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ON THE DIVISOR SUM FUNCTION

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In memory of Robert A. Smith and Ernst G. Straus

1. Introduction. The function $\tau_k(n)$ representing the number of ways of expressing n as a product of k factors (the order of the factors being taken into account) has been studied since the time of Dirichlet. In contrast to this well-established function, the corresponding sum function $\sigma(n, k)$, which we define as the sum of the divisors corresponding to such factorizations of n, does not seem to have appeared in the literature. Indeed the only reference the authors can submit is their preliminary report [**9**].

We here formally define the divisor sum function $\sigma_r(n, k)$ for the rth powers of these divisors and obtain some identities (including two of a well-known Ramanujan type), and as an application obtain an asymptotic estimate for $\sum_{n\leq x} \sigma_a(n, 3)\sigma_b(n, 3)$ which may be new. We extend the definition of $\sigma_r(n, k)$ to the case when k is complex and obtain some asymptotic estimates for its summatory function. Towards the end, we introduce the notation of k-ply perfect numbers and raise some open problems.

2. Preliminaries. Let

$$\tau_k(n) = \sum_{d_1 d_2 \cdots d_k = n} 1$$

for k a positive integer, so that $\tau_k(n)$ denotes the number of ways of expressing n as a product of k factors, the order of the factors being taken into account. In particular, let

$$\tau(n) = \tau_2(n) = \sum_{d_1 d_2 = n} 1 .$$

It is clear that if $\zeta(s)$ stands for the Riemann zeta function, we have

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}, s = \sigma + it, \sigma > 1.$$

$\tau_k(n)$ is multiplicative in n, and if

$$n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}$$

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where the p's are distinct primes, then [19]

$$\tau_k(n) = \frac{(k + m_1 - 1)!}{m_1!(k - 1)!} \cdots \frac{(k + m_r - 1)!}{m_r!(k - 1)!}.$$

Though the function $\tau_k(n)$ has been studied since the time of Dirichlet, some authors, for example Beumer [1] do not seem to realize this fact. If $*$ denotes the Dirichlet product of two arithmetic functions $f(n)$ and $g(n)$ defined by

$$(f * g)(n) = \sum_{d_1 d_2 = n} f(d_1) g(d_2)$$

then defining $i_0(n) = 1, n = 1, 2, 3, \ldots$ we have

$$\tau_k(n) = \sum_{d \mid n} 1 \cdot \tau_{k-1}\left(\frac{n}{d}\right),$$

so that

$$\tau_k(n) = (i_0 * i_0 * \cdots * i_0)(n)$$

to k factors.

It is obvious that for all integers $k \geqq 2$, $\tau_k(n) = 0(n^\varepsilon)$, $\varepsilon > 0$. Define

$$T_k(x) = \sum_{n \leqq x} \tau_k(n).$$

When $k = 2$, we write $T(x)$ for $T_2(x)$. It is well known [19, Chapter 12] that

$$T_k(x) = x\, P_{k-1}(\log x) + \Delta_k(x).$$

where $P_{k-1}(\log x)$ is a polynomial in $\log x$ of degree $k - 1$ with constant coefficients. The exact order of the error function $\Delta_k(x)$ is still unknown. If we define α_k to be the least number such that for every positive ε we have

$$\Delta_k(x) = O(x^{\alpha_k + \varepsilon})$$

and if this is to be true for every $\varepsilon > 0$, then Kolesnik [13] showed (what is probably the best result for $k = 2$ so far) that $\alpha_2 \leqq 35/108$.

Probably, the best Ω_+ and Ω_- results for $\Omega_2(x)$ so far are due respectively to Hafner [7], Corrádi and Kátai [4], namely

$$\Delta_2(x) = \Omega_+\{(x \log x)^{1/4} (\log\log x)^{(3+2\log 2)/4} \cdot \exp(-c\sqrt{\log(\log\log x)})\}$$

and

$$\Delta_2(x) = \Omega_-\{x^{1/4} \exp c\,(\log\log x)^{1/4} (\log\log\log x)^{-3/4}\}.$$

Hafner gave more general Ω_+ results in [7].

