DENSITY OF M-TUPLES WITH RESPECT TO POLYNOMIALS

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1. **Introduction.** Let h, k and m be positive integers with $m \geq 2$, x real and ≥ 1, and f_1, $f_2, \ldots, f_m$ arbitrary nonconstant polynomials with integer coefficients. Let $M(x; f_1, f_2, \ldots, f_m; h; k)$ denote the number of m-tuples $\langle x_1, x_2, \ldots, x_m \rangle$ of positive integers such that $x_i \leq x$ for $1 \leq i \leq m$ and $(f_1(x_1), f_2(x_2), \ldots, f_m(x_m))_k = h$. Here the symbol $(a_1, a_2, \ldots, a_m)_k$ stands for the greatest k-th power common divisor of $a_1, a_2, \ldots, a_m$ with the convention that $(0, 0, \ldots, 0)_k = 0$. We also write $d(f_1, f_2, \ldots, f_m; h; k) = \lim_{x \to \infty} x^{-m} M(x; f_1, f_2, \ldots, f_m; h; k)$ and call this the density of the m-tuples $\langle x_1, x_2, \ldots, x_m \rangle$ with $(f_1(x_1), f_2(x_2), \ldots, f_m(x_m))_k = h$. In the special case when $f_1(x) = f_2(x) = \ldots, f_m(x) = x$, $h = k = 1$, it is known due to Césaro [2], J. J. Sylvester [7], D. N. Lehmer [5] and J. E. Nymann [6] that this density is $1/\zeta(m)$, $\zeta(s)$ being Riemann's ζ-function. Recently, R. N. Buttsworth [1] determined this density in the general case with $k = 1$ but his proof contains some serious errors—for example, his lemma 2.2 basic to his work is fallacious (See Section 4). In this paper we evaluate this density in the following cases: (1) $k = 1$ and at least one of the polynomials is of degree less than m (2) $k \geq 2$ and at least one of the polynomials is linear and (3) $k \geq 1$ and the polynomials are primitive and irreducible. In fact, in each of the cases we obtain an asymptotic formula for $M(x; f_1, f_2, \ldots, f_m; h; k)$ with an 0-estimate for the error term (see Theorems 1, 2, and 3.)

2. **Preliminaries.** We denote by $\rho_i(n)$ the number of solutions mod n of the congruence

$$f_i(x) \equiv 0(\text{mod } n), \quad \rho(n) = \prod_{i=1}^{m} \rho_i(n), \quad \rho_i^*(n) = \max\{1, \rho_i(n)\}$$

and

$$\rho^*(n) = \prod_{i=1}^{m} \rho_i^*(n), \quad D_i = \deg f_i(x), \quad D = \prod_{i=1}^{m} D_i \text{ and } u = \min_{1 \leq i \leq m} D_i.$$

Also $\mu(n)$ denotes the Möbius function and $\omega(n)$ the number of distinct prime factors of n.

It is not hard to see that there exists a positive integer C such that $\rho_i(p) \leq C$ for $1 \leq i \leq m$ and for all primes p. Further, let

$$(2.1) \qquad a_k(n) = \mu^2(n)\rho^*(n^k).$$

Also we shall use Vinogradov notation wherever convenient.

LEMMA 2.1 *As* $x \to \infty$

$$(2.2) \qquad \sum_{n \leq x} a_k(n) \ll x^{(k-1)m+1}(\log x)^{C^m-1},$$

$$(2.3) \qquad \sum_{n > x} \frac{a_k(n)}{n^{km}} \ll x^{-m+1}(\log x)^{C^m-1}$$

and for $0 \leq r \leq m - 1$,

$$(2.4) \qquad \sum_{n \leq x} \frac{a_k(n)}{n^{kr}} \ll \begin{cases} x^{(k-1)m+1-kr}(\log x)^{C^m}, & \text{if } (k-1)m \geq kr - 1, \\ 1, & \text{otherwise.} \end{cases}$$

