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# WELL HEIGHTS, NÉRON PAIRINGS AND V-METRICS ON CURVES

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ABSTRACT. In this paper we supplement Néron's theory of height pairings on curves by attaching to each non-constant rational function $f$ on a curve $C$ defined over a field $K$, endowed with an absolute value $|\ |_v$, a "height pairing" $\hat{\lambda}_{f,v}$ on $\mathrm{Div}(C) \times \mathrm{Div}(C)$. It is shown that the stipulation that these height pairings be "as functorial as possible" forces them to be unique (up to a constant); in particular, their restrictions to $\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C)$ reduce to Néron's height pairing. We also show that $\hat{\lambda}_{f,v}$ may be computed explicitly in the non-archimedean (discrete) case, via the Lichtenbaum-Shafarevich intersection theory on a suitable two-dimensional scheme over $\mathfrak{O}_v$, and in the archimedean case via Arakelov's theory of Green's functions on Riemann surfaces attached to a suitable (1, 1)-form.

**0. Introduction.** The theory of height pairings, which was created in 1965 by A. Néron **[19]** as a refinement of A. Weil's theory of distributions (Weil **[23]**–**[25]**), is important not only as a practical tool in proving diophantine statements (e.g. theorems of Mordell-Weil, Mumford, Manin etc.), but also as an intrinsic concept reflecting the finer quantitative nature of the diophantine problem in question (e.g., *Conjecture of Birch and Swinnerton-Dyer* and the recent *Theorem of Gross and Zagier*). Although Néron's theory mainly concerns abelian varieties (in fact, it is only in this case that Néron's theory completely refines Weil's theory), he does obtain (by appealing to the theory of Picard varieties) a similar (but weaker) theory of height pairings in the case of an arbitrary (smooth, complete) variety.

The purpose of this paper is to reconsider Weil's and Néron's theory of (local) heights and height pairings in the special case of curves. In doing so, we have two principal aims in mind.

The first aim is to demonstrate that it is possible to give a direct treatment of Néron's theory for curves without recourse to the theory of abelian varieties (or Jacobians). The main idea here is the observation that $v$-metrics on curves (as defined below in §3) yield "crude Néron

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pairings" and that, hence, by sharpening Tate's averaging process (cf. Manin **[15]**, Néron **[19]** and §4), it is possible to refine any $v$-metric on the curve $C$ to the Néron pairing $\lambda_{c,v}$ on $(\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C))'$; here, $\mathrm{Div}^\circ(C)$ denotes the group of divisors of degree 0 on $C$ and $'$ denotes the subset of the product set consisting of all disjoint divisors.

The second aim is to extend Néron's pairing to a pairing on $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ and thus obtain a "refined" theory of Weil heights. Such extensions (to $(\mathrm{Div}(C) \times \mathrm{Div}^\circ(C))'$, at least) were already considered by Néron **[19]** who showed that these are unique once one imposes the condition that they be functorial with respect to morphisms between curves, and Parshin **[20]**, **[21]** subsequently used these to derive interesting diophantine statements about modular curves. Unfortunately, as Manin **[16]** pointed out, no such functorial extensions can exist on the category of curves of genus $g \geqq 2$; as he shows, the "obstruction to functoriality" is tied up with the existence of non-trivial endomorphisms on the Jacobian $J_C$ of the curve $C$.

Now, while a completely functorial theory of heights (or height pairings) cannot exist, it is possible to advance a theory of heights "with limited functoriality". To this end we consider on each curve $C$ not only one but a whole family of extensions of Néron's height pairing to $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$. To be precise, we attach to each non-constant rational function $f \in K(C)$ on $C$ an extension $\hat{\lambda}_{f,v}$ of $\lambda_{C,v}$ to $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ and stipulate that these be "as functorial as possible", by which we mean the following. First, if $\phi\colon C' \to C$ is any finite covering of curves and $f \in K(C)\backslash K$, then we require (as does Néron) that the projection formula holds for $\hat{\lambda}_{\phi^*f,v}$, $\hat{\lambda}_{f,v}$ and $\phi$. Secondly, we require that we have $\hat{\lambda}_{\alpha(f),v} = \hat{\lambda}_{f,v}$ for all $f \in K(C)\backslash K$ and all $\alpha \in \mathrm{Aut}(v)$, where $\mathrm{Aut}(v) \subset PGl_2(K)$ is a certain group of fractional linear transformations associated to $v$, namely Aut $(v) = PGl_2(\mathfrak{O}_v)$, if $v$ is non-archimedean and $\mathfrak{O}_v$ its valuation ring, and $\mathrm{Aut}(v) = PU(2)$, if $v$ is archimedean (*wlog* $K = \mathbf{C}$) and $PU(2)$ denotes the image of the unitary group $U(2) \subset Gl_2(\mathbf{C})$ in $PGl_2(\mathbf{C})$. It then turns out that these properties of "limited functoriality", together with the usual properties of height pairings, uniquely characterize the functions $\hat{\lambda}_{f,v}$ up to an arbitrary additive constant (which may be fixed by the normalization condition $\hat{\lambda}_{f,v}((f)_0, (f)_\infty) = 0$) and therefore give rise to a "canonical" theory of Weil heights.

Finally, we show that these height pairings $\hat{\lambda}_{f,v}$ have natural interpretations in terms of "intersection numbers" on a (suitable) arithmetic surface; i.e., they can be computed via the Lichtenbaum **[14]**-Shafarevich **[22]** intersection theory on two-dimensional schemes in the non-archimedean (discrete) case (cf. §9) and via Arakelov's **[2]** theory of Green's functions on compact Riemann surfaces in the archimedean case (cf. §10).

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Mathematics at Harvard University, whose hospitality I greatly appreciated. I would like to thank B. Mazur for the interest which he took in this work and for his stimulating comments. Also, I have greatly profited from conversations with B. Gross and, above all, from his extremely interesting manuscript **[4]**, for which I am very grateful. Finally, I would like to thank P. Roquette, R. Rumely and E. Viehweg for helpful conversations.

**1. Weil heights (local distributions).** Let $K$ be a field, $|\ |_v$ an absolute value of $K$ (archimedean or not), and $v = -\log |\ |_v$ its associated "valuation". Moreover, let $V$ be a complete, smooth, irreducible variety defined over $K$. We denote by:

$F = K(V)$ its field of rational functions,

$X^1(V)$ its set of prime divisors ($=$ set of irreducible subvarietes of codim 1),

$\mathrm{Div}(V)$ its divisor group ($=$ free abelian group generated by $X^1(V)$),

$\mathrm{Div}_\mathscr{l}(V)$ the subgroup of $\mathrm{Div}(V)$ of principal divisors,

$V(K)$ its set of $K$-rational points,

$Z'(V)$ its group of 0-cycles, all of whose components are $K$-rational ($=$ free abelian group generated by $V(K)$),

$Z'_0(V)$ the subgroup of $Z'(V)$ consisting of cycles of degree 0.

By the approximation theorem for valuations, any divisor $D$ or $V$ has a representation as a minimum-maximum of principal divisors, i.e.,

$$D = \min_i \max_j (f_{ij}), \tag{1}$$

where $\mathbf{f} = \{f_{ij}\} \subset F^\times$ is some suitable finite set. To each such representation (1) of $D$ we can attach a function, $h_{\mathbf{f},v}$: $V(K)\backslash\mathrm{supp}(D) \to \mathbf{R}$, called a Weil height (or, more correctly, a local distribution) by the formula

$$h_{\mathbf{f},v}(P) = \min_i \max_j v(f_{ij}(P)). \tag{2}$$

The major drawback of Weil's theory is that the Weil height $h_{\mathbf{f},v}$ depends not only on the divisor $D$ but in fact also on the particular choice of the representation (1) of the divisor. Thus, to each divisor we have associated not just a single function but rather a whole family of Weil heights. The crucial fact, therefore, in Weil's theory is that under suitable hypotheses these Weil heights do not substantially differ from each other, i.e., that they are equivalent in the following sense.

DEFINITION. Two real-valued functions $f$ and $g$ defined on a set $S$ are said to be equivalent (notation: $f \sim g$) if their difference is bounded on $S$.

One then has the following fundamental fact which we state for simplicity only in the case that $V$ is a curve. (It is also true for dim $V > 1$ provided that we impose a further condition on the sets $\mathbf{f}$ attached to a divisor $D$; cf. Weil **[25]**).

PROPOSITION 1. *If* $\dim V = 1$, *then any two Weil heights* $h_{\mathfrak{f},v}$ *and* $h_{\mathfrak{g},v}$ *attached to the same divisor D on V are equivalent on* $V(K)\backslash\mathrm{supp}(D)$.

*Notation.* We denote by $\mathscr{H}_{D,v}$ the set of all (real-valued) functions on $V(K)\backslash\mathrm{supp}(D)$ which are equivalent to some (and hence all) Weil heights $h_{\mathfrak{f},v}$ attached to the divisor $D$ on $V$.

**2. Néron pairings and Néron heights.** One of the principal aims of Néron's theory is to single out in each equivalence class $\mathscr{H}_{D,v}$ of height functions attached to a divisor $D$ on a variety $V$ a "canonical" representative $h_{D,v}$ (now called a Néron height) which is unique up to an additive constant. In order to achieve this aim, Néron was guided by the following observation.

Let $D = (f)$ be the divisor of the rational function $f \in F^\times$ (i.e., $D$ is a principal divisor) and let $P_0 \in V(K)$ be a "base point". Then, while the height function $h_{f,v}$ on $V(K)\backslash\mathrm{supp}(D)$ defined by $h_{f,v}(P) = v(f(P))$ depends on the choice of $f$, the function $h_{D,P_0,v}$ defined by

$$h_{D,P_0,v}(P) = v\left(\frac{f(P)}{f(P_0)}\right)$$

does not. More generally, if $\mathfrak{a} = \sum n_i P_i \in Z_0'(V)$ is any 0-cycle of $V$ of degree 0 with $\mathrm{supp}(D) \cap \mathrm{supp}(\mathfrak{a}) = \varnothing$, then the element

$$f(\mathfrak{a}) = \prod_P f(P_i)^{n_i} \in K \tag{3}$$

does not depend on the choice of $f$ but only on the divisor $D = (f)$, so that we may define the symbol $\lambda_{V,v}(D, \mathfrak{a}) \in \mathbf{R}$ by

$$\lambda_{V,v}(D, \mathfrak{a}) = v(f(\mathfrak{a})), \tag{4}$$

when $D = (f)$. This suggests, therefore, that the proper context for viewing height functions on $V$ is via pairings

$$\lambda_{V,v}\colon (\mathrm{Div}(V) \times Z_0'(V))' \to \mathbf{R} \tag{5}$$

which extend the basic pairing on $(\mathrm{Div}_l(V) \times Z_0'(V))'$ defined by (4) above; here, the $'$ denotes the subset of the product set consisting of all pairs $(D, \mathfrak{a})$ such that $\mathrm{supp}(D) \cap \mathrm{supp}(\mathfrak{a}) = \varnothing$. One can then recover Weil's theory by fixing a base point $P_0 \in V(K)$ and putting

$$h_{D,P_0,v}(P) = \lambda_{V,v}(D, P - P_0). \tag{6}$$

In general, there exist many pairings (5) which extend the basic pairing (4). In order to pin down a preferred choice, Néron imposes on the pairing the condition that it is functorial with respect to morphisms between varieties and also satisfies a certain "topological property". However, as already mentioned in the introduction, such functorial pairings can only exist on the category of abelian varieties; in the case of an arbitrary variety, one has to replace $\mathrm{Div}(V)$ by its subgroup $\mathrm{Div}_a(V)$ consisting of all divisors algebraically equivalent to 0 in order to preserve the functoriality property.

Suppose from now on that $V = C$ is a curve. In that case $\mathrm{Div}_a(C) = \mathrm{Div}^0(C)$ is the group of divisors of degree 0 which, in the case of an algebraically closed ground field, may be identified with $Z_0'(C)$. Because of this identification, it is possible to restate Néron's principal theorem (specialized to the case of curves) in the following, possibly more natural, way.

**Theorem 1.** (Néron) *Suppose $K$ is an algebraically closed field. Then on each curve $C$ defined over $K$ there exists a unique real-valued function $\lambda_{C,v}$ on* $(\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C))'$ *satisfying*:

(i) *$\lambda_{C,v}$ is bi-additive (when defined)*;

(ii) $\lambda_{C,v}((f), D) = v(f(D))$, *if* $(f) \in \mathrm{Div}_\ell(C)$ *and* $D \in \mathrm{Div}^\circ(C)$ *are disjoint*;

(iii) *For any* $D \in \mathrm{Div}^\circ(C)$, *and* $P_0 \in C(K)\setminus \mathrm{supp}(D)$, *the function* $h_{D,P_0,v}$ *defined on* $C(K)\setminus \mathrm{supp}\,(D)$ *by*

$$(7) \qquad h_{D,P_0,v}(P) = \lambda_{C,v}(P - P_0, D)$$

*is a height function associated to $D$*; *i.e.*, $h_{D,P_0,v} \in \mathscr{H}_{D,v}$.