There is a good deal of literature concerning $\Delta_k(x)$, into which we shall not go. We content ourselves by mentioning the conjecture [19], p. 270] that $\alpha_k = (k - 1)/2k$, $k = 2, 3, \ldots$, it is known [**19**], p. 273] that $\alpha_k \geqq (k - 1/2k$ for $k = 2, 3, \ldots$.

Significant results involving extension of the definition of $\tau_k(n)$ for real and complex values of k have been obtained by Oppenheim [**14**] in 1926 and by Iseki [**11**] and A. Selberg [**16**] in 1954. Since Selberg's results go beyond those of Iseki and since we use them later in our paper, we shall quote his main results below.

Define $\tau_z(n)$ for any complex z by the generating function

$$\sum_{n=1}^{\infty} \frac{\tau_z(n)}{n^s} = \zeta^z(s), \text{ Re } s > 1, \tag{2.1}$$

and set

$$T_z(x) = \sum_{n \leqq x} \tau_z(n). \tag{2.2}$$

Then Selberg proved the following:

THEOREM A.

$$T_z(x) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O[x(\log x)^{z-2}] \tag{2.3}$$

uniformly for $|z| \leqq A$, $x \geqq 2$, *the constant in O depending only on A.*

Selberg also proved:

THEOREM B. *Let*

$$f(s, z) = \sum_{n=1}^{\infty} \frac{b_z(n)}{n^s} \text{ for } \sigma > 1 \tag{2.4}$$

(*so that* $f(s, z)$ *is defined for* $\sigma > 1$); *let*

$$\sum_{n=1}^{\infty} \frac{|b_z(n)|}{n} (\log 2n)^{B+3}$$

be uniformly bounded for $|z| \leqq B$. *Further, put*

$$\{\zeta(s)\}^z f(s, z) = \sum_{n=1}^{\infty} \frac{a_z(n)}{n^s}, \sigma > 1. \tag{2.5}$$

Then we have

$$A_z(x) \overset{\text{def}}{\equiv} \sum_{n \leqq x} a_z(n) = \frac{f(1, z)}{\Gamma(z)} x(\log x)^{z-1} + O\{x(\log x)^{z-2}\}, \tag{2.6}$$

uniformly for $|z| \leqq B$, $x \geqq 2$.

3. The function σ_r(n, k). First we define this functions when k is a positive integer and later consider for complex k. Consider a factorization of n as a product of k factors:

$$n = d_1 d_2 \cdots d_k. \tag{3.1}$$

We define

$$\sigma_r(n, k) = \sum_{d_1 d_2 \cdots d_k = n} d_1^r, \; k = 1, 2, \cdots. \tag{3.2}$$

Additionally we define

$$\sigma_r(n, 0) = \begin{cases} 1, & n = 1 \\ 0, & \text{otherwise} \end{cases}. \tag{3.3}$$

In (3.2), d_1^r occurs as many times as there are $(k - 1)$ fold factorizations of n/d_1. Hence

$$\begin{aligned} \sigma_r(n, k) &= \sum_{d_1 | n} d_1^r \tau\left(\frac{n}{d_1}, k - 1\right) \\ &= \sum_{d | n} d^r \tau\left(\frac{n}{d}, k - 1\right). \end{aligned} \tag{3.4}$$

Let us define $i_r(n) = n^r$ for all n. Recall that $\tau(n, k) = i_0 * i_0 * \cdots * i_0(n)$, to k factors. Hence

$$\sigma_r(n, k) = i_r * i_0 * i_0 * \cdots * i_0(n), \; ((k - 1) \text{ factors } i_0). \tag{3.5}$$