PROOF. First we prove that as $x \to \infty$

$$(2.5) \qquad \sum_{n \leq x} k^{\omega(n)} \ll x(\log x)^{k-1}.$$

The proof is by induction on k. For $k = 1$, (2.5) is readily seen to be true. Now assuming (2.5) for k and observing that

$$(k + 1)^{\omega(n)} = \sum_{d \mid n} \mu^2(d)k^{\omega(d)} \leq \sum_{d \mid n} k^{\omega(d)},$$

we have

$$\sum_{n \leq x}(k+1)^{\omega(n)} \leq \sum_{n \leq x}\sum_{d \mid n} k^{\omega(d)} \leq \sum_{d \leq x} k^{\omega(d)} \left[\frac{x}{d}\right] \leq x \sum_{d \leq x} \frac{k^{\omega(d)}}{d} \ll x(\log x)^k$$

where in the last step we used partial summation in addition to the inductive hypothesis. This proves (2.5).

We observe that $(m, n) = 1$ implies $\rho^*(mn) \leq \rho^*(m)\rho^*(n)$ and so for $k = 1$, we have for square free n,

$$a_1(n) = \prod_{i=1}^{m} \rho_i^*(n) \leq \prod_{i=1}^{m} \prod_{p \mid n} \rho_i^*(p) \leq (C^m)^{\omega(n)}$$

and (2.2) follows in this case by virtue of (2.5).

For $k > 1$, we have

$$\rho_i^*(n^k) = \max\{1, \rho_i(n^k)\} \leq \max\{1, n^{k-1}\rho_i(n)\} \leq n^{k-1}\max\{1, \rho_i(n)\} = n^{k-1}\rho_i^*(n)$$

and so again for square free n

$$(2.6) \qquad a_k(n) = \prod_{i=1}^{m} \rho_i^*(n^k) \leq n^{(k-1)m} \rho^*(n).$$

Now, (2.6), (2.2) for $k = 1$, and partial summation give (2.2) for $k > 1$. Finally, (2.3) and (2.4) follow from (2.2) by partial summation.

LEMMA 2.1*. *If the polynomials* $f_i(x)$ *are primitive and irreducible, then*

$$(2.2)^* \qquad \sum_{n \leq x} a_k(n) \ll x(\log x)^{D-1}$$

$$(2.3)^* \qquad \sum_{n > x} a_k(n)/n^{km} \ll x^{1-km}(\log x)^{D-1}$$

and for $0 < r \leq m - 1$

$$(2.4)^* \qquad \sum_{n \leq x} \frac{a_k(n)}{n^{kr}} \ll \begin{cases} (\log x)^D, & \text{if } k = r = 1 \\ 1 & \text{otherwise.} \end{cases}$$

PROOF. It is well known [4] that $\rho_i(n) \ll D_i^{\omega(n)}$ and so $\rho_i^*(n) \ll D_i^{\omega(n)}$ and $\rho^*(n) \ll D^{\omega(n)}$. Now since $\omega(n^k) = \omega(n)$, (2.2)* results from (2.5) and (2.3)* and (2.4)* result from (2.2)* by partial summation.

LEMMA 2.2. *Let* $f_1, f_2, \ldots, f_m$ *have positive leading coefficients and* $u = \min_{1 \leq i \leq m} \deg f_i(x)$. *Then for any* y *satisfying* $x^{u/k} \ll y \ll x^{u/k}$,

$$M(x; f_1, f_2, \ldots, f_m; H^k; k) = \frac{x^m}{H^{km}} \sum_{n=1}^{\infty} \frac{\mu(n)\rho(n^k H^k)}{n^{km}}$$

$$+ O\left(x^m \sum_{n > y} \frac{a_k(n)}{n^{km}}\right) + O\left(\sum_{r=0}^{m-1} x^r \sum_{n \geq y} \frac{a_k(n)}{n^{kr}}\right) + O(x^{m-1}).$$