*Moreover, $\lambda_{C,v}$ also satisfies*:

(iv) $\lambda_{C,v}(E, D) = \lambda_{C,v}(D, E)$, *if* $D, E \in \mathrm{Div}^\circ(C)$ *are disjoint*;

(v) *If $\phi: C' \to C$ is any finite covering of curves, then the "projection formula" holds*, i.e.,

$$(8) \qquad \lambda_{C',v}(\phi^*D, E') = \lambda_{C,v}(D, \phi_*(E')),$$

*where* $D \in \mathrm{Div}^\circ(C)$ *and* $E' \in \mathrm{Div}^\circ(C')$ *are divisors which are "$\phi$-disjoint", i.e., $\phi^*D$ and $E'$ are disjoint.*

Remark. Note that in the statement of property (iii) above we have made use of the identification $\mathrm{Div}^\circ(C) = Z_0'(C)$; in particular, property (iii) cannot be generalized to higher dimensional varieties. On the other hand, if we use the "correct" definition of $h_{D,P_0,v}$ as given by (6), then that property does generalize to arbitrary varieties. (Observe that because of (iv), the two definitions coincide for curves.) In fact, it it a weakening of this property (namely, that $\lambda_{C,v}$ be "$v$-bounded") together with properties (i), (ii) and (v) which Néron uses to characterize his pairing.

In the above formulation of Néron's theorem, we had for convenience assumed that the ground field $K$ is algebraically closed. It is, however, possible to give a similar characterization of Néron's pairing over an arbitrary ground field if one considers Weil heights not only on $C(K)$

but also on the sets $C^{(n)}(K)$, $n = 1, 2, \ldots$, consisting of all positive divisors $D$ on $C$ of degree $n$.

To define these Weil heights, we shall first extend the definition of the symbol $f(D)$ (cf. (3)) to all divisors $D \in \mathrm{Div}(C)$. For this, let $P \in X^1(C)$ be an arbitrary prime divisor of $C$ (not necessarily of degree 1), which we view as an equivalence class of places of $F/K$. If $\tilde{P}$ is any representative of this class, and $F\tilde{P} \supset K$ denotes its residue field, then for any $f \in F$ with $f\tilde{P} \neq \infty$, we put

$$f(P) = \mathcal{N}_{F\tilde{P}/K}(f\tilde{P}), \tag{9}$$

where $\mathcal{N}_{F\tilde{P}/K}$ denotes the field norm of the finite extension $F\tilde{P}/K$. Clearly, the right hand side of (9) does not depend on the choice of the place $\tilde{P}$ in its equivalence class $P$; this therefore justifies the notation $f(P)$. By multiplicativity, we thus have the symbol $f(D)$ defined for every $f \in F^\times$ and $D \in \mathrm{Div}(C)$ with $\mathrm{supp}((f)) \cap \mathrm{supp}(D) = \emptyset$.

We can now define the Weil heights on $C^{(n)}(K)$ in a similar manner as before. If $\mathbf{f} = \{f_{ij}\}$ is a finite set representing the divisor $D$ by an equation (1), then putting

$$h_{f,v}^{(n)}(E) = \min_i \max_j v(f_{ij}(E)) \tag{10}$$

defines a function on $C^{(n)}(K)\backslash\mathrm{supp}^{(n)}(D)$, where $\mathrm{supp}^{(n)}(D)$ denotes the set of positive divisors of degree $n$ which are not disjoint from $D$. As before, any two Weil heights on $C^{(n)}(K)\backslash\mathrm{supp}^{(n)}(D)$ belonging to the same divisor $D$ are equivalent; we denote the set of functions on $C^{(n)}(K)\backslash\mathrm{supp}^{(n)}(D)$ which are equivalent to some (hence all) $h_{f,v}^{(n)}$ by $\mathcal{H}_{D,v}^{(n)}$.

We then have

THEOREM 1′. *On each curve* C *defined over* K *exists a unique real-valued function* $\lambda_{C,v}$ on $(\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C))'$ *satisfying*:

(i) $\lambda$ *is bi-additive* (*when defined*);

(ii) $\lambda((f), D) = v(f(D))$, *if* $(f) \in \mathrm{Div}(C)$ *and* $D \in \mathrm{Div}^\circ(C)$ *are disjoint*;

(iii) *for any divisor $E$ on $C$ of degree $n > 0$ and any* $D \in \mathrm{Div}^\circ(C)$ *disjoint from $E$, the real-valued function $h_{D,E,v}^{(n)}$ defined on* $C^{(n)}(K)\backslash\mathrm{supp}^{(n)}(D)$ *by*

$$h_{D,E,v}^{(n)}(E') = \lambda_{C,v}(E' - E, D) \tag{11}$$

*is a height function on $C^{(n)}(K)$ associated to $D$, i.e., $h_{D,E,v}^{(n)} \in \mathcal{H}_{D,v}^{(n)}$.*

*Moreover, $\lambda_{C,v}$ also satisfies properties* (iv) *and* (v) *of Theorem* 1 *as well as*:

(vi) *If $K'$ is an algebraic extension of $K$, $v'$ an extension of $v$ to $K'$, $C' = C \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$ the curve $C$ lifted to $K'$, and $b: C' \to C$ the "base—change" morphism, then for every pair $D$, $E \in \mathrm{Div}^\circ(C)$ of disjoint divisors we have*

$$\lambda_{C,v}(D, E) = \lambda_{C',v'}(b^*D, b^*E). \tag{12}$$

REMARKS. 1) In the above characterization of $\lambda_{C,v}$, it is possible to weaken property (iii) by requiring that the condition "$h^{(n)}_{D,E,v} \in \mathscr{H}^{(n)}_{D,v}$" holds only for $n = 1, \ldots, g = \text{genus}(C)$.

2) Although Theorem 1′ appears to be more general than Theorem 1, it is actually quite easy to deduce the former from the latter. (Use equation (12) as the definition of $\lambda_{C,v}$!)

**3. v-metrics on curves.** As before, let $C$ be a curve defined over a field $K$ endowed with an absolute value $|\ |_v$. In this section we shall consider certain metrics on the point-set $C(K)$ (assumed tacitly to be non-empty to avoid trivialities) which may be used to define the $v$-topology on $C(K)$.

For any metric $d$ on $C(K)$, we define the function $\lambda_d$ on $C(K) \times C(K)\backslash$ diagonal by

$$\lambda_d(P, Q) = -\log d(P, Q). \tag{13}$$

In what follows, we shall be interested in the following class of metrics on $C(K)$.

DEFINITION. A metric $d$ on $C(K)$ is called a $v$-metric if $\lambda_d$ is a Weil height associated to the diagonal divisor $\Delta_C$ on $C \times C$ with respect to $v$; i.e., if $\lambda_d \in \mathscr{H}_{\Delta_C, v}$.

REMARK. Clearly (by Proposition 1), any two $v$-metrics induce the same uniformity on $C(K)$; this uniformity will be called the $v$-uniformity on $C(K)$.

EXAMPLE. $C = \mathbf{P}^1$. Fix an identification $K_\infty = K \cup \{\infty\} \leftrightarrow P^1(K)$ by choosing a generator $t \in F = K(\mathbf{P}^1)$ of $F/K$ and putting $t(P_a) = a$, $a \in K_\infty$. We then define the function $\chi_v$ on $\mathbf{P}^1(K) \times \mathbf{P}^1(K)$ by

$$\chi_v(P_x, P_y) = \frac{|x - y|_v}{\max(1, |x|_v) \cdot \max(1, |y|_v)}, \tag{14a}$$

if $x, y \in K$, and by

$$\chi_v(P_x, P_\infty) = \chi_v(P_\infty, P_x) = \frac{1}{\max(1, |x_v|)}, \tag{14b}$$

if $x \in K$. We also put

$$\chi_v(P_\infty, P_\infty) = 0. \tag{14c}$$

It is then easy to see that $\chi_v$ is a metric on $\mathbf{P}^1(K)$, and, hence, a $v$-metric because we have the formula

$$\Delta = (t \otimes 1 - 1 \otimes t) - \min(0, (1 \otimes t)) - \min(0, (t \otimes 1)) \tag{15}$$

(Note that $K(\mathbf{P}^1 \times \mathbf{P}^1) = \text{Quot}(K(t) \otimes K(t))$.)

In order to construct $v$-metrics on an arbitary curve $C$, we proceed as follows. Choose a finite open affine covering $\mathscr{U} = \{U_i\}_{1 \leq i \leq n}$ of $C$, and, for each $U_i$, fix a closed immersion $\phi_i : U_i \to \mathbf{A}^m$, for some $m$. Then, if $X_1, \ldots, X_m$ denote the coordinate functions on $\mathbf{A}^m$, put $f_{ij} = \phi_i \circ X_j \in F$ and set

$$\chi_{\mathscr{U},v}(P, Q) = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \chi_v(f_{ij}(P), f_{ij}(Q)), \qquad \text{if } P, Q \in C(K), \tag{16}$$

where we now view each $f_{ij}$ as a function $f_{ij} : C(K) \to K_\infty \leftrightarrow \mathbf{P}^1(K)$. We shall call each such function $\chi_{\mathscr{U},v}$ the $v$-metric associated to the covering. $\mathscr{U}$. The term "$v$-metric" is justified by the following fact.

PROPOSITION 2. *Each $\chi_{\mathscr{U},v}$ as defined above is a $v$-metric on $C(K)$.*

PROOF. (SKETCH). Clearly, $\chi_{\mathscr{U},v}$ is pesudo-metric on $C(K)$ since $\chi_v$ is one on $\mathbf{P}^1(K)$. Next, observe that we have the formula (cf. Kani [7])

$$\begin{aligned} \Delta_C = \min_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} & ((f_{ij} \otimes 1 - 1 \otimes f_{ij}) \\ & - \min(0, (f_{ij} \otimes 1)) - \min(0, (1 \otimes f_{ij})) \end{aligned} \tag{17}$$

which generalizes (15). From (17) we can conclude on the one hand that $\chi_{\mathscr{U},v}$ is a metric and on the other hand that $\lambda_{\mathscr{U},v} = \lambda_{\chi_{U,v}} \in \mathscr{H}_{\Delta_C, v}$.

COROLLARY. *The $v$-uniformity on $C(K)$ is the weakest uniformity such that all $f \in F = K(C)$, viewed as maps $f : C(K) \to \mathbf{P}^1(K)$, are uniformly continuous when we endow $\mathbf{P}^1(K)$ with its $v$-uniformity (given by $\chi_v$ above).*

PROOF. For a fixed choice of a covering $\mathscr{U}$, the $\chi_{\mathscr{U},v}$—uniformity (i.e., the $v$-uniformity) is by construction the weakest uniformity on $C(K)$ such that all $f_{ij}$ are uniformly continuous. Since every $f \in F$ appears as a coordinate function of some suitable open cover $\mathscr{U}$, the assertion follows.

REMARKS. 1) Recall (cf. Lang [**10**]) that the $v$-topology on $C(K)$ is by definition the weakest topology on $C(K)$ such that all $f \in F$ are continuous. From the above corollary we therefore see that the topology on $C(K)$ induced by the $v$-uniformity is the $v$-topology.

2) In the non-archimedean case, Néron [**18**] constructed, on any variety $V$, a metric (which he called the $p$-adic metric). It is easy to see that in the case of a curve $V = C$, Néron's metric is a $v$-metric in the above sense. The advantage of the metric constructed above is, however, that it does not depend on a choice of a model over $\mathfrak{O}_v$ and hence is applicable also in the archimedean case (and also "globalizes" nicely).

If $d$ is any metric on $C(K)$, then the function $\lambda_d$ extends by bilinearity uniquely to a bilinear map (also denoted by $\lambda_d$) on $(Z'(C) \times Z'(C))'$. As we shall see in §5, it is possible to obtain Néron's pairing on $(\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C))'$ from any $v$-metric $d$ by applying a simple "averaging process" to $\lambda_d$. In order to be able to apply this process, however, we need to know the following crucial fact.

THEOREM 2. *Let $d$ be any $v$-metric on $C(K)$, where (for simplicity) $K$ is algebraically closed. Then*:

a) *For each $n \geqq 1$ there exists a constant $c_n \geqq 0$ such that the inequality*

$$|\lambda_d((f), D) - v(f(D))| \leqq c_n \tag{18}$$

*holds for all $f \in F^{\times}$ and $D \in \mathrm{Div}^{\circ}(C)$ disjoint from $(f)$ with* $\max(\deg(f)_{\infty}, \deg D_{\infty}) \leqq n$.

b) *For each $D \in \mathrm{Div}(C)$, the function $h_{D,d}$ defined on $C(K)\backslash\mathrm{supp}(D)$ by*

$$h_{D,d}(P) = \lambda_d(D, P) \tag{19}$$

*is a height function associated to $D$, i.e., $h_{D,d} \in \mathscr{H}_{D,v}$.*

PROOF. Easy by "nonstandard methods" (cf. Kani **[8]**).

**4. The averaging process.** In this section we present a general "averaging process" which will be used in the next section to refine a given $v$-metric to Néron's pairing. More precisely, we shall show that, given real-valued homomorphisms $\alpha$ and $\beta$ defined on an abelian group A and on a subgroup $B \subset A$, respectively, such that the restriction of $\alpha$ to $B$ "almost coincides" with $\beta$, it is possible (under suitable hypotheses) to refine $\alpha$ to a homomorphism $\hat{\alpha}$ on $A$ which "almost coincides" with $\alpha$ and whose restriction to $B$ does coincide with $\beta$.

In order to explain the term "almost coincides" which was used above, we shall consider the abelian group $A$ to be endowed with a filtration $\mathscr{S}$; by this we mean a sequence $\mathscr{S} = \{S_n\}_{n\geq 1}$ of increasing subsets of $A$ (i.e., $S_1 \subset S_2 \subset \cdots \subset A$) with the property that $\bigcup_n S_n = A$ and $S_n + S_m \subset S_{n+m}$, for all $n, m \geqq 1$. In that case we shall refer to the pair $(A, \mathscr{S})$ as a filtered abelian group.