We have immediately

$$\sigma_r(n, k + 1) = \sum_{d | n} \sigma_r(d, k), \tag{3.6}$$

Since the generating function for $i_r(n)$ is

$$\sum_{n=1}^{\infty} \frac{i_r(n)}{n^s} = \zeta(s - r), \; \mathrm{Re}(s - r) > 1,$$

we have

$$\sum_{n=1}^{\infty} \frac{\sigma_r(n, k)}{n^s} = \zeta(s - r)\zeta^{k-1}(s), \; \mathrm{Re}(s - r) > 1, \; R(s) > 1. \tag{3.7}$$

Using the product formulation of ζ, we get, after a routine computation, that

$$\begin{aligned} \sigma_r(p^i, k) = p^{ri} &+ \binom{k - 1}{1} p^{r(i-1)} \\ &+ \binom{k}{2} p^{r(i-2)} + \cdots + \binom{k + i - 2}{i}, \; i \geqq 0. \end{aligned} \tag{3.8}$$

When z is complex, we define $\sigma_r(n, z)$ by utilizing (3.4) thus:

$$\sigma_r(n, z) = \sum_{d | n} d^r \tau(n/d, z - 1). \tag{3.9}$$

This is equivalent to defining $\sigma_r(n, z)$ by the generating function

(3.10) $$\sum_{n=1}^{\infty} \frac{\sigma_r(n, z)}{n^s} = \zeta^{z-1}(s)\zeta(s - r).$$

4. Identities. Several arithmetical identities involving $\sigma_r(n, k)$ arise from manipulations of the generating function. We state three of these here without proof:

(4.1) $$\sum_{d|n} \sigma_r(d, k)\, \sigma_r\left(\frac{n}{d}, k\right) = \sum_{d|n} d^r\, \tau(d, 2)\, \tau\left(\frac{n}{d}, 2k - 2\right)$$

(4.2) $$\sum_{d|n} J_r(d)\sigma_r\left(\frac{n}{d}, k\right) = \sum_{d|n} d^r\, \tau(d, 2)\, \tau\left(\frac{n}{d}, k - 2\right) = \sum_{d|n} d^r \sigma_r\left(\frac{n}{d}, k - 1\right)$$

(4.3) $$\sum_{1 \le j \le n} \sigma_r(j, k)\left[\frac{n}{j}\right] = \sum_{j=1}^{n} j^r \sum_{1 \le \ell \le n/j} \tau(\ell, k - 1)\left[\frac{n}{\ell j}\right].$$

The last identity can also be obtained on utilizing an identity given by Buschman [**2**].

5. An extension of an indentity of Ramanujan. Ramanujan gave his now-famous identity ([**15**], equation 15)

(5.1) $$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s - a)\zeta(s - b)\zeta(s - a - b)}{\zeta(2s - a - b)}$$

provided the real parts of s, $s - a$, $s - b$, $s - a - b$, $2s - a - b$ are all > 1. This has been generalized by many authors and in several directions. However, a similar identity involving

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n, k)\sigma_b(n, k)}{n^s}$$

has not been considered before. Such an identity for any general value of k will be very complicated. However, we give below identities for the special cases $k = 3$, 4 and then give three applications.

THEOREM 5.2. *For the real parts of* s, $s - a$, $s - b$, $s - a - b$, *all* > 1,

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n, 3)\sigma_b(n, 3)}{n^s} = \zeta^3(s)\zeta^2(s - a)\zeta^2(s - b)\zeta(s-a-b) \prod_{p \text{ prime}} F(p^{-s})$$

where

$$F(x) = 1 + x - (2p^b + 2p^a + 4p^{a+b})x^2 + (4p^{a+b} + 2p^{a+2b} + 2p^{2a+b})x^3 - p^{2a+2b}x^4 - p^{2a+2b}x^5.$$

PROOF. From the definitions, we have

$$\sum_{n=1}^{\infty} \sigma_a(n, 3)x^{n-1} = \prod_{p \text{ prime}} \frac{1}{(1 - p^a x)(1 - x)^2}$$

and

$$\sum_{n=1}^{\infty} \sigma_b(n, 3)x^{n-1} = \prod_{p \text{ prime}} \frac{1}{(1 - p^b x)(1 - x)^2}.$$