PROOF. First we note that there exists an $X \geq 1$ such that $f_i(X) \geq 1$ and each $f_i(x)$ is increasing on $[X, \infty)$. For $x \geq X$, let

$$\lambda_k(x) = \min_{1 \leq i \leq m} \frac{(f_i(x))^{1/k}}{H}.$$

Now

$$M(x, f_1, f_2, \ldots, f_m, H^k, k) = \sum_{\substack{1 \leq x_i \leq x \\ (f_1(x_1), f_2(x_2), \ldots, f_m(x_m))_k = H^k}} 1$$

$$= \sum_{\substack{1 \leq x_i \leq x \\ H^k | f_i(x_i), 1 \leq i \leq m}} \sum_{\substack{d^k | (f_i(x_i)/H^k) \\ 1 \leq i \leq m}} \mu(d)$$

$$= \sum_{\substack{1 \leq x_i \leq x \\ f_i(x_i) \equiv 0 (d^k H^k) 1 \leq i \leq m}} \mu(d)$$

$$= \sum_{\substack{X \leq x_i \leq x \\ f_i(x_i) \equiv 0 (d^k H^k) 1 \leq i \leq m}} \mu(d) + \sum_{\substack{1 \leq x_i \leq X, 1 \leq i \leq m \\ f_i(x_i) \equiv 0 (d^k H^k) \text{ and} \\ x_i < X \text{ for at least one } i}} \mu(d)$$

$$= \sum_{d \leq \lambda_k(x)} \mu(d) \prod_{i=1}^{m} \left(\sum_{\substack{X \leq x_i \leq x, \\ f_i(x_i) \equiv 0 (d^k H^k) \\ 1 \leq i \leq m}} 1\right) + O(x^{m-1})$$

$$= \sum_{d \leq \lambda_k(x)} \mu(d) \prod_{i=1}^{m} \left\{ \frac{x}{d^k H^k} \rho_i(d^k H^k) + O(\rho_i^*(d^k H^k)) \right\}$$

$$+ O(x^{m-1})$$

$$= \sum_{d \leq \lambda_k(x)} \mu(d) \left\{ \frac{x^m}{d^{km} H^{km}} \rho(d^k H^k) \right.$$

$$\left. + O\left(\sum_{r=0}^{m-1} \frac{x^r}{d^{kr} H^{kr}} \binom{m}{r} \rho^*(d^k H^k) \right) \right\} + O(x^{m-1})$$

$$= \frac{x^m}{H^{km}} \sum_{d=1}^{\infty} \frac{\mu(d) \rho(d^k H^k)}{d^{km}}$$

$$+ O\left(x^m \sum_{d > \lambda_k(x)} \frac{\mu^2(d) H^k \rho^*(d^k)}{d^{km}} \right)$$

$$+ O\left(\sum_{r=0}^{m-1} x^r \sum_{d \leq \lambda_k(x)} \frac{\mu^2(d) H^k \rho^*(d^k)}{d^{kr}} \right) + O(x^{m-1}).$$

Since $\lambda_k(x) \ll y \ll \lambda_k(x)$, the lemma follows.

3. Main results.

THEOREM 1. *Let $f_1, f_2, \ldots, f_m$ be arbitrary nonconstant polynomials with integer coefficients and let $\min_{1 \leq i \leq m} \deg f_i(x) < m$. Then*

$$M(x; f_1, f_2, \ldots, f_m; h; 1) = \frac{x^m}{h^m} \sum_{n=1}^{\infty} \frac{\mu(n) \rho(nh)}{n^m} + O(x^{m-1} (\log x)^{Cm}).$$

PROOF. Without loss of generality we can suppose that the leading coefficients of the polynomials $f_i(x)$ are all positive. Now by Lemma 2.2 with $k = 1$, we have

$$M(x; f_1, f_2, \ldots, f_m; h; 1)$$

$$= \frac{x^m}{h^m} \sum_{n=1}^{\infty} \frac{\mu(n) \rho(nh)}{n^m} + O\left(x^m \sum_{n \gg x^u} \frac{a_1(n)}{n^m} \right)$$