DEFINITION. A real-valued function $f$ defined on a filtered abelian group $(A, \mathscr{S})$ is said to be weakly bounded (or $\mathscr{S}$-bounded) if $f$ is bounded on each subset $S_n$, $n \geqq 1$.

Two real-valued functions $f$ and $g$ on A are said to be weakly equivalent (or to "almost coincide") if their difference $f - g$ is weakly bounded on $A$. (Notation: $f \approx g$ or $f \approx_{\mathscr{S}} g$.)

REMARK. Note that if we give $A$ the trivial filtration ($S_n = A$, for all $n$), then the notions of weak boundedness and weak equivalence reduce to the usual notions of boundedness and equivalence (the latter as defined in §1 above).

PROPOSITION 3. ("Averaging Lemma") *Let $B \subset A$ be a subgroup of* *a filtered abelian group* $(A, \mathscr{S})$ *satisfying the following "co-compactness property":*

$$B + S_N = A, \qquad \text{for some } N \geqq 1. \tag{20}$$

*If* $\alpha$ *and* $\beta$ *are real-valued homomorphisms on* $A$ *and* $B$ *respectively such that* $\alpha$ *is a "weak extension" of* $\beta$ *in the sense that we have*

$$\alpha|_B \approx \beta \tag{21}$$

*then there exists a unique homomorphism* $\hat{\alpha} = \hat{\alpha}_\beta$ *on* $A$ *in the weak equivalence class of* $\alpha$ *which extends* $\beta$; *i.e., we have*

$$\hat{\alpha} \approx \alpha \tag{22a}$$

$$\hat{\alpha}|_B = \beta. \tag{22b}$$

PROOF (Sketch.) Uniqueness is clear, for if $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are two such homomorphisms, then $\hat{\alpha}_1 - \hat{\alpha}_2$ is a bounded homomorphism on $A/B$ (and hence is 0).

To prove existence, observe first that by (20) we have a retraction map $s : A \to B$ with the property that $(s - \mathrm{id}_A)(A) \subset S_N$. Thus, if for $a \in A$, $n \in N$ we put

$$\alpha_n(a) = \alpha(a) + \frac{1}{2^n}(\beta(s(2^n a)) - \alpha(s(2^n a))), \tag{23}$$

then one easily sees that

$$\hat{\alpha}(a) = \lim_{n \to \infty} \alpha_n(a) \tag{24}$$

converges and that $\hat{\alpha}$ satisfies the required properties.

REMARK. A more careful analysis of the proof shows that if (for each $n \geqq 1$) $\alpha$ and $\beta$ "differ by" $c_n$ on $S_n \cap B$, i.e., if $|\alpha(b) - \beta(b)| \leqq c_n$, for $b \in S_n \cap B$, then $\hat{\alpha}$ and $\alpha$ differ by $c_{N+n} + c_{3N}$ on $S_n$. (This fact will be important later on.)

COROLLARY. *If* $\alpha_i$ *and* $\beta_i$ $(i = 1, 2)$ *are real-valued homomorphisms on* $A$ *and* $B$ *respectively such that* $\alpha_i|_B \approx \beta_i$, $i = 1, 2$, *then*

$$(\alpha_1 \pm \alpha_2)^{\wedge}_{\beta_1 \pm \beta_2} = (\alpha_1)^{\wedge}_{\beta_1} \pm (\alpha_2)^{\wedge}_{\beta_2}. \tag{25}$$

**5. The Néron pairing via v-metrics.** We shall now show how the Néron pairing on $(\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C))'$ may be constructed by applying the averaging process of the previous section to the function $\lambda_d$, where $d$ is any $v$-metric on $C(K)$. Here, $K$ is (without loss of generality) an algebraically closed field.

To do this, fix a divisor $E \in \mathrm{Div}^\circ(C)$ and let $\mathrm{Div}^\circ(C)_E$ (resp. $\mathrm{Div}_\ell(C)_E$) denote the group of all divisors $D \in \mathrm{Div}^\circ(C)$ (resp. of all divisors $D \in$

$\mathrm{Div}_f(C)$) which are disjoint from $E$. On $\mathrm{Div}^\circ(C)$ (and hence on any subgroup) we have the "natural filtration" given by

$$S_n = \{D \in \mathrm{Div}^\circ(C)\colon \deg D_\infty \leqq n\}. \tag{26}$$

We observe then that by Riemann—Roch we have

$$\mathrm{Div}_f(C)_E + S_{2g} = \mathrm{Div}^\circ(C)_E, \tag{27}$$

where $g$ denotes the genus of $C$, so that the "co-compactness property" (20) is satisfied for $A = \mathrm{Div}^\circ(C)_E$ and $B = \mathrm{Div}_f(C)_E$.

Next, define the real-valued homomorphisms $\alpha_E$ and $\beta_E$ on $A$ and $B$ respectively by:

$$\alpha_E(D) = \lambda_d(D, E); \tag{28a}$$

$$\beta_E((f)) = v(f(E)). \tag{28b}$$

Now Theorem 2a) states precisely that $\alpha_E$ is a "weak extension" of $\beta_E$ (i.e., condition (21) of the Averaging Lemma is satisfied) so that we can conclude that there exists a unique homomorphism $\hat{\alpha}_E$ on $\mathrm{Div}^\circ(C)_E$ such that property (22) holds. Thus, if we put

$$\lambda_{C,v}(D, E) = \hat{\alpha}_E(D), \tag{29}$$

then $\lambda_{C,v}$ is the desired Néron pairing since it clearly satisfies properties (i) and (ii) of Theorem 1 and also satisfies property (iii) by Theorem 2b). We also observe that the uniqueness assertion of Theorem 1 is obvious since for any $\lambda$ satisfying (i)–(iii) we must have $\lambda(\cdot, E) \approx \alpha_E$.

Finally, let us prove properties (iv) and (v) of Theorem 1. For this, we first note that if $\phi\colon C' \to C$ is any finite covering of curves, then we have the following "projection formulae":

$$f(\phi_* D') = (\phi^* f)(D'), \qquad \text{if } f \in F,\ D' \in \mathrm{Div}(C'), \tag{30a}$$

$$f'(\phi^* D) = (\phi_* f')(D), \qquad \text{if } f' \in F' = K(C'),\ D \in \mathrm{Div}(C). \tag{30b}$$

From these we obtain on the one hand Weil's reciprocity formula:

$$f((g)) = g((f)), \tag{31}$$

if $f, g \in F^\times$ are such that $(f)$ and $(g)$ are disjoint (for by (30), we can reduce the problem to $C = \mathbf{P}^1$, where it is easily verified), from which property (iv) is immediate since $\lambda_d$ is symmetric. On the other hand, combining (30b) with the trivial fact

$$\mathscr{H}_{\phi^* E, v} \supset \mathscr{H}_{E,v} \circ \phi \tag{32}$$

yields property (v).

REMARK. The above (sketched) proof of Weil's reciprocity formula is much simpler than other proofs of the formula found in the literature (cf. e.g. Igusa [**5**]; Lang [**11**, p. 172]; Griffiths—Harris [**3**, p. 242]).

**6. Height pairings on** Div(**C**) $\times$ Div(**C**). We now turn to the problem of extending Néron's pairing $\lambda_{C,v}$ defined on $(\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C))'$ to $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$. Among the many such possible extensions we shall be interested only in those satisfying the conditions of the following definition.

DEFINITION. A $v$-height pairing on the curve $C$ is any real-valued map $\lambda$ defined on $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ satisfying:

(i) $\lambda$ is bi-additive (when defined);

(ii) $\lambda((f), D) = v(f(D))$, if $(f) \in \mathrm{Div}_\ell(C)$ and $D \in \mathrm{Div}^\circ(C)$ are disjoint;

(iii) for each $D \in \mathrm{Div}(C)$ and $n \geqq 1$, the function $h_{D,\lambda}^{(n)}$ defined on $C^{(n)} \setminus \mathrm{supp}^{(n)}(D)$ by

$$h_{D,\lambda}^{(n)}(E) = \lambda(D, E) \tag{33}$$

is a height function attached to $D$; i.e., $h_{D,\lambda}^{(n)} \in \mathscr{H}_{D,v}^{(n)}$;

(iv) $\lambda(D, E) = \lambda(E, D)$, if $D, E \in \mathrm{Div}(C)$ are disjoint.

REMARKS. 1) Clearly (by Theorem 1), any such height pairing is an extension of Néron's pairing $\lambda_{C,v}$.

2) If $K$ is algebraically closed, then property (iii) has to be verified only for $n = 1$.

We shall now show that such height pairings actually exist. For this, we first observe that we can reduce to the case of an algebracially closed field $K$.

LEMMA 1. *Let $K'$ and $v'$ be an extension of $K$ and $v$, respectively, and let $b$: $C' = C \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K') \to C$ be the base change morphism. If $\lambda'$ is any $v'$–height pairing on $C'$, then the function $\lambda$ defined on* $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ *by*

$$\lambda(D, E) = \lambda'(b^*D, b^*E) \tag{34}$$

*is a $v$–height pairing on $C$.*

PROOF. This is immediate from the formula

$$(b^*f)\,(b^*D) = f(D), \qquad f \in F = K(C),\ D \in \mathrm{Div}(C). \tag{35}$$

Next, we verify the existence of $v$-height pairings in the case $C = \mathbf{P}^1$, where we can exhibit such a pairing explicitly.

LEMMA 2. *Let $K$ be algebraically closed, and let $\chi_v$ be the $v$-metric on $\mathbf{P}^1$ defined by* (14). *Then the bilinear extension of $\lambda_v = -\log \chi_v$ to* $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ *is a $v$-height pairing on $\mathbf{P}^1$.*

REMARK. Recall that $\lambda_v = \lambda_{t,v}$ depends on an identification $\mathbf{P}^1(K) \leftrightarrow K_\infty$ (given by a rational function $t \in K(\mathbf{P}^1)$)!

Finally, to prove the existence of height functions for a general curve $C$, we shall fix a finite morphism $\phi: C \to \mathbf{P}^1$ and pull the height pairing back to a height pairing $\lambda_{\phi,v}$ on $C$. Such a pullabck exists and is unique in the following sense.

LEMMA 3. *If $\phi: C' \to C$ is a finite covering of curves and $\lambda$ is any height pairing on $C$, then there exists a unique height pairing $\lambda' = \lambda_{\phi,\lambda}$ on $C'$ such that the following "projection formula" holds:*

$$\lambda'(\phi^*D, E') = \lambda(D, \phi_*E'), \tag{36}$$

*for all $D \in \mathrm{Div}(C)$ and $E' \in \mathrm{Div}(C')$ which are $\phi$-disjoint.*

PROOF. (Sketch.) Suppose $\lambda'$ is a height pairing on $C'$ satisfying (36), If $D', E' \in \mathrm{Div}(C')$ are two disjoint divisors on $C'$, choose two disjoint divisors $D, E \in \mathrm{Div}(C)$ of positive degrees such that $\phi^*D$ and $E'$ (respectively, $\phi^*E$ and $D'$) are disjoint. Then, if $n = \deg \lambda$, $d = \deg D$, $e = \deg E$, $d' = \deg D'$, and $e' = \deg E'$, we have

$$\begin{aligned} \lambda'(D', E') &= \frac{1}{n^2de}\lambda_{C',v}(ndD' - d'\phi^*D, neE' - e'\phi^*E') \\ &+ \frac{e'}{ne}\lambda(\phi_*D', E) + \frac{d'}{nd}\lambda(D, \phi_*E') - \frac{d'e'}{n^2de}\lambda(D, E). \end{aligned} \tag{37}$$

which proves uniqueness.

To prove existence, define $\lambda'$ by (37) and check that the definition is independent of the choice of $D$, $E$ and that it satisfies the required properties. (Use the fact that $\phi_*\phi^*D = nD$.)

This, therefore, settles the existence of height pairings on curves. For later applications, however, we observe that in Lemma 3 we can "weaken" the hypothesis that $\lambda_{\phi,\lambda}$ be a heght pairing in the following way.