Following the technique in Subbarao [17], we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_a(n, 3)\sigma_b(n, 3)x^{n-1} \\
&= \prod_{p \text{ prime}} \left[\left\{ \frac{1}{0!} \frac{\partial^0}{\partial t^0} \frac{t^2 \cdot 1 \cdot \dfrac{1}{(1 - p^b t)(1 - t)^2}}{(t - x)^2} \right\}_{t = p^a x} \right. \\
&\qquad \left. + \frac{1}{1!} \frac{\partial^1}{\partial t^1} \left\{ \frac{t^2 \cdot 1 \cdot \dfrac{1}{(1 - p^b t)(1 - t)^2}}{(t - p^a x)} \right\}_{t = x} \right] \qquad (5.3) \\
&= \prod_{p \text{ prime}} \left\{ \frac{p^{2a} x^2}{(1 - p^{a+b} x)(1 - p^a x)^2 (p^a - 1)^2 x^2} \right\} \\
&\qquad + \frac{\partial}{\partial t} \left\{ \frac{t^2}{(1 - p^b t)(1 - t)^2 (t - p^a x)} \right\}_{t = x}.
\end{aligned}$$

This, simplified, with p^{-s} replacing x, establishes the result given in the theorem.

THEOREM 5.4. *For real parts of* s, $s - a$, $s - b$, $s - a - b$, *all* > 1, *we have*

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n, 4)\sigma_b(n, 4)}{n^s} = \zeta^5(s)\zeta^3(s - a)\zeta^3(s - b)\zeta(s - a - b) \prod_{p \text{ prime}} f(p^{-s})$$

where

$$\begin{aligned}
f(x) = 1 &+ 4x + (1 - 9p^b - 9p^a - 9p^{a+b})x^2 \\
&+ (-3p^b + 3p^{2b} - 3p^a + 18p^{a+b} + 9p^{a+2b} + 3p^{2a} + 9p^{2a+b})x^3 \\
&+ (3p^{2b} + 9p^{a+b} - 3p^{a+3b} + 3p^{2a} - 9p^{2a+2b} - 3p^{3a+b})x^4 \\
&+ (-9p^{a+2b} - 3p^{a+3b} - 9p^{2a+b} - 18p^{2a+2b} + 3p^{2a+3b} \\
&\qquad - 3p^{3a+b} + 3p^{3a+2b})x^5 \\
&+ (9p^{2a+2b} + 9p^{2a+3b} + 9p^{3a+2b} - p^{3a+3b})x^6 \\
&- 4p^{3a+3b}x^7 - p^{3a+3b}x^8.
\end{aligned}$$

PROOF. To obtain the result of the theorem we again use the same technique as in the proceding theorem. We here compute

$$\prod_{p \text{ prime}} \left[\left\{ \frac{t^3 \cdot 1 \cdot \dfrac{1}{(1-p^b t)(1-t)^3}}{(t-x)^3} \right\}_{t = p^a x} + \frac{1}{2!} \left\{ \frac{\partial^2}{\partial t^2} \frac{t^3 \cdot 1 \cdot \dfrac{1}{(1-p^b t)(1-t)^3}}{(t-p^a x)} \right\}_{t = x} \right].$$

We omit the details of the computations as they are too long for insertion.

6. Applications of the identities (5.1) and (5.2). The identity (5.1) was utilized by Ramanujan and later by Wilson **[22]** and others to estimate sums such as $\sum_{n \leqq x} \sigma^2(n)$ and $\sum_{n \leqq x} \tau(n)\sigma_a(n)$. In a like manner, we can utilize the identities (5.2) and (5.4) to obtain estimates for

$$\sum_{n \leqq x} \sigma_a(n, k)\, \sigma_b(n, k),\ k = 3, 4.$$

Here we consider only the simplest cases, namely $a = b = 0$. The case $a > 0$, $b < 0$ is considered later in (7.3). We need the following:

LEMMA 6.1. *Let*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} (\sigma > 1),$$

where $a_n = 0(n^\varepsilon)$ *for every positive* ε. *Let*

$$f(s) = \zeta^k(s) g(s)$$

where k is a positive integer and $g(s)$ is regular and bounded in the half plane $\sigma \geqq \sigma_0 (\sigma_0 < 1)$. *Then*

$$\sum_{n \leqq x} a_n \sim \frac{g(1)}{(k-1)!} x(\log x)^{k-1}, (x \to \infty).$$

This follows from a result of Van der Corput **[20]** or even as a special case of Theorem B of Selberg given in (2.4).

With the help of this lemma and the identities (5.2) and (5.4), we obtain by standard arguments the following:

THEOREM 6.2.

$$\sum_{n \leqq x} \tau_3^2(n) \sim \frac{A_1}{8!} x \log^8 x$$

where

$$A_1 = \prod_{p \text{ prime}} \left\{ \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right) \right\}.$$

THEOREM 6.3.

$$\sum_{n\leq x} \tau_4^2(n) \sim \frac{A_2}{15!} x \log^{15} x$$

where

$$A_2 = \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^8 \left(1 - \frac{1}{p^2}\right)\left(1 + \frac{8}{p} + \frac{1}{p^2}\right).$$

REMARKS 6.4. The results (6.2) and (6.3) may not be new. (6.2) is certainly known. In fact, it was derived earlier by several authors, for example, Vinogradov **[21]** in 1938, Van der Corput **[20]** in 1939, and Titchmarsh **[18]** in 1942. In this connection it is interesting to recall the observation of Titchmarsh that he also calculated $\sum_{n\leq x} \tau_3^2(n)$ by the circle method by forming the appropriate generating function, then replacing it by its dominant part on each Farey are and integrating around a circle without estimating the error terms. What he got for $\sum_{n=1}^{\infty} \tau_3^2(n)e^{-2n\delta}$ was $(17A_1/2^{16}\cdot 21\delta)\log^8(1/\delta)$ whereas, from Theorem 6.2, it should be $(A_1/2\cdot 8!\delta)$ $\log^8(1/\delta)$. (Use partial summation to the result of Theorem 6.2.) Thus what he had was 255/256th of the correct value. In his words [18] p. 130 "it seems curious that the results should be so nearly right without being right!"

7. Some asymptotic estimates. For z complex we already defined $\sigma_r(n, z)$ in (3.9) and (3.10).

THEOREM 7.1. *For $x \geqq 2$, $r > 0$, we have uniformly for $|z| \leqq B$, (B arbitrary but positive and finite)*

$$\sum_{n\leq x} \sigma_{-r}(n, z + 1) = \frac{\zeta(r + 1)}{\Gamma(z)} x(\log x)^{z-1} + O\{x(\log x)^{z-2}\}.$$

PROOF. We apply Theorem 2.4 of Selberg, setting $b_n(z) = n^{-r} (r > 0)$, so that the function $f(s, z) = \sum_{n=1}^{\infty} n^{-s-r}$ is defined for Re $s > 1$.

The conditions of Theorem 2.4 are satisfied for $r > 0$. Utilizing (3.10), (2.5) and (2.6), we get the stated result.

In the next theorem we confine ourself to real values of r (though it can be extended to r complex).

THEOREM 7.2. *Let $r \geqq 0$ and $k \geqq 2$. Then*

$$\sum_{n\leq x} \sigma_r(n, k) = \frac{x^{r+1}}{r + 1} \zeta^{k-1}(r + 1) + \begin{cases} O(x^r) & \text{if } r > 1 \\ O(x(\log x)^{k-1}) & \text{if } r = 1. \end{cases}$$

PROOF.

