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# THE SCHUR AND HAHN THEOREMS FOR OPERATOR MATRICES

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In [7] (and [8]), I. Maddox has generalized a classical summability result of Schur to infinite matrices whose elements are continuous linear operators between $B$-spaces. Another classical summability result which is closely related to the Schur summability