LEMMA 4. *Let $C$ be a curve defined over a field $K$ endowed with an absolute value $| \, |_v$, and fix a $v$-height pairing $\lambda_0$ on $\mathbf{P}^1$. Then for each non-constant morphism $\phi: C \to \mathbf{P}^1$ there exists a unique function $\lambda_\phi = \lambda_{\phi,\lambda_0}$ on* $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ *such that we have:*

(i) *$\lambda_\phi$ is bi-additive (when defined);*

(ii) *$\lambda_\phi((f), D) = v(f(D))$, if $(f) \in \mathrm{Div}_f(C)$ and $D \in \mathrm{Div}^\circ(C)$ are disjoint;*

(iii) *for any two morphisms $\phi, \phi': C \to \mathbf{P}^1$ we have $\lambda_\phi \approx \lambda_{\phi'}$ with respect to the filtration $\mathfrak{S} = \{S_n\}_{n\geq 1}$ on $\mathrm{Div}(C) \times \mathrm{Div}(C)$ given by*

$$S_n = \{(D_1, D_2): \deg(D_i)_0 \leqq n, \deg(D_i)_\infty \leqq n, i = 1, 2\}: \tag{38}$$

(iv) $\qquad \lambda_\phi(E, D) = \lambda_\phi(D, E)$, *if $D$, $E$ are disjoint;*

(v) *for any two $\phi$-disjoint divisors $D \in \mathrm{Div}(\mathbf{P}^1)$, $E \in \mathrm{Div}(C)$ we have the "projection formula"*

(39) $$\lambda_\phi(\phi^*D, E) = \lambda_0(D, \phi_*E).$$

*Moreover, each* $\lambda_\phi$ *is a* $v$*-height pairing on* $C$ *and also satisfies*

(vi) *If* $\phi'\colon C' \to C$ *is a covering of curves and* $\phi\colon C \to \mathbf{P}^1$ *is a finite morphism, then we have for* $\phi'$*-disjoint divisors* $D \in \mathrm{Div}(C)$ *and* $E' \in \mathrm{Div}(C')$ *the "projection formula":*

(40) $$\lambda_{\phi\cdot\phi'}(\phi'^*D, E') = \lambda_\phi(D, \phi'_*E').$$

PROOF. Existence is assured by Lemma 3. To prove uniqueness, it is enough to show that properties (i)–(v) above imply that each $\lambda_\phi$ is a $v$-height pairing of $C$, i.e., that for each prime divisor $P \in X^1(C)$ of $C$ and each $n \geqq 1$ we have $h^{(n)}_{P,\lambda_\phi} \in \mathscr{H}^{(n)}_{P,v}$. Now by property (iii) we see that (for $P$ fixed) it is enough to verify this for one morphism $\phi\colon C \to \mathbf{P}^1$. Thus, if (by Riemann-Roch) we choose $\phi$ such that $\phi^*P_0 = mP$, for some $P_0 \in \mathbf{P}^1(K)$ and $m > 0$, and observe that by property (v) we have $h^{(n)}_{mP,\lambda_\phi} = h_{P_0,\lambda_0} \circ \phi^{(n)}$, then, since $\lambda_0$ is a $v$-height pairing on $\mathbf{P}^1$, it follows by (32) that $h^{(n)}_{mP,\lambda_\phi} \in \mathscr{H}^{(n)}_{mP,v}$ and hence also that $h^{(n)}_{P,\lambda_\phi} \in \mathscr{H}^{(n)}_{P,v}$, as claimed.

We may summarize the above construction of $v$-height pairings as follows.

THEOREM 3. *Let* $K$ *be an algebraically closed field endowed with an absolute value* $|\ |_v$, *and let* $C$ *be a curve defined over* $K$. *Then for each* $f \in F \setminus K$ *there exists a unique map* $\lambda_{f,v}$ *on* $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ *such that:*

(i) $\lambda_{f,v}$ *is bi-additive (when defined);*

(ii) $\lambda_{f,v}((g), D) = v(g(D))$, *if* $(g) \in \mathrm{Div}_\ell(C)$ *and* $D \in \mathrm{Div}^\circ(C)$ *are disjoint;*

(iii) *for any pair* $f, f' \in F \setminus K$ *we have* $\lambda_{f,v} \approx \lambda_{f',v}$ *(with respect to the filtration defined by* (38));

(iv) $\lambda_{f,v}(E, D) = \lambda_{f,v}(D, E)$, *if* $D, E \in \mathrm{Div}(C)$ *are disjoint;*

(v) *We have*

(41a) $$\lambda_{f,v}((f)_\infty, P) = h_v(f(P)), \qquad \text{if } P \in C(K)\backslash \mathrm{supp}\,((f)_\infty)$$

(41b) $$\lambda_{f,v}((f)_0, P) = h_v(f(P)^{-1}), \qquad \text{if } P \in C(K)\backslash \mathrm{supp}((f)_0),$$

*where* $h_v$ *denotes the "basic height function" on* $K$ *defined by*

(42) $$h_v(x) = -\min(0, v(x)), \qquad \text{for } x \in K.$$

*Moreover, each* $\lambda_{f,v}$ *is a height pairing on* $C$ *and satisfies the following projection formula.*

(vi) *If* $\phi\colon C' \to C$ *is a covering of curves and* $f \in F \setminus K$, *then for* $\phi$*-disjoint divisors* $D \in \mathrm{Div}(C)$ *and* $E' \in \mathrm{Div}(C')$ *we have*

(43) $$\lambda_{\phi^*f,v}(\phi^*D, E') = \lambda_{f,v}(D, \phi_*E').$$

REMARKS. 1) When one is dealing with a "concrete" curve (e.g., modular curves, Fermat curves etc.), there is usually a distinguished morphism $\phi: C \to \mathbf{P}^1$. Thus, "concrete" curves carry a preferred height pairing $\lambda_{f,v}$.

2) As was already indicated in the introduction, we therefore obtain for each $f \in F \setminus K$ a system of "canonical heights" on $C$ which are defined by

$$(44) \qquad h_{D,f,v}(P) = \lambda_{f,v}(D, P), \qquad \text{if } P \in C(K) \setminus \operatorname{supp}(D).$$

These satisfy the following "rule of functoriality"

$$(45) \qquad h_{\phi^*D,\phi^*f,v} = h_{D,f,v} \circ \phi,$$

if $\phi: C' \to C$ is a covering of curves, $D \in \operatorname{Div}(C)$, and $f \in F \setminus K$.

Note, however, that in the case of an elliptic curve, none of these coincides with Néron's canonical height on $(\operatorname{Div}(C) \times \operatorname{Div}^\circ(C))'$ since Néron's is functorial with respect to morphisms between elliptic curves but not with respect to morphisms to $\mathbf{P}^1$.

**7. Properties of height pairings.** In this section we shall derive some (elementary) properties of height pairings. We begin by classifying the set of height pairings on a curve $C$.

PROPOSITION 4. *If $\lambda$ is any height pairing on $C$ and $b: X^1(C) \to \mathbf{R}$ is any weakly bounded function on the set $X^1(C)$ of prime divisors on $C$ (i.e., for each $n \geqq 1$, the restriction of $b$ to $S_n = \{P \in X^1(C): \deg P \leqq n\}$ is bounded), then the function $\lambda_b$ defined by*

$$(46) \qquad \lambda_b(P, Q) = \lambda(P, Q) + \deg(Q) \cdot b(P) + \deg(P) \cdot b(Q),$$

*on $X^1(C) \times X^1(C) \setminus$ diagonal, and extended by bilinearity to* $(\operatorname{Div}(C) \times \operatorname{Div}(C))'$ *is also a height pairing on $C$. Conversely, every height pairing $\lambda'$ on $C$ is of the form $\lambda' = \lambda_b$ for a suitable* (*unique*) *weakly bounded function $b$ on $X^1(C)$.*

This follows easily from the following general fact.

LEMMA. *Let $C$ be a curve defined over $K$ and let $a$ be a symmetric, bilinear, real-valued function on* $(\operatorname{Div}(C) \times \operatorname{Div}(C))'$ *satisfying*

$$(47) \qquad a(D, E) = 0, \qquad \text{if } D, E \in \operatorname{Div}^\circ(C) \text{ are disjoint.}$$

*Then there exists a unique real-valued function $b$ on $X^1(C)$ such that we have*

$$(48) \qquad a(P, Q) = \deg(Q) \cdot b(P) + \deg(P) \cdot b(Q),$$

*for all $P, Q \in X^1(C)$, $P \neq Q$.*

PROOF. (Sketch.) Choose three distinct prime divisors $P_1, P_2, P_3 \in X^1(C)$, and let $n_i = \deg(P_i)$, for $i = 1, 2, 3$. If we put

$$b(P) = \begin{cases} \frac{1}{n_1}(a(P, P_1) + \deg(P) \cdot c & \text{if } P \neq P_1 \\ -c & \text{if } P = P_1, \end{cases}$$

where $c = (1/(2n_2))(a(P_3, P_2 - P_1) - (1/n_3)a(P_1, P_2))$, then it is easy to check that (48) holds.

PROPOSITION 5. *If $\lambda$ is any $v$-height pairing and $d$ any $v$-metric on $C$, then $\lambda - \lambda_d$ is bounded on $(C(K) \times C(K)) \setminus$ diagonal.*

PROOF. It is enough to prove this for $\lambda = \lambda_{f,v}$. Pick $P_0, Q_0 \in \mathbf{P}^1(K)$. Then by (32) and Theorem 2b) we see that $\lambda(\cdot, \phi_f^* Q_0) - \lambda_d(\cdot, \phi_f^* Q_0)$ is bounded on $C(K) \setminus \operatorname{supp}(\phi_f^* Q_0)$ (and similarly for $\lambda(\phi_f^* P_0, \cdot) - \lambda_d(\phi_f^* P_0, \cdot))$, so it is enough to show that $\theta(P, Q) = \lambda_{C,v}(nP - \phi_f^* P_0, nQ - \phi_f^* Q_0) - \lambda_d(nP - \phi_f^* P_0, nQ - \phi_f^* Q_0)$ is bounded on $(C(K) \times C(K)) \setminus$ diagonal. This, however, is immediate from the construction of $\lambda_{C,v}$ and the Remark of §4, since $N = 2g$ is independent of $E$.

COROLLARY 1. *If $\lambda$ is any height pairing on $C$, then there exists a constant $c_1 = c_1(\lambda)$ such that we have*

$$\lambda(P, R) \geqq \min(\lambda(P, Q), \lambda(Q, R)) - c_1, \tag{49}$$

*for all $P, Q, R \in C(K)$.*

PROOF. If $d$ is a $v$-metric, then (49) holds for $\lambda_d$ in place of $\lambda$ with $c_1 = \log 2$. Applying Proposition 5 therefore yields the result.

COROLLARY 2. *If $\lambda$ is any height pairing on $C$, then there exist a constant $c_2 = c_2(\lambda)$ such that we have*

$$\lambda(P, Q) \geqq -c_2, \tag{50}$$

*for all $P, Q \in C(K)$ with $P \neq Q$.*

PROOF. First observe that if $C = \mathbf{P}^1$ and $\lambda = \lambda_v$, then (50) holds with $c_2 = 0$ (resp. $c_2 = -v(2)$) if $v$ is non-archimedean (resp. archimedean). Thus, on an arbitrary curve $C$, (50) holds for $\lambda_{\mathscr{U},v}$ with the same $c_2$, and hence also (with a different $c_2$) for any $\lambda$ by Proposition 5.

REMARK. The last corollary is useful in the proof of Mumford's theorem (Mumford [17]) on the sparseness of rational points on curves of genus $g \geqq 2$ (cf. Kani [6]).

**8. Canonical height pairings on curves.** In §6 we had established the existence of height pairings by exhibiting for each $f \in F \setminus K$ a height pairing $\lambda_{f,v}$ which is functorial with respect to the morphism $\phi_f$: $V \to \mathbf{P}^1$. Although these height pairings appear at a first glance to be quite "natural" (cf. Theorem 3) and certainly suffice for a precise theory of Weil heights, there actually exist "more refined" height pairings $\hat{\lambda}_{f,v}$ which (at least in the archimedean case) are more functorial than the $\lambda_{f,v}$. (In the non-archimedean case we shall see that $\hat{\lambda}_{f,v} = \lambda_{f,v}$.) We shall now construct these refined height pairings and characterize them by their properties.

In order to state the main theorem, it will be convenient to introduce a certain subgroup $\mathrm{Aut}(v) \subset PGl_2(K)$ which is naturally associated to the absolute value $\|_v$ on $K$. To define $\mathrm{Aut}(v)$, consider the "standard norm" $\mathbf{n}_v$ defined on the affine plane $\mathbf{A}^2(K)$ by

$$(51) \qquad \mathbf{n}_v(x) = \begin{cases} \max(|x_1|_v, |x_2|_v), & \text{if } v \text{ is non-archimedean} \\ \sqrt{|x_1|_v^2 + |x_2|_v^2}, & \text{if } v \text{ is archimedean} \end{cases}$$

and let

$$(52) \qquad \mathrm{Aut}(\mathbf{n}_v) = \{\alpha \in Gl_2(K) \colon \mathbf{n}_v(\alpha(x)) = \mathbf{n}_v(x)\ \forall x \in \mathbf{A}^2(K)\}$$

denote the automorphism group of $\mathbf{n}_v$. Then the image of the group $\mathrm{Aut}(\mathbf{n}_v)$ under the projection map $Gl_2(K) \to PGl_2(K)$ is the desired group $\mathrm{Aut}(v)$.

REMARK. It is easy to verify that

$$(53) \qquad \mathrm{Aut}(\mathbf{n}_v) = \begin{cases} Gl_2(\mathfrak{O}_v), & \text{if } v \text{ is non-archimedean} \\ U(2), & \text{if } v \text{ is archimedean and } K = \mathbf{C}. \end{cases}$$

In what follows, we shall always view $PGl_2(K)$ as acting on $K_\infty$ via fractional linear transformations; i.e., if $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Gl_2(K)$, then $\alpha(\infty) = a/c$ and $\alpha(x) = (ax+b)/(cx+d)$, if $x \in K$. If $C$ is any curve defined over $K$, and $F = K(C)$ denotes the function field of $C$, then this action clearly extends to an action of $F_\infty$ such that we have $\alpha(f)\,(P) = \alpha(f(P))$, for $P \in C(K)$. We observe that under this action, $F \setminus K$ is mapped into itself.

We are now ready to state the main theorem of this section.