$$\begin{aligned}\sum_{n\leqq x}\sigma_r(n,k) &= \sum_{n\leqq x}\sum_{d\delta=n} d^r\tau_{k-1}(\delta) = \sum_{d\delta\leqq x} d^r\tau_{k-1}(\delta)\\ &= \sum_{\delta\leqq x}\tau_{k-1}(\delta)\sum_{d\leqq x/\delta} d^r\\ &= \sum_{\delta\leqq x}\tau_{k-1}(\delta)\frac{\left(\frac{x}{\delta}\right)^{r+1}}{r+1} + O\left(\frac{x}{\delta}\right)^5\\ &= \frac{x^{r+1}}{r+1}\sum_{\delta\leqq x}\frac{\tau_{k-1}(\delta)}{\delta^{r+1}} + O\left(x^r\sum_{\delta\leqq x}\frac{\tau_{k-1}(\delta)}{\delta^r}\right)\\ &= \frac{x^{r+1}}{r+1}\sum_{\delta=1}^{\infty}\frac{\tau_{k-1}(\delta)}{\delta^{r+1}} + O\left(x^{r+1}\sum_{\delta>x}\frac{\tau_{k-1}(\delta)}{\delta^{r+1}}\right)\\ &\quad + O\left(x^r\sum_{\delta\leqq x}\frac{t_{k-1}(\delta)}{\delta^r}\right).\end{aligned}$$

Now, we can see by induction on s, (or as a corollary of theorem 7.3) that for $s \geqq 1$,

$$T_s(x) = \sum_{n\leqq x}\tau_s(n) = O(x(\log x)^{s-1}),\ x \geqq 2.$$

(For another proof see Lemma 12, Cohen [3]). Hence by partial summation we get

$$\begin{aligned}\sum_{n>x}\frac{\tau_s(n)}{n^{r+1}} &= \frac{T_s([x]+1)}{([x]+1)^{r+1}} - \sum_{n>x} T_s(n)\left(\frac{1}{n^{r+1}} - \frac{1}{(n+1)^{r+1}}\right)\\ &= O\left(\frac{(\log x)^{s-1}}{x}\right) + O\left(\sum_{n>x} n(\log n)^{s-1}\frac{1}{n^{r+2}}\right)\\ &= O(x^{-1}(\log x)^{s-1}) + O(x^{-r}(\log x)^{s-1}) = O(x^{-1}(\log x)^{s-1}).\end{aligned}$$

Note that the constant implied by the O term may depend on r and s. Also

$$\sum_{\delta\leqq x}\frac{\tau_s(\delta)}{\delta^r} = O(1)\ \textit{if}\ r > 1$$

while if $r = 1$ we get

$$\begin{aligned}\sum_{\delta\leqq x}\frac{\tau_s(\delta)}{\delta} &= O\left\{\frac{x(\log x)^{s-1}}{x} + \int_1^x \frac{t(\log t)^{s-1}}{t^2}\,dt\right\}\\ &= O(\log x)^s.\end{aligned}$$

Hence for $k \geqq 2$ and $r \geqq 1$ we have

$$\sum_{n\leq x}{}' \sigma_r(n, k) = \frac{x^{r+1}}{r+1}\zeta^{k-1}(r+1) + O(x^{r+1}x^{-1}(\log x)^{k-2})$$
$$+ \begin{cases} O(x^r) & \text{if } r > 1 \\ O(x(\log x)^{k-1}) & \text{if } r = 1 \end{cases}$$
$$= \frac{x^{r+1}}{r+1}\zeta^{k-1}(r+1) + \begin{cases} O(x^r (\log x)^{k-2}) & \text{if } r > 1 \\ O(x(\log x)^{k-1}) & \text{if } r = 1 \end{cases}.$$

where the constants in the O-terms depend on k (and r if $r > 1$).

THEOREM 7.3. *Let* $G(s)$ *be given by*

$$G(s) = \prod_{p \text{ prime}} F(p^{-s})$$

where $F(p^{-s})$ *is defined in the statement of theorem* (5.2). *If* $G(s)$ *is analytic for* Re $s > a + b + 1$, *where a and b are positive, then*

$$\sum_{n\leq x} \sigma_a(n, 3)\sigma_b(n, 3) \sim Cx^{a+b+1} \tag{7.4}$$

where $C = G(a + b + 1)\zeta^3(a + b + 1)\zeta^2(a + 1)\zeta^2(b + 1)$.

