(n)}(z)$.

We can also use Theorem 4.1 to prove that $f(z) = K(a_n(z)/b_n(z))$ is meromorphic in a domain $\mathscr{D}$, and thereafter combine this with results on boundedness of $f(z)$.

COMMENT 6. If we have, in addition to the conditions of Theorem 4.1, that both $K(a_n(z)b_n/(z))$ and $K(\tilde{a}_n(z)/\tilde{b}_n(z))$ converge in a domain $\mathscr{D}_0 \subseteq \mathscr{D}$, that $F^{(0)}(z) = f(z)$ in $\mathscr{D}_0$, and that the analytic continuation $w_n(z)$ of $\tilde{f}^{(n)}(z)$ to $\mathscr{D}$ satisfies the conditions (ii) and (iii) of Theorem 4.1 in $\mathscr{D}$, then $F^{(0)}(z)$ is the meromorphic continuation of $f(z)$ to $\mathscr{D}$.

If, in particular, $N(z) \equiv 0$, and (4.6) is satisfied for $N = 0$, then $F^{(0)}(z)$ is the analytic continuation of $f(z)$ to $\mathscr{D}$.

However, in this situation, the computation of $F(z)$ by means of $S_n(w_n, z)$ can be unstable. This is not a big problem, since it is easily solved by means of continued fractions. The method is presented in [**8**]. In short, it consists of constructing the continued fraction which has $S_n(w_n, z)$ as its ordinary approximants, and of computing its approximants by one of the stable methods for computing approximants of continued fractions.

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