THEOREM 4. *Let $K$ be an algebraically closed field endowed with an absolute value $|\ |_v$. (In the archimedean case, assume that $K = \mathbf{C}$ and that $|\ |_v$ is the usual absolute value.) Then on the category of curves defined over $K$, there is a unique way of assigning to each non-constant rational function $f \in K(C)\backslash K$ on a curve $C$ a real-valued map $\hat{\lambda}_{f,v}$ defined on* $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ *such that the following properties hold*:

(i) $\hat{\lambda}_{f,v}$ *is bi-additive* (*when defined*);

(ii) $\hat{\lambda}_{f,v}((g), D) = v(g(D))$, *if* $(g) \in \mathrm{Div}_\ell(C)$ *and* $D \in \mathrm{Div}^\circ(C)$ *are disjoint*;

(iii) *for any pair* $f, f' \in K(C)\backslash K$ *we have* $\hat{\lambda}_{f,v} \approx \hat{\lambda}_{f',v}$ *with respect to the filtration defined by* (38);

(iv) $\hat{\lambda}_{f,v}(E, D) = \hat{\lambda}_{f,v}(D, E)$, *if* $D, E \in \mathrm{Div}(C)$ *are disjoint*;

(v) $\hat{\lambda}_{\alpha(f),v} = \hat{\lambda}_{f,v}$, *for all* $\alpha \in \mathrm{Aut}(v)$;

(vi) $\hat{\lambda}_{f,v}((f)_0, (f)_\infty) = 0$;

(vii) *for each covering* $\phi$: $C' \to C$ *of curves and each* $f \in K(C)\backslash K$ *we have the projection formula*

$$\hat{\lambda}_{\phi^* f, v}(\phi^* D, E') = \hat{\lambda}_{f,v}(D, \phi_* E'), \tag{54}$$

*if* $D \in \mathrm{Div}(C)$ *and* $E' \in \mathrm{Div}(C')$ *are* $\phi$*-disjoint divisors.*

*Moreover, each* $\hat{\lambda}_{f,v}$ *is a height pairing on C and hence satisfies in addition*

(viii) *For any divisor* $D \in \mathrm{Div}(C)$ *and any* $n > 0$, *the real-valued function* $\hat{h}^{(n)}_{D,f,v}$ *defined on* $C^{(n)}(K) - \mathrm{supp}^{(n)}(D)$ *by*

$$\hat{h}^{(n)}_{D,f,v}(E) = \hat{\lambda}_{f,v}(D, E) \tag{55}$$

*is a height function on* $C^{(n)}(K)$ *associated to D, i.e.,* $\hat{h}^{(n)}_{D,f,v} \in \mathscr{H}^{(n)}_{D,v}$.

In order to prove this theorem, we shall first consider the case $C = \mathbf{P}^1$. For this, we shall prove the following proposition which is of some interest in itself:

PROPOSITION 6. *There exists a unique real-valued function* $\hat{\chi}_v$ *on* $K_\infty \times K_\infty$ *satisfying*:

a) $\hat{\chi}_v(x, y) \geqq 0$, *for all* $x, y \in K_\infty$, *with equality if and only if* $x = y$.

b) *For any four distinct points* $x, y, z, w \in K_\infty$ *we have*

$$\frac{\hat{\chi}_v(x, z) \cdot \hat{\chi}_v(y, w)}{\hat{\chi}_v(x, w) \cdot \hat{\chi}_v(y, z)} = |\mathrm{cr}(x, y, z, w)|_v, \tag{56}$$

*where* cr $(x, y, z, w)$ *denotes the* cross-ratio *of* $x, y, z, w$ *(i.e.,* cr $(x, y, z, w) = ((x - z)(y - w))/((x - w)(y - z))$, *if* $x, y, z, w \in K$, *with the usual conventions if one of* $x, y, z$, *or* $w = \infty$.)

c) $\hat{\chi}_v(\alpha(x), \alpha(y)) = \hat{\chi}_v(x, y)$, *for all* $x, y \in K_\infty$ *and* $\alpha \in \mathrm{Aut}(v)$.

d) $\hat{\chi}_v(y, x) = \hat{\chi}_v(x, y)$, *for all* $x, y \in K_\infty$.

e) $\hat{\chi}_v(0, \infty) = 1$.

REMARK. If $K$ is a local, non-archimedean field, then it is possible to prove a similar theorem for any quasi-character $\psi$ in place of the unramified quasi-character $\psi = |\ |_v$. In that case one has to replace $Gl_2(\mathfrak{O}_v)$ by a suitable congruence subgroup attached to $\psi$ (and the function $\mathbf{n}_v$ in formula (57) below by the new form of $Gl_2(K)$ attached to an induced representation associated with $\psi$); cf. Kani [**9**].

PROOF. (Uniqueness.) Let $\hat{\chi}_1$, $\hat{\chi}_2$ be two functions satisfying properties a) – e) and define the function a on $K_\infty \times K_\infty \setminus$ diagonal by $a(x, y) = \hat{\chi}_1(x, y)/\hat{\chi}_2(x, y)$. Then by properties b) and d) we see (as in the proof of Proposition 4) that there is a non-negative function $b$ on $K_\infty$ such that we have $a(x, y) = b(x) \cdot b(y)$, for all $x, y \in K_\infty$ with $x \neq y$. We want to show that $b(x) = 1$, for all $x \in K_\infty$; for this we distinguish two cases.

*Case* 1. ($v$ non-archimedean). If $x \in \mathfrak{O}_v$, then $\alpha = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in Gl_2(\mathfrak{O}_v)$.

So, by c) we have (since $\alpha(\infty) = \infty$, $\alpha(0) = x$) that $b(x)b(\infty) = b(0)b(\infty)$, i.e., that $b(x) = b(0)$. Similarly, if $x^{-1} \in \mathfrak{O}_v$, then $\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1/x \end{smallmatrix}\right) \in Gl_2(\mathfrak{O}_v)$ and so $b(x)b(0) = b(0)b(\infty)$ or $b(x) = b(\infty)$. Putting $x = 1$ yields $b(0) = b(\infty)$ and so by e) we have $b(0) = 1$. Thus, in all cases we obtain $b(x) = 1$, as claimed.

*Case* 2. ($v$ archimedean (i.e., $K = \mathbf{C}$)). Let $z \in \mathbf{C}^\times$. Then $\alpha = \left(\begin{smallmatrix} z/|z| & 0 \\ 0 & 1 \end{smallmatrix}\right) \in U(2)$ so that, by c), we have $b(z)b(\infty) = b(|z|)b(\infty)$, i.e.,

$$(*) \qquad b(z) = b(|z|), \quad \text{for all } z \in \mathbf{C}^\times.$$

Next, if $r \in \mathbf{R}_{+'}$, then putting $s = (1 + r^2)^{-1/2}$, we see that $\beta_r = \left(\begin{smallmatrix} s & rsi \\ rsi & s \end{smallmatrix}\right) \in U(2)$ and so by applying c) with $\alpha = \beta_r$, $x = \infty$, $y = 0$ we have $b(1/ri) \cdot b(ri) = b(\infty)b(0)$. From (*) and e) we therefore obtain

$$(**) \qquad b(1/z)b(z) = 1, \qquad \text{for all } z \in \mathbf{C}_\infty.$$

On the other hand, applying c) with $\alpha = \beta_r$, $x = \infty$, $y = -ri$ yields $b(1/ri)$ $b(0) = b(\infty)b(-ri)$ or $b(1/z)b(0) = b(\infty)b(z)$ by (*). Combining this with (**) and e), we can conclude $b(z)^2 = b(0)^2$, i.e.,

$$(***) \qquad b(z) = b(0), \text{ for all } z \in \mathbf{C}.$$

Finally, applying (**) with $z = 1$ gives $b(1) = 1$ and so we obtain that $b(0) = 1$ (by (***)) and $b(\infty) = 1$ (by e)). Thus, in all cases we have $b(z) = 1$, for $z \in \mathbf{C}_\infty$, as claimed.

(Existence.) Define the function $\hat{\chi}_v$ on $(\mathbf{A}^2(K)\backslash\{0\}) \times (\mathbf{A}^2(K)\backslash\{0\})$ by the rule

$$(57) \qquad \hat{\chi}_v(\boldsymbol{x},\boldsymbol{y}) = \frac{|\det(\boldsymbol{x}|\boldsymbol{y})}{n_v(\boldsymbol{x}) \cdot n_v(\boldsymbol{y})}, \qquad \text{if } \boldsymbol{x}, \boldsymbol{y} \in \mathbf{A}^2(K)\backslash\{0\},$$

where $\boldsymbol{x} \mid \boldsymbol{y}$ denotes the $2 \times 2$ matrix

$$(58) \qquad \boldsymbol{x} \mid \boldsymbol{y} = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}.$$

Then clearly $\hat{\chi}_v(c\,\boldsymbol{x}, \boldsymbol{y}) = \hat{\chi}_v(\boldsymbol{x}, c\boldsymbol{y}) = \hat{\chi}_v(\boldsymbol{x},\boldsymbol{y})$, if $c \in K^\times$, and so $\hat{\chi}_v$ defines a function on $K_\infty \times K_\infty$ (also denoted by $\hat{\chi}_v$) via the identification $(\mathbf{A}^2(K)\backslash\{0\})/K^\times \to K_\infty$ given by $(x_1, x_2) \to x_2/x_1$.

It is immediate that $\hat{\chi}_v$ satisfies properties a), d) and e). Moreover, we see that property c) holds since we have for $\alpha \in Gl_2(K)$ that

$$(59) \qquad \alpha(\boldsymbol{x})|\alpha(\boldsymbol{y}) = \alpha \cdot (\boldsymbol{x} \mid \boldsymbol{y}),$$

and that $|\det(\alpha)|_v = 1$ when $\alpha \in \mathrm{Aut}(\mathbf{n}_v)$. Finally, to prove b), we first observe that since both sides of (56) are $Gl_2(K)$—invariant (the right hand side by well-known properties of the cross-ratio, and the left hand side by construction and (59)), we may assume that $x, y, w, z \in K$. In that case we have

$$\frac{\hat{\chi}_v(x, z) \cdot \hat{\chi}_v(y, w)}{\hat{\chi}_v(x, w) \cdot \hat{\chi}_v(y, z)} = \frac{|x - z|_v \cdot |y - w|_v}{|x - w|_v \cdot |y - z|_v} = |\mathrm{cr}(x, y, z, w)|_v,$$

as claimed.

REMARK. From the above proof we see that in the non-archimedean case, $\hat{\chi}_v$ is just the $v$-metric defined earlier in §3 (cf. (14)); i.e., we have

$$\text{(60a)} \qquad \hat{\chi}_v = \chi_v, \qquad \text{if } v \text{ is non-archimedean.}$$

On the other hand, if $v$ is archimedean (and hence $K = \mathbf{C}$ and $| \, |_v$ is the usual absolute value by our conventions), then $\hat{\chi}_v$ is just the chordal metric on the Riemann sphere $\mathbf{C}_\infty$. Note that the chordal metric does not differ substantially from $\chi_v$; more precisely, we have

$$\text{(60b)} \qquad 1 \leqq \frac{\chi_v(x, y)}{\hat{\chi}_v(x, y)} \leqq 2, \qquad \text{for all } x, y \in \mathbf{C}_\infty \text{ with } x \neq y.$$

COROLLARY. *Let $f \in K(C)$ be a rational function with* $\deg(f)_\infty = 1$ *(and so in particular $C \simeq \mathbf{P}^1$). Then there exists a unique real-valued function $\hat{\lambda}_{f,v}$ on* $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ *satisfying properties* (i), (ii), (iv), (vi) *of Theorem* 4 *and the property*

(v′) $\hat{\lambda}_{f,v}(\alpha^* D, E) = \hat{\lambda}_{f,v}(D, \alpha_* E)$, *for all* $\alpha \in \mathrm{Aut}(v)$ *and all $\alpha$-disjoint divisors $D$, $E \in$* $\mathrm{Div}(C)$, *where we view* $PGL_2(K) \supset \mathrm{Aut}(v)$ *as acting on $C$ via the rule $f(\alpha(P)) = \alpha(f)(P)$, for $P \in C(K)$. Moreover, the function $\hat{\lambda}_{f,v}$ is a height pairing on $C$ and hence also satisfies property* (viii) *of Theorem* 4.

PROOF. Define $\hat{\lambda}_{f,v}$ on $C(K) \times C(K) \setminus$ diagonal by

$$\text{(61)} \qquad \hat{\lambda}_{f,v}(P, Q) = -\log \hat{\chi}_v(f(P), f(Q))$$

and extend to $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$ by bilinearity. Then clearly $\hat{\lambda}_{f,v}$ satisfies property (i) and also properties (iv), (v′) and (vi), since $\hat{\chi}_v$ satisfies properties c), d) and e) of Proposition 6, respectively. Moreover, $\hat{\lambda}_{f,v}$ also satisfies property (ii), since $\hat{\chi}_v$ satisfies property b) and we have

$$\text{(62a)} \quad \left(\frac{f-c}{f-d}\right)(P_a - P_b) = \mathrm{cr}(a, c, b, d), \qquad \text{if } a, b \in K_\infty,\ c, d \in K; \text{ and}$$

$$\text{(62b)} \qquad (f - c)\,(P_a - P_b) = \mathrm{cr}(a, b, c, \infty), \qquad \text{if } a, b, c \in K.$$

Finally, $\hat{\lambda}_{f,v}$ satisfies property (viii) since $\lambda_{f,v}$ does and we have $\hat{\lambda}_{f,v} \approx \lambda_{f,v}$ by (60).

Uniqueness is immediate from Proposition 6.

PROOF OF THEOREM 4. (Existence.) Let $C$ and $f \in K(C) \backslash K$ be given. Then there exists a unique morphism

(63a) $$\phi_f: C \to \mathbf{P}^1 \overset{\text{def}}{=} \operatorname{Proj}(K[X_0, X_1])$$

such that

(63) $$f = \phi_f^* X (\overset{\text{def}}{=} X \circ \phi_f),$$

where $X \in K(\mathbf{P}^1)$ denotes the rational function defined by $X = X_1/X_0$.