In particular, the result holds for $a = b = 1$. The proof of this which uses standard contour integration methods is omitted. Note that the series

$$\sum_{n-1}^{\infty} \sigma_a(n, 3)\sigma_b(n, 3)/n^s$$

converges absolutely for Re $s > a + b + 1$. The order of the error term involved in (7.4) will be considered in a separate paper.

8. A limiting value.

THEOREM 8.1

$$\lim_{n\to\infty}\left\{\frac{\sigma_n(n + b, k + 1)}{\sigma_n(n, k + 1)}\right\} = e^b.$$

PROOF. From (8) with t replacing p^{-r} we get

$$\frac{\sigma_r(p^a, k+1)}{t^{-a}} = 1 + kt + \frac{k(k+1)}{1\cdot 2}t^2 \cdots + \frac{k(k+1)\cdots(k+a-2)}{(a-1)!}t^{a-1}$$
$$+ \frac{k(k+1)\cdots(k+a-1)}{a!}.$$

From this we get at once

$$1 < \frac{\sigma_r(p^a, k+1)}{p^{ra}} < \frac{1}{(1-p^{-r})^k}.$$

Hence if $n = \prod_{p^a\|n} p^a$, we have

$$(8.2) \qquad 1 < \frac{\sigma_r(n, k+1)}{n^r} < \frac{1}{\prod_{p|n} (1 - p^{-r})^k} < \zeta^k(r).$$

This gives

$$\left(1 + \frac{b}{n}\right)^r \zeta^{-k}(r) < \frac{\sigma_r(n + b, k+1)}{\sigma_r(n, k+1)} < \left(1 + \frac{b}{n}\right)^r \zeta^k(r).$$

By setting $r = n$ we get

$$\left(1 + \frac{b}{n}\right)^n \zeta^{-k}(n) < \frac{\sigma_n(n + b, k+1)}{\sigma_n(n, k+1)} < \left(1 + \frac{b}{n}\right)^n \zeta^k(n).$$

Noting that $\lim_{n\to\infty} \zeta(n) = 1$ and $\lim_{n\to\infty}(1 + (b/n))^n = e^b$ we get the result

$$e^b \leqq \lim_{n\to\infty} \frac{\sigma_n(n + b, k+1)}{\sigma_n(n, k+1)} \leqq e^b.$$

Hence

$$\lim_{n\to\infty} \frac{\sigma_n(n + b, k+1)}{\sigma_n(n, k+1)} = e^b.$$

9. k-ply perfect and k-ply multiperfect numbers. The authors' interest in perfect numbers prompted them to extend the same as follows:

Let us say that an integer n is k-ply multiperfect if

$$(9.1) \qquad \sigma(n, k) \equiv 0 \pmod{n}$$

for a given $k \geqq 2$; i.e., $\sigma(n, k) = \lambda n$ for some integer $\lambda > 1$. If $\lambda = k$, the number will simply be called k-ply perfect. The usual perfect numbers are thus 2-ply perfect.

Some examples of k-ply perfect numbers are the following:

$$\sigma(4, 4) = 16; \ \sigma(10, 4) = 40; \ \sigma(14, 6) = 84;$$
$$\sigma(15, 8) = 120; \ \sigma(105, 3) = 315; \ \sigma(5487, 29) = 29.5487.$$

Proceeding as in Erdös [6] one is able to prove that the density of k-ply multiperfect numbers is zero.

As problems presenting themselves we mention.

(9.2) Is there any simple formula for k-ply perfect numbers?

(9.3) Are there infinitely many k-ply perfect numbers for any integer $k > 2$?

(9.4) Are there infinitely many k-ply multiperfect numbers?