(3.1)

$$+ O\left(\sum_{r=0}^{m-1} x^r \sum_{n \leq x^u} \frac{a_1(n)}{n^r} \right) + O(x^{m-1})$$

$$= \frac{x^m}{h^m} \sum_{n=1}^{\infty} \frac{\mu(n) \rho(nh)}{n^m} + O_1 + O_2 + O(x^{m-1}).$$

Now by (2.3) and (2.4),

(3.2) $$O_1 \ll x^{m-u(m-1)} (\log x)^{Cm-1}$$

and

(3.3)
$$O_2 \ll \sum_{r=0}^{1} x^r (x^u)^{1-r} (\log x)^{Cm} + \sum_{r=2}^{m-1} x^r \cdot 1$$

$$\ll x^u (\log x)^{Cm} + x^{m-1}.$$

Thus the theorem follows from (3.1), (3.2) and (3.3).

THEOREM 2. *Let $f_1, f_2, \ldots, f_m$ be arbitrary nonconstant polynomials with integer coefficients. If $k \geq 2$ and at least one of the polynomials is linear, then*

$$M(x; f_1, f_2, \ldots, f_m; H^k; k) = \frac{x^m}{H^{mk}} \sum_{n=1}^{\infty} \frac{\mu(n)\rho(n^k H^k)}{n^{mk}}$$

$$+ O(x^{m-1} + x^{((k-1)m+1)/k}(\log x)^{Cm}).$$

PROOF. Without loss of generality, we can suppose that the leading coefficients of the polynomials $f_i(x)$ are all positive. Then by Lemma 2.2

$$M(x; f_1, f_2, \ldots, f_m; h; k)$$

(3.4)
$$= \frac{x^m}{H^{km}} \sum_{n=1}^{\infty} \frac{\mu(n)\rho(n^k H^k)}{n^{km}} + O\left(x^m \sum_{n>x^{1/k}} \frac{a_k(n)}{n^{km}}\right)$$

$$+ O\left(\sum_{r=0}^{m-1} x^r \sum_{n \leq x^{1/k}} \frac{a_k(n)}{n^{kr}}\right) + O(x^{m-1}).$$

By (2.3), the first O-term on the right side of (3.4) is $O(x^{((k-1)m+1)/k}$ $(\log x^{Cm-1})$ while the second O-term is, by (2.4), $O(x^{((k-1)m+1)/k} (\log x)^{Cm} + x^{m-1})$. This completes the proof of Theorem 2.

THEOREM 3. *Let $f_1, f_2, \ldots, f_m$ be nonconstant polynomials with integer coefficients. If the polynomials are primitive and irreducible, then for $k \geq 1$*

$$M(x; f_1, f_2, \ldots, f_m; H^k; k) = \frac{x^m}{H^{km}} \sum_{n=1}^{\infty} \frac{\mu(n)\rho(n^k H^k)}{n^{km}} + O(x^{m-1}(\log x)^p).$$

PROOF. The proof of this theorem is similar to that of Theorems 1 and 2 except that we use Lemma 2.1* instead of Lemma 2.1, and we omit th details.

THEOREM 4. *Let $f_1, f_2, \ldots, f_m$ and the integer k satisfy the hypotheses of either Theorem 1, 2 or 3. Then $d(f_1, f_2, \ldots, f_m; h; k)$*

(3.5)
$$= \frac{1}{h^m} \sum_{n=1}^{\infty} \frac{\mu(n)\rho(n^k h)}{n^{km}}$$

(3.6)
$$= \frac{1}{h^m} \prod_{p^l \| h} \left\{\rho(p^l) - \frac{\rho(p^{l+k})}{p^{km}}\right\} \prod_{p \nmid h} \left\{1 - \frac{\rho(p^k)}{p^{km}}\right\}$$

(3.7)
$$= \frac{1}{h^m} \left(\sum_{d \mid h} \frac{\mu(d)\rho(d^k h)}{d^{km}}\right) \prod_{p \nmid h} \left\{1 - \frac{\rho(p^k)}{p^{km}}\right\}$$

if h is a kth power and $= 0$ otherwise.