Let $\hat{\lambda}_{X,x}$ be the unique height pairing on $\mathbf{P}^1$ defined by the above corollary (associated to $X$). Then by Lemma 3 of §6 there exists a unique height pairing

(64) $$\hat{\lambda}_{f,v} = \lambda_{\phi_f, \hat{\lambda}_{X,v}}$$

on $C$ such that the projection formula (34) holds for $\hat{\lambda}_{f,v}$, $\hat{\lambda}_{X,v}$ and $\phi_f$. Then, since $\hat{\lambda}_{f,v}$ is height pairing on $C$, properties (i), (ii), (iii), (iv) and (viii) are immediate. Moreover, since $\hat{\lambda}_{\alpha(f),v} = \lambda_{\phi_f, \hat{\lambda}_{X\circ\alpha,v}}$ and $\hat{\lambda}_{X\circ\alpha,v} = \hat{\lambda}_{X,v}$, if $\alpha \in \operatorname{Aut}(v)$, we conclude the validity of (v). Finally, property (vi) follows from the projection formula and the fact that $\phi_f^*(X)_0 = (f)_0$, $\phi_f^*(X)_\infty = (f)_\infty$, and property (vii) follows since we have $\hat{\lambda}_{\phi'^* f,v} =^{\text{def}} \lambda_{\phi_f\circ\phi', \hat{\lambda}_{X,v}} = \lambda_{\phi', \hat{\lambda}_{f,v}}$ because $\lambda_{\phi', \hat{\lambda}_{f,v}}$, $\hat{\lambda}_{X,v}$ and $\phi_f \circ \phi'$ satisfy the projection formula.

(Uniqueness.) By Lemma 4 of §6 it is enough to show that $\hat{\lambda}_{f,v}$ is uniquely determined in the case $C = \mathbf{P}^1$ and $\deg (f)_\infty = 1$. This, however, follows from the uniqueness assertion of the above corollary, since properties (v) and (vii) together imply property (v′). (Use the fact that $\hat{\lambda}_{\alpha(X),v} = \hat{\lambda}_{X,v} \circ (\alpha \times \alpha)$.)

**9. Explicit construction of $\hat{\lambda}_{f,v}$: non-archimedean (discrete) case.** In this section we shall consider the special case that the field $K$ is endowed with a discrete absolute value $|\ |_v$ which, for convenience, we assume to be normalized; i.e., we assume that we have $v(t) = 1$ for any uniformizing element $t$. We shall denote the valuation ring and residue field of $v$ by $\mathfrak{O}$ and $k$, respectively.

As before, let $C$ be a curve defined over $K$ and let $f \in K(C)\backslash K$ be a nonconstant rational function on $C$. Our aim here is to show that the $v$-height pairing $\hat{\lambda}_{f,v} = \lambda_{f,v}$ on $C$, which was constructed "axiomatically" in the previous sections, may be explicitely calculated in terms of intersection numbers on a suitable model of $C/\mathfrak{O}$. In order to explain this more precisely, let us first make the following definition.

DEFINITION. If $C$ is a curve defined over $K$, then a (regular, resp. normal) model of $C/\mathfrak{O}$ is a pair $(\tilde{C}, j)$ where $\tilde{C}$ is a (regular resp. normal) two-dimensional scheme which is proper over $\mathfrak{O}$ and $j : C \to \tilde{C}$ is an immersion such that the following diagram commutes:

$$(65)\qquad \begin{tikzcd} C \arrow[r, "j"] \arrow[d, "\pi"'] & \tilde{C} \arrow[d, "\tilde{\pi}"] \\ \mathrm{Spec}(K) \arrow[r, "i"] & \mathrm{Spec}(\mathfrak{O}) \end{tikzcd}$$

Here, $\pi: C \to \mathrm{Spec}(K)$ and $\tilde{\pi}: \tilde{C} \to \mathrm{Spec}(\mathfrak{O})$ denote the structure morphisms and $i : \mathrm{Spec}(K) \to \mathrm{Spec}(\mathfrak{O})$ the map induced by the inclusion $\mathfrak{O} \subset K$.

On any regular two-dimensional scheme $\tilde{C}$ over $\mathfrak{O}$ we have an intersection pairing $(\ \cdot\ )_{\tilde{C}} = (\ \cdot\ )_{C/\mathfrak{O}}$ which is defined as follows (cf. Lichtenbaum [**14**], Shafarevich [**22**]):

1. If $\mathfrak{D}$, $\mathfrak{E} \in \mathrm{Div}(\tilde{C})$ are effective divisors which intersect properly in the sense that $\mathrm{supp}(\mathfrak{D}) \cap \mathrm{supp}(\mathfrak{E})$ is finite, i.e., $\mathfrak{D}$ and $\mathfrak{E}$ have no common components, then

$$(66\mathrm{a})\qquad (\mathfrak{D}\cdot\mathfrak{E})_{\tilde{C}} = \sum_x (\mathfrak{D}\cdot\mathfrak{E})_x.$$

where the sum extends over all points $x \in \mathrm{supp}(\mathfrak{D}) \cap \mathrm{supp}(\mathfrak{E})$ and

$$(66\mathrm{b})\qquad (\mathfrak{D}\cdot\mathfrak{E})_x = \dim_k \mathfrak{O}_x/(f, g),$$

where $\mathfrak{O}_x$ denotes the local ring of the (closed) point $x \in \tilde{C}$ and $f \in \mathfrak{O}_x$ and $g \in \mathfrak{O}_x$ denote the local equations at $x$ of the (Cartier) divisors $\mathfrak{D}$ and $\mathfrak{E}$, respectively.

2. If $\mathfrak{F}_i$ is a component of the special fibre $\tilde{\pi}^{-1}(s) = e_1\mathfrak{F}_1 + \cdots + e_r\mathfrak{F}_r = \mathfrak{F}$, then the self-intersection number $(\mathfrak{F}_i\cdot\mathfrak{F}_i)_{\tilde{C}}$ is defined uniquely by the rule

$$(67)\qquad (\mathfrak{F}_i\cdot\mathfrak{F}) = 0.$$

Extending the above definition by bilinearity, we therefore obtain a symmetric, bilinear map

$$(\ \cdot\ )_{\tilde{C}/\mathfrak{O}}: (\mathrm{Div}(\tilde{C}) \times \mathrm{Div}(\tilde{C}))'' \to \mathbf{Z},$$

where the $''$ denotes the subset of $\mathrm{Div}(\tilde{C}) \times \mathrm{Div}(\tilde{C})$ consisting of all pairs of divisors $\mathfrak{D}$, $\mathfrak{E} \in \mathrm{Div}(\tilde{C})$ which intersect suitably, i.e., whose common components lie entirely in the special fibre $\tilde{\pi}^{-1}(s)$.

Remark. From the above discussion it is by no means evident that the intersection pairing $(\ \cdot\ )_{\tilde{C}}$ assumes only integral values, since (67) merely gives $(\mathfrak{F}_i\cdot\mathfrak{F}_i) \in \mathbf{Q}$. One way to see that in fact $(\mathfrak{F}_i\cdot\mathfrak{F}_i)_{\tilde{C}} \in \mathbf{Z}$ is to make use of the rule (cf. Shafarevich [**22**]):

$$(68)\qquad ((f)_{\tilde{C}}\cdot\mathfrak{F}_i) = 0, \qquad \text{for all } f \in F = K(C) = K(\tilde{C}).$$

(Here, $(f)_{\tilde{C}} \in \mathrm{Div}(\tilde{C})$ denotes the principal divisor on $\tilde{C}$ defined by $f \in F$.)

Another, possibly more direct way, to see this is to use Lichtenbaum's definition of intersection numbers (cf. Lichtenbaum **[14]**).

In order to be able to compare the intersection pairing with the $\nu$-height pairing $\lambda_{f,v}$ on $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$, we need a homomorphism $h$: $\mathrm{Div}(C) \to \mathrm{Div}(\tilde{C})$ which transports divisors on $C$ to divisors on $\tilde{C}$. Now if $(\tilde{C}, j)$ is a model of $C/\mathfrak{O}$, then such a homomorphism is induced by $j$ by the rule

$$(69) \qquad j_*(P) = \overline{j(P)},$$

where $P \in X^1(C)$ denotes a prime divisor on $C$ and the $\overline{\phantom{x}}$ denotes the closure of the subscheme $j(P)$ of $\tilde{C}$ in $\tilde{C}$.

It is tempting to hope that the formula

$$(70) \qquad \lambda_{f,v}(D, E) = (j_*D, j_*E)_{\tilde{C}}$$

holds for a suitable choice of a model $(\tilde{C}, j)$ (depending on $f$). This hope seems to be supported by the following well-known fact.

LEMMA. *Let $(\tilde{C}, j)$ be model of $C/\mathfrak{O}$. Then*:

(i) *For any $f \in F = K(C)$ and any prime divisor $P \in X^1(C)$ on $C$ we have*

$$(71) \qquad ((f)_{\tilde{C}} \cdot j_*P)_{\tilde{C}} = v(f(P)).$$

(ii) *For any $D \in \mathrm{Div}(C)$, the function $h^{(n)}_{D,\tilde{C},j}$ defined by*

$$(72) \qquad h^{(n)}_{D,\tilde{C},j}(E) = (j_*D \cdot j_*E)_{\tilde{C}}$$

*on $C^{(n)} = \mathrm{supp}^{(n)}(D)$ is a height function associated to $D$, i.e., $h^{(n)}_{D,\tilde{C},j} \in \mathcal{H}^{(n)}_{D,v}$.*

As we shall see below, however, the above formula (70) cannot hold except in special cases (viz., when $C$ has good reduction at $v$) since $\lambda_{f,v}$ will not be integral-valued in general. Nevertheless, the above formula turns out to be "almost correct" since, by slightly modifying the homomorphism $j_*$ it is possible to arrive at a correct formula. Our aim, therefore, is two-fold; firstly, to identify the correct model of $C/\mathfrak{O}$ for our purposes and secondly, to explain how to redefine $j_*$ such that formula (70) holds.

Let us first consider the special case $C = \mathbf{P}^1$. In order to single out among the many $\mathfrak{O}$-isomorphism classes of models of $\mathbf{P}^1/\mathfrak{O}$ a preferred choice, let us fix a rational function $x \in F = K(\mathbf{P}^1)$ with $\deg(x)_\infty = 1$. Then there exists a unique morphism

$$(73a) \qquad j = j_x\colon \mathbf{P}^1_K \overset{\mathrm{def}}{=} \mathrm{Proj}(K[X_0, X_1]) \to \mathbf{P}^1_{\mathfrak{O}} \overset{\mathrm{def}}{=} \mathrm{Proj}(\mathfrak{O}[X_0, X_1])$$

such that diagram (65) commutes and such that we have

(73b)
$$x = j^*X \overset{\text{def}}{=} X \circ j,$$

where $X \in K(\mathbf{P}^1)$ denotes the rational function $X = X_1/X_0$ on $\mathbf{P}^1_{\mathfrak{O}}$. We call the pair $(\mathbf{P}^1_{\mathfrak{O}}, j_x)$ the $x$-model of $\mathbf{P}^1_K/\mathfrak{O}$.

PROPOSITION 7. *If $f \in K(\mathbf{P}^1)$ is a rational function on $\mathbf{P}^1$ with* $\deg\,(f)_\infty = 1$, *and $(\tilde{C}, j)$ denotes its f-model, then formula* (70) *holds.*

PROOF. First observe that both sides coincide on $(\mathbf{P}^1(K) \times \mathbf{P}^1(K)) \setminus$ diagonal, since both satisfy (the additive versions of) properties a) – e) of Proposition 6 when we identify $\mathbf{P}^1(K) \leftrightarrow K_\infty$ via $f$. Thus, by bi-additivity, they coincide on $(Z'(\mathbf{P}^1) \times Z'(\mathbf{P}^1))'$. Finally, since the notion of an $f$-model of $\mathbf{P}^1$ is compatible with base change, both sides of (70) "commute with base-change", and hence we see that both sides agree on $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$, as claimed.

We now pass to the case of a general curve $C$ defined over $K$ and a non-constant rational function $f \in F \setminus K$. As in the proof of Theorem 4, we let $\phi_f\colon C \to \mathbf{P}^1_K = \mathrm{Proj}\,(K[X_0, X_1])$ denote the unique morphism such that $f = \phi_f^* X$. Then there exists a unique normal model $(\tilde{C}_f, j_f)$ of $C/\mathfrak{O}$ and finite morphism

(74a)
$$\tilde{\phi}_f\colon \tilde{C}_f \to \mathbf{P}^1_{\mathfrak{O}} = \mathrm{Proj}(\mathfrak{O}[X_0, X_1])$$

such that

(74b)
$$\tilde{\phi}_f \circ j_f = j_X \circ \phi_f.$$

(In fact, $\tilde{C}_f$ is just the normalization of $\mathbf{P}^1_{\mathfrak{O}}$ in $F$.) We shall call $(\tilde{C}_f, j_f)$ the $f$-model of $C/\mathfrak{O}$. (Note that this generalizes the previous use of the term "$f$-model".)