10. The unitary analog. We recall that

$$\sigma_r^*(n) = \sum_{\substack{d_1 d_2 = n \\ (d_1, d_2) = 1}} d_1^r = \text{the sum of } r\text{-th powers of the unitary divisors of } n.$$

Let $n = p_1^{a_1} p_2^{a_2} \ldots p_\nu^{a_\nu}$ where $p_1, \ldots, p_\nu$ are distinct primes. The number of unitary divisros of n is $\tau^*(n) = 2^{\omega(n)}$ where $\omega(n)$ is the number of distinct prime divisors of n. Now $\sigma_r^*(n)$ can be generalized to the k-ply case by defining

$$\sigma_r^*(n, k) = \sum_{\substack{d_1 d_2 \cdots d_k = n \\ (d_i, d_j)=1, i \neq j, \\ (i, j=1, 2, \ldots, k)}} d_1^r. \tag{10.1}$$

Write $\sigma^*(n, k) = \sigma_1^*(n, k)$ and $\tau^*(n, k) = \tau_k^*(n) = \sigma_0^*(n, k)$. Clearly,

$$\tau_k^*(n) = k^{\omega(n)} \tag{10.2}$$

$$\sigma^*(n, k) = \prod_{p^a \| n} (p^a + k - 1) \tag{10.3}$$

These definitions can be extended to the case when k is a complex number, say z. Thus

$$\tau_z^*(n) = z^{\omega(n)} \tag{10.4}$$

$$\sigma^*(n, z) = \prod_{p^a \| n} (p^a + z - 1). \tag{10.5}$$

Obviously σ^* (n, z) is a polynomial in z of degree $\omega(n)$.
Now the sum $\sum_{n \leq x} z^{\omega(n)}$ has been estimated by Delange **[4]** as

$$\sum_{n \leq x} z^{\omega(n)} = x(\log x)^{z-1} F(z) + O(x(\log x)^{z-2})$$

where $x > 0$, z is an arbitrary complex number and large, and F is the entire function

$$F(z) = \frac{1}{\Gamma(z)} \prod_{p \text{ prime}} \left(1 - \frac{1}{p}\right)^z \left(1 + \frac{1}{p-1}\right).$$

The O-term is uniform in z for $|z| < B$, B any finite number. Using this one can obtain an asymptotic estimate for $\sum_{n \leq x} \sigma^*(n, k)$ but we shall not go into details.

We say that n is unitary perfect if $\sigma^*(n) = 2n$. So far only five such integers are known: 6, 60, 90, 87360 and a 24-digit number $2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$. It is trivial to show that there are no odd unitary perfect numbers. Whether the number of these numbers is finite or infinite is not inown.

A unitary multiperfect number n is one which, by definition, satisfies $\sigma^*(n) = kn$ for some integer $k > 2$. This is the same as asking if there exist a finite number of primes $p_1, \ldots, p_\nu$, and positive integers $a_1, \ldots, a_\nu$ such that

$$\prod_{i=1}^{\nu} \left(1 + \frac{1}{p_i^{a_i}}\right)$$

is an integer > 2. In 1974 the authors in **[10]** raised the problem of the existence of such numbers. They showed, apart from some theoretical results, that if there is such an integer n, it must be very very large, [9] and even.

Recently Hagis [9] continued our work and, in a recent paper, proved that if there is such an integer, it must be $> 10^{102}$ and must have > 44 distinct odd prime factors.

We shall not go into the question of k-ply unitary perfect numbers.

1. Some more open questions. Among the open problems that may be of interest we mention the following:

(11.1) to improve the order of the error term in theorem 7.2; and

(11.2) to obtain estimates for $\sum_{n \leq x} \sigma_r^t(n, k)$, at least in some simple cases: when $k = 3$, $t = 2$ (Formulas (3.4) and (3.6) should come in handy for this purpose). For $t = k = 2$, Ramanujan gave an estimate [**15**, eq. 19]. Theorem 7.3 is a result in this direction. However it is only a first attempt, and the true order of the error term is open.

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