PROOF. If h is not a kth power, then clearly $M(x; f_1, \ldots, f_m; h; k) = 0$ for all $x \geq 1$ and the assertion follows. So let h be a kth power. Then

(3.5) is an immediate consequence of Theorems 1, 2 and 3. To prove (3.6), we observe the following generalization of the familiar Euler's infinite product factorization Theorem (cf. [3], Theorem 286): If $f(n)$ and $g(n)$ are multiplicative arithmetical functions, h a positive integer with $h = \prod p^l$ and the series $\sum_{n=1}^{\infty} f(n)g(hn)$ converges absolutely, then

$$(3.8) \qquad \sum_{n=1}^{\infty} f(n)g(hn) = \prod_{p} \left\{ \sum_{m=0}^{\infty} f(p^m)g(p^{m+l}) \right\}.$$

In fact, a special case of this appears in E. C. Titchmarsh (cf. [8], p. 9). Now (3.6) follows from (3.5) and (3.8). Also (3.7) could be deduced from (3.6) in view of the following: If $F(n)$ and $G(n)$ are multiplicative functions and $h = \prod_{p} p^l$, then

$$(3.9) \qquad \sum_{d \mid h} \mu(d)F(d)G(dh) = \prod_{p^l \| h} \{ G(p^l) - F(p)G(p^{l+1}) \}.$$

It may be of interest to note that (3.9) could be deduced from (3.8) above by taking $\mu(n) F(n) \varepsilon_h(n)$ for $f(n)$ and $G(n)$ for $g(n)$ where the function ε_h is defined by

$$\varepsilon_h(n) = \begin{cases} 1 & \text{if } n \mid h, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of the theorem.

COROLLARY 4.1. *Let* $f_1, f_2, \ldots, f_m$ *and* k *satisfy the hypothesis of Theorem 1, 2 or 3. Then if* h *is a* k^{th} *power, the density* $d(f_1, f_2, \ldots, f_m; h; k) = O$ *iff either* $\rho(p^l)p^{km} = \rho(p^{l+k})$ *for some* $p \nmid h$, *i.e., either* $\rho_i(p^l) = 0$ *for some* i *and some* $p^l \| h$ *or* $\rho_i(p^{l+k}) = p^k \rho_i(p^l)$ *for* $1 \leq i \leq m$ *and some* $p^l \| h$ *or* $\rho_i(p^k) = p^k$ *for* $1 \leq i \leq m$ *and for some* $p \nmid h$.

4. REMARKS. Buttsworth [1] mentions in the introduction of his paper that his results are valid for arbitrary non-constant polynomials with integer coefficients. But in course of his proof, he uses the result (his Lemma 2.5) that $\rho_i(n) = O(d_i^{\omega(n)})$ which is not true generally. For example, for $f_i(x) = x^2$, $\rho_i(n^2) = n$ for all n whereas $d_i^{\omega(n)} = 2^{\omega(n)} = O(n^\varepsilon)$ for each $\varepsilon > 0$. His Lemma 2.5 is (IV) of Lemma 4 of C. Hooley [4] and he refers to this fact. But Hooley clearly states, at the outset of the section in which this Lemma appears, that throughout that section the polynomial f is primitive and irreducible.

Even if we restrict the polynomials f_i to be primitive and irreducible, Buttsworth's Lemma 2.2, basic to his work, is fallacious. Specifically, the upper limit for the summation over j in his Lemma 2.2, as stated by him, is not valid generally. In fact, if we take $m = 2$, $f_1(x) = f_2(x) = x^4 + 1$, then the upper limit in the sum should be $[(x^4 + 1)/h]$ but not $[x/h]$ as stated by Buttsworth. Also it is not hard to see that the proof of his main

theorem (Theorem 2.7) breaks down in this special case. In fact, equation (2.7.3) of his paper cannot be justified.

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