(75)
$$\begin{tikzcd}
 & & \tilde{C} \arrow[d, "\phi'"] \arrow[dd, bend left=50, "\tilde{\phi}"] \arrow[ddd, bend left=70, "\tilde{\pi}"] \\
C \arrow[urr, hook, bend left, "j"] \arrow[rr, hook, "j_f"] \arrow[d, "\phi_f"'] \arrow[dd, bend right=50, "\pi"'] & & \tilde{C}_f \arrow[d, "\tilde{\phi}_f"] \\
\mathbf{P}^1_K \arrow[rr, hook, "j_X"] \arrow[d, "\rho"'] & & \mathbf{P}^1_{\mathfrak{O}} \arrow[d, "\tilde{\rho}"] \\
\mathrm{Spec}(K) \arrow[rr, hook, "i"] & & \mathrm{Spec}(\mathfrak{O})
\end{tikzcd}$$

Since $\tilde{C}_f$ need not be a regular scheme, we let $\phi'\colon \tilde{C} \to \tilde{C}_f$ be a desingularization of $\tilde{C}_f$ which always exists by Abhyankar [**1**], at least if the residue field $k$ of $\mathfrak{O}$ is perfect. (This will be tacitly assumed henceforth.) We thus have the following situation:

We shall now show how to express $\lambda_{f,v}$ in terms of intersection numbers on the model $(\tilde{C}, j)$. In order to do this, we shall construct a homomorphism $h = h_{\tilde{C},j,\tilde{\phi}}\colon \mathrm{Div}(C) \to \mathrm{Div}(C) \otimes \mathbf{Q}$ such that the formula

$$\lambda_{f,v}(D, E) = (h(D) \cdot h(E))_C \tag{76}$$

holds. Here, the symbol $(\ \cdot\ )_{\tilde{C}}$ denotes the unique bilinear extension of the intersection pairing to $((\mathrm{Div}(\tilde{C}) \otimes \mathbf{Q}) \times (\mathrm{Div}(\tilde{C}) \otimes \mathbf{Q}))''$.

In order to define $h$, we shall make use of the direct image map $\tilde{\phi}_*\colon \mathrm{Div}(\tilde{C}) \to \mathrm{Div}(\mathbf{P}^1_{\mathfrak{O}})$ induced by $\tilde{\phi} = \tilde{\phi}_f \circ \phi'$ (cf. Shafarevich [**22**], p. 97) which, for a prime divisor $\mathfrak{P}$ on $\tilde{C}$, is defined by

$$\tilde{\phi}_*(\mathfrak{P}) = \begin{cases} [\kappa(\mathfrak{P}); \kappa(\mathfrak{p})]\mathfrak{p}, & \text{if } \tilde{\phi}(\mathfrak{P}) = \mathfrak{p} \text{ is a prime divisor on } \mathbf{P}^1_{\mathfrak{O}} \\ 0 & \text{otherwise} \end{cases} \tag{77}$$

where $\kappa(\mathfrak{P})$ (resp. $\kappa(\mathfrak{p})$) denotes the residue field of $\mathfrak{P}$ (resp. of $\mathfrak{p}$). Note that since $\tilde{\phi}$ is not necessarily finite, it is possible that we actually have $\tilde{\phi}_*(\mathfrak{P}) = 0$ for some prime divisors $\mathfrak{P}$ on $\tilde{C}$.

For what follows, it will be convenient to decompose each divisor $\mathfrak{D} \in \mathrm{Div}(\tilde{C})$ into its fibre part and its non-fibre part according to the decomposition

$$\mathrm{Div}(\tilde{C}) = \mathrm{Div}_f(\tilde{C}) \oplus \mathrm{Div}_{nf}\tilde{C}), \tag{78}$$

where $\mathrm{Div}_f(\tilde{C})$ (resp. $\mathrm{Div}_{nf}(\tilde{C})$) denotes the subgroup of $\mathrm{Div}(\tilde{C})$ generated by the prime divisors $\mathfrak{P}$ on $\tilde{C}$ which are (resp., are not) contained in the special fibre $\pi^{-1}(s)$ of $\tilde{C}$. The projection maps of $\mathrm{Div}(\tilde{C})$ onto $\mathrm{Div}_f(\tilde{C})$ and $\mathrm{Div}_{nf}(\tilde{C})$ are denoted by $pr_f$ and $pr_{nf}$, respectively.

PROPOSITION 8. *There exists a unique homomorphism* $h = h_{\tilde{C},j,\tilde{\phi}}\colon \mathrm{Div}(C) \to \mathrm{Div}(\tilde{C}) \otimes \mathbf{Q}$ *satisfying*:

(i) $pr_{nf} \circ h = pr_{nf} \circ j_*$;

(ii) $\tilde{\phi}_* \circ h = (j_X)_* \circ (\phi_f)_*$;

(iii) *For any* $D \in \mathrm{Div}(C)$ *and any* $\mathfrak{A} \in \mathrm{Div}_f(\tilde{C})$ *we have*

$$(h(D) \cdot \mathfrak{A})_{\tilde{C}} = \frac{1}{n}(\tilde{\phi}_* h(D) \cdot \tilde{\phi}_*\mathfrak{A})_{\mathbf{P}^1_{\mathfrak{O}}} \tag{79}$$

*where* $n = \deg \phi_f$.

REMARK. Not that formula (79) represents a partial "sharpening" of the usual projection formula for $\phi$ (cf, Shafarevich [**22**, p. 97]) since we have $\tilde{\phi}_*\tilde{\phi}^*\mathfrak{A} = n\mathfrak{A}$, for $\mathfrak{A} \in \mathrm{Div}(\mathbf{P}^1_{\mathfrak{O}})$.

PROOF. Let $D \in \mathrm{Div}(C)$ be of degree $d$. Then, by property (i), we must have

$$h(D) = j_*D + \mathfrak{A}_D, \tag{80}$$

for some divisor $\mathfrak{A}_D \in \mathrm{Div}_f(\tilde{C}) \otimes \mathbf{Q}$ which by property (ii) satisfies the condition

$$\tilde{\phi}^*\mathfrak{A}_D = 0. \tag{81}$$

Finally, let us analyse property (iii). For this let (as before) $\mathfrak{F}_1, \ldots, \mathfrak{F}_r$ denote the components of the special fibre

$$\mathfrak{F} = e_1\mathfrak{F}_1 + \cdots + e_r\mathfrak{F}_r = \tilde{\pi}^{-1}(s); \tag{82}$$

note that if $\mathfrak{f} = \tilde{\rho}^{-1}(s)$ denotes the special fibre of $\tilde{\rho}$ on $\mathbf{P}^1_{\mathfrak{D}}$, then $\mathfrak{f}$ is reduced and irreducible and we have

$$\mathfrak{F} = \tilde{\phi}^*\mathfrak{f}. \tag{83}$$

If we now define the integers $f_i \geqq 0$, $1 \leqq i \leqq r$, by the rule

$$\tilde{\phi}_*\mathfrak{F}_i = f_i\mathfrak{f}, \tag{84}$$

then (in view of property (ii)) property (iii) is equivalent to the conditions

$$(h(D)\cdot\mathfrak{F}_i)_{\tilde{C}} = \frac{f_i d}{n}, \qquad 1 \leqq i \leqq r, \tag{85}$$

since $((j_X)_*(\phi_f)_*D\cdot\mathfrak{f})_{\mathbf{P}^1_{\mathfrak{D}}} = d$. Thus, if we write $A_D = x_1\mathfrak{F}_1 + \cdots + x\mathfrak{F}_r$, with $x_i \in \mathbf{Q}$, and let $d_i = (j_*D \cdot \mathfrak{F}_i)_{\tilde{C}}$ and $a_{ij} = (\mathfrak{F}_i \cdot \mathfrak{F}_j)_{\tilde{C}}$, then (81) and (85) are equivalent to the following system of $r + 1$ linear equations in $r$ unknowns.

$$\begin{aligned} f_1x_1 + \cdots + f_rx_r &= 0 \\ a_{11}x_1 + \cdots + a_{r1}x_r &= \frac{f_1 d}{n} - d_1 \\ \vdots \qquad\qquad & \qquad \vdots \qquad\quad \vdots \\ a_{1r}x_1 + \cdots + a_{rr}x_r &= \frac{f_r d}{n} - d_r \end{aligned} \tag{86}$$

We therefore see that the proof of the proposition will be complete once we have shown that the system (86) has a unique solution. To see this, we first observe that, since the special fibre $\pi^{-1}(s)$ is connected, the intersection pairing $(\,\cdot\,)_{\tilde{C}}$ is negative definite and non-degenerate on the quotient space $W =^{\mathrm{def}} (\mathrm{Div}_f(\tilde{C}) \otimes \mathbf{Q})/\mathbf{Q}\mathfrak{F}$ (cf. Shafarevich **[22]**, p. 92 or Gross **[4]**) and that, hence (by elementary linear algebra) the system

$$Ax = y \tag{87}$$

has a solution $\boldsymbol{x}$ if and only if we have

$$\sum_{i=1}^{r} e_i y_i = 0. \tag{88}$$

In that case the general solution of (87) has the form

$$\boldsymbol{x} = \boldsymbol{x}_0 + s\boldsymbol{e}, \tag{89}$$

where $x_0$ is a particular solution of (87), $s \in \mathbf{Q}$, and $e = (e_1, \ldots e_r)^t$. Thus, since we have

$$\sum e_i d_i = d, \tag{90}$$

$$\sum e_i f_i = n, \tag{91}$$

we see that condition (88) holds for $y_i = (f_i d)/n - d_i$ and that, hence, (87) has a solution $\boldsymbol{x}'$. Moreover, if we let $s = (\sum f_i x_i')/n$, then we see by (91) that $\boldsymbol{x} = \boldsymbol{x}' + s\boldsymbol{e}$ is the unique solution of (86).

REMARK. From the above proof we see that we have in fact $h(D) \in \mathrm{Div}_{nf}(\tilde{C}) + (\mathrm{Div}_f(\tilde{C}) \otimes \mathbf{Q})$ and that if we let $\alpha = \alpha_{ij} \neq 0$ denote any $(i,j)$-th cofactor of the matrix $A = (a_{ij}) = ((\mathfrak{F}_i \cdot \mathfrak{F}_j)_{\tilde{C}})$, then we have

$$n\alpha \cdot h(D) \in \mathrm{Div}(\tilde{C}). \tag{92}$$

COROLLARY 1. *If $\mathfrak{F} = \mathfrak{F}_1$ is reduced and irreducible, then $h = j_*$. In particular, if $C$ has good reduction with respect to $f$ (i.e., if $\tilde{\rho} \circ \tilde{\phi}_f$ is smooth), then $h_{\tilde{C}_f, j_f, \tilde{\phi}_f} = (j_*)_f$.*

REMARK. It is possible to have $\mathfrak{F} = \mathfrak{F}_1$ without $C$ having good reduction at $v$. For example, this happens in the case of the "Fermat quotients"

$$C: \qquad y^p = x^s(1 - x), \qquad 0 < s < p - 1,$$

when $K = \mathbf{Q}$, $v = v_p$ and $\beta_s^2 \not\equiv \beta_s \pmod p$, where $\beta_s = s^s/(s + 1)^{(s+1)}$.

PROOF (of Corollary 1). Clearly $j_*$ satisfies properties (i) and (ii) of Proposition 8. Moreover, $j_*$ also satisfies (iii), since $f_1 = n$ (by hypothesis), and we have $(j_* D \cdot \mathfrak{F}) = \deg D$.

COROLLARY 2. *For any $g \in F = K(C)$ we have*

$$h((g)) = (g)_{\tilde{C}} - s\mathfrak{F}, \tag{93}$$

*where $s = s_g \in \mathbf{Q}$ is defined by*

$$s_g = \frac{1}{n} v_f(\mathcal{N}_{F/K(f)}(g)). \tag{94}$$

*Here, $v_f$ denotes the valuation on $K(\mathbf{P}^1)$ defined by the prime divisor $f$ on $\mathbf{P}^1_{\mathfrak{O}}$.*

PROOF. Put $\mathfrak{D} = (g)_C - s\mathfrak{F}$. By the uniqueness assertion of Proposition 8, it is enough to verify the following three properties:

(i) $pr_{nf}\mathfrak{D} = j_*(g)$.

(ii) $\tilde{\phi}_*\mathfrak{D} = (j_X)_*(\phi_f)_*(g)$.

(iii) $(\mathfrak{D} \cdot \mathfrak{A})_{\tilde{C}} = 0, \quad$ if $\mathfrak{A} \in \mathrm{Div}_f(\tilde{C})$.

Of these, property (i) is trivial, and (iii) follows from (67) and (68). For (ii), we observe that $\tilde{\phi}_*\mathfrak{D} = (\mathcal{N}_{F/K(f)}(g))_{\tilde{C}} - \phi_*(s\mathfrak{F}) = [(j_X)_*(\mathcal{N}_{F/K(f)}(g)) + ns\mathfrak{f}] - ns\mathfrak{f} = (j_X)_*(\phi_f)_*(g)$.

COROLLARY 3. *We have*

$$h \circ \phi_f^* = \tilde{\phi}^* \circ (j_X)_*. \tag{95}$$

PROOF. Let $D \in \mathrm{Div}(\mathbf{P}_K^1)$. Then:

i) $pr_{nf}\tilde{\phi}^*(j_X)_*D = pr_{nf}h(\phi_f^*D) = j_*\phi_f^*D$.

ii) $\tilde{\phi}_*(\tilde{\phi}^*j_*D) = n(j_X)_*D = (j_X)_*(\phi_f)_*\phi_f^*D$.

iii) By the projection formula (cf. Shafarevich [**22**], p. 97]) we have $(\tilde{\phi}^*(j_X)_*D\cdot\mathfrak{F}_i)_{\tilde{C}} = ((j_X)_*D\cdot\tilde{\phi}_*\mathfrak{F}_i) = \deg D \cdot f_i = f_i\cdot\deg(\phi_f^*D)/n$. From the uniqueness assertion of Proposition 8, we therefore obtain $h(\phi_f^*D) = \tilde{\phi}_*(j_X)_*D$, as claimed.

We are now ready to state and prove the main result of this section.

THEOREM 5. *Let $C$ be a curve defined over $K$ and let $f \in F\backslash K$. If $(\tilde{C}, j)$ is a desingularization of the $f$-model $(\tilde{C}_f, j_f)$ of $C$ over $\mathfrak{O}$ and $h = h_{\tilde{C},j,\tilde{\phi}}$:* $\mathrm{Div}(C) \to \mathrm{Div}(\tilde{C}) \otimes \mathbf{Q}$ *is as defined in proposition* 8, *then we have*

$$\lambda_{f,v}(D, E) = (h(D)\cdot h(E))_{\tilde{C}}, \tag{96}$$

*for any two disjoint divisors $D$, $E \in$* $\mathrm{Div}(C)$.

PROOF. We first observe that if $D, E \in \mathrm{Div}(C)$ are disjoint, then $h(D)$ and $h(E)$ intersect suitably (by the construction of $h$) and, hence, $\lambda(D, E) =^{\mathrm{def}} (h(D)\cdot h(E))_{\tilde{C}}$ is defined.

Next, we check that $\lambda$ is a $v$-height pairing on $C$. For this we note that properties (i) and (iv) of the defintion of a height pairing (cf. §6) are trivial, since $h$ is a homomorphism and $(\,\cdot\,)_C$ is symmetric. Property (ii) is a special case of the following more general formula

$$\lambda((g), D) = v(g(D)) - s_g\cdot\deg D, \tag{97}$$

where $s_g$ is as defined by (94). To prove (97), we observe that by Lemma i) above, formulae (67) and (68) and Corollary 2 we have

$$\begin{aligned}\lambda((g), D) &= ([(g)_{\tilde{C}} - s_g\mathfrak{F}]\cdot[j^*D + A_D])_C \\ &= ((g)_{\tilde{C}} \cdot j_*D)_{\tilde{C}} - s_g\cdot(\mathfrak{F}\cdot j_*D)_{\tilde{C}} \\ &= v(g(D)) - s_g\cdot\deg D.\end{aligned}$$

Finally, to prove property (iii), we observe that in view of Lemma ii), it is enough to verify

$$(98) \qquad \lambda \approx \lambda',$$

where $\lambda'$ is defined by $\lambda'(D, E) = (j_*D \cdot j_*E)_{\tilde{C}}$. To prove (98), write $h(D) = x_1\mathfrak{F}_1 + \cdots + x_r\mathfrak{F}_r$, where the $x_i$ satisfy (86) with $d_i = (j_*D \cdot \mathfrak{F}_i)$ and $d = \deg D$. Similarly, write $h(E) = y_1\mathfrak{F}_1 + \cdots + y_r\mathfrak{F}_r$ and $e_i = (j_*E \cdot \mathfrak{F}_i)$. Then

$$(99) \qquad \lambda(D, E) - \lambda'(D, E) = \sum_{i=1}^{r} (x_ie_i + y_id_i) + \sum_{i,j} a_{ij}x_iy_j.$$

Now if $\deg D_0 \leqq m$, and $\deg D_\infty \leqq m$, then we have $-m \leqq d_i \leqq m$, and hence only finitely many $r$-tuples $(d_1, \cdots, d_r)$ and $(x_1, \cdots, x_r)$ can occur. Similarly, if also $\deg E_0 \leqq m$ and $\deg E_\infty \leqq m$, then we see that the right hand side of (99) assumes only finitely many values; in particular, we obtain $\lambda \approx \lambda'$, as claimed.

This therefore proves that $\lambda$ is a $v$-height pairing. By definition of $\lambda_{f,v}$ (cf. (64) and (60a)), the proof of Theorem 5 will be complete once we have shown that the projection formula $\lambda(\phi_f{}^*D, E) = \lambda_v(D. (\phi_f)_*E)$ holds. To see this, we use Corollary 3, the projection formula for $(\,\cdot\,)_{\tilde{C}}$ (cf. Shafarevich [**22**, p. 97]), property (ii) of $h$, and Proposition 7 to obtain

$$\begin{aligned} \lambda(\phi_f^*D, E) &= (h(\phi_f^*D)\cdot h(E))_{\tilde{C}} \\ &= (\tilde{\phi}^*(j_X)_*D\cdot h(E))_{\tilde{C}} \\ &= ((j_X)_*D\cdot \tilde{\phi}_*h(E))_{\mathbf{P}^1_{\mathfrak{D}}} \\ &= ((j_X)_*D\cdot (j_X)_*(\phi_f)_*E)_{\mathbf{P}^1_{\mathfrak{D}}} \\ &= \lambda_v(D, (\phi_f)_*E). \end{aligned}$$

REMARKS. 1) It is possible to simplify equation (99) slightly so that we obtain

$$(99') \quad \lambda_{f,v}(D,E) - (j_*D\cdot j_*E)_{\tilde{C}} = (j_*D\cdot\mathfrak{A}_E)_{\tilde{C}} = (\mathfrak{A}_D\cdot j_*E)_{\tilde{C}} = -(\mathfrak{A}_D\cdot\mathfrak{A}_E)_{\tilde{C}},$$

for any two disjoint divisors $D, E \in \mathrm{Div}(C)$. This is immediate from the observation that, by using property (iii) of Proposition 8 and (81), we have $(h(D) \cdot \mathfrak{A}_E)_{\tilde{C}} = 0$, or $(j_*D\cdot\mathfrak{A}_E)_{\tilde{C}} = -(\mathfrak{A}_D\cdot\mathfrak{A}_E)_{\tilde{C}}$.

2) If $D = P$ and $E = Q$ are effective divisors of degree 1 (i.e., $P, Q \in C(K)$, then (99′) may be simplified still further; we have

$$(99'') \qquad \lambda_{f,v}(P. Q) = (j_*D\cdot j_*E)_{\tilde{C}} + x_k^{(k')},$$

where $\boldsymbol{x}^{(j)} = (x_1^{(j)}, \cdots, x_r^{(j)})^t$ denotes the solution of (86) with $d_i = f_i/n - \delta_{ij}$ ($\delta_{ij}$ denoting the Kronecker delta), and $k$ and $k'$ are defined by the conditions $(j_*P \cdot \mathfrak{F}_i) = \delta_{ij}$, $(j_*Q \cdot \mathfrak{F}_i) = \delta_{ik'}$, $1 \leqq i \leqq r$, respectively.

3) From (99″) we obtain an explicit version of Corollary 2 of Proposition 5 in the case $\lambda = \lambda_{f,v}$; viz., we see that (50) holds with $c_2 = -\min_{i,j} x_i^{(j)}$. Note also that, since $x_k^{(k)} = (j_*P \cdot \mathfrak{A}_P) = -(\mathfrak{A}_P \cdot \mathfrak{A}_P) \geqq 0$, we have $\lambda_{f,v}(P, Q) \geqq 0$ when $(j_*P \cdot \mathfrak{F}_i) = (j_*Q \cdot \mathfrak{F}_i)$, for all $i$. (In fact, we even have in that case that $\lambda_{f,v}(P, Q) > 0$ unless $C$ has good reduction with respect to $f$ !)

4) We also see from (99″) that for a suitable choice of an intersection maxtrix $A = (a_{ij})$ we have $\lambda_{f,v}(P, Q) \notin \mathbf{Z}$, as was claimed at the beginning of this section.

**10. Explicit construction of $\hat{\lambda}_{f,v}$: archimedean case.** In this section we shall consider the case that $K$ is endowed with an archimedean absolute value $|\ |_v$, so that without loss of generality we may assume that $K = \mathbf{C}$ and that $|\ |_v$ is the usual absolute value on $\mathbf{C}$. If $C$ is a curve defined over $K$, then we may view $C$ as a compact Riemann surface, and so by the "potential theory" of compact Riemann surfaces (cf., e.g., Arakelov [**2**, §3]), one can associate to each positive (1, 1)–form $d\mu$ on $C$ a Green's function $G = G_{d\mu}$ defined on $(C(K) \times C(K))\backslash$diagonal. We shall show here that the canonical $v$-height pairing $\hat{\lambda}_{f,v}$ on $C$ which was constructed "axiomatically" in §6 is (up to a constant term) the bilinear extension $\lambda_G$ of the Green's function $G = G_{d\mu}$ to $(\mathrm{Div}(C) \times \mathrm{Div}(C))'$, for a suitable choice of a (1, 1)–form $d\mu$ (depending on $f$).

Let us begin by recalling Arakelov's definition of a Green's function on a compact Riemann surface.

Proposition 9. (Arakelov [**2**]) *Let $d\mu$ be a* (1, 1)*–form on $C$ which is positive on $C$ except for the presence of a finite number of zeros and which satisfies $\int_C d\mu = 1$. Then there exists a unique function $G = G_{d\mu}$ on $(C(K) \times C(K))\backslash$diagonal such that we have*:

1) *The function $\mathcal{G} = \exp G$ is smooth on $(C(K) \times C(K))\backslash$diagonal*;

2) *$\mathcal{G}$ has a first order zero on the diagonal*;

3) *For each point $P \in C(K)$, we have*

$$(100) \qquad -\frac{1}{2\pi}\Delta G(P, \cdot) = d\mu,$$

*where $\Delta$ denotes the Laplacian on $C$ (i.e., if $f$ is a function on $C$ and $z = x + iy$ is a local parameter, then $\Delta f = ((\partial^2 f)/(\partial^2 x) + (\partial^2 f)/(\partial^2 y))dxdy$).*

4) *$G$ is symmetric in $P$ and $Q$: $G(Q, \mathbf{P}) = G(P,Q)$, for all $P \neq Q$.*

5) *For each $P \in C(K)$ we have $\int_C G(P, z)d\mu(z) = 0$.*

Definition. The function $G = G_{d\mu}$ constructed in Proposition 9 above is called the Green's function on $C$ associated to the (1, 1)–form $d\mu$.

*Warning.* In Arakelov [**2**], a different sign convention was taken in (100); as a result, the Green's function which Arakelov associates to $d\mu$ is $-G_{d\mu}$.

COROLLARY. *Let* $D \in \mathrm{Div}(C)$ *be a divisor on* $C$. *Then there exists a unique function* $h_D = h_{D,d\mu}$ *on* $C(K)\setminus\mathrm{supp}(D)$ *such that*:

(i) $\exp h_D$ *is smooth on* $C(K)\setminus\mathrm{supp}(D)$.

(ii) *In a neighbourhood of any point* $P \in C(K)$ *we have*

$$\exp(-h_D) = |z|^{v_P(D)} \cdot u(z),$$

*where* $z$ *is a local parameter at* $P$ *and* $u(z)$ *a suitable smooth function with* $u(P) \neq 0$.

(iii) $-1/2\pi\, \Delta h_D = (\deg D) d\mu$.

(iv) $\int_C h_D d\mu = 0$.

Let us first consider the case $C = \mathbf{P}^1$. In that case $C = \mathrm{Proj}\,(K[X_0, X_1])$ is the Riemann sphere $\mathbf{C}_\infty$, on which we have a "canonical" positive (1, 1)–form $d\mu_X$ (namely, the Fubini-Study metric on $\mathbf{P}^1$) which is given in affine cordinates $z = x + iy$ by

$$d\mu_X = \frac{1}{2\pi} \frac{dx\, dy}{(1 + |z|^2)^2}. \tag{101}$$

It is then easily checked that we have $\int_{\mathbf{C}_\infty} d\mu_X = 1$ and that the function $\hat{\lambda}_v = -\log \hat{\chi}_v$, where $\hat{\chi}_v$ denotes the chordal metric on $\mathbf{C}_\infty$ (cf. Proposition 6 and the remark following its proof), satisfies properties 1)–4) of Proposition 9. On the other hand, in place of property 5) we find, for $P \in \mathbf{C}_\infty$, that

$$\int_{\mathbf{C}_\infty} \hat{\lambda}_v(P, z) d\mu_X(z) = \frac{1}{2}, \tag{102}$$

and thus we obtain

$$\hat{\lambda}_v = G_{d\mu_X} + \frac{1}{2}. \tag{103}$$

If $C$ is an arbitrary curve defined over $K = \mathbf{C}$ and if $f \in F \setminus K$, then we can pull the positive (1, 1)–form $d\mu_X$ on $\mathbf{P}^1 = \mathrm{Proj}(K[X_0, X_1])$ back to the (1, 1)–form

$$d\mu_f = \frac{1}{n} \phi_f^* d\mu_X \tag{104}$$

on $C$; here, as before, $\phi_f \colon C \to \mathbf{P}^1$ denotes the morphism attached to $f$ and $n = \deg \phi_f$. Note that $d\mu_f$ is positive on $C$ except for finitely many zeros located at the ramification points of $\phi_f$ and that we have $\int_C d\mu_f = 1$.

THEOREM 6. *If* $G_f = G_{d\mu_f}$ *denotes the Green's function attached to* $d\mu_f$ *defined by* (104), *then we have for any two disjoint divisors* $D, E \in \mathrm{Div}(C)$ *the formula*

$$\hat{\lambda}_{f,v}(D, E) = \lambda_{G_f}(D, E) + \frac{1}{2}\deg D \cdot \deg E. \tag{105}$$

PROOF. By Arakelov [2] we know that $\lambda_{G_f}$ is a height pairing on $C$, so it is enough to verify that $\lambda_{G_f}$ satisfies the projection formula, i.e., that we have

$$h_{\phi_f{}^*D, d\mu_f} = h_{D, d\mu_X} \circ \phi_f, \qquad \text{for } D \in \mathrm{Div}(\mathbf{P}^1), \tag{106}$$

in the notation of the above corollary. But this is immediate since $h_{D, d\mu_X} \circ \phi_f$ satisfies the same properties (i)–(iv) of the corollary as does $h_{\phi_f{}^*D, d\mu_X}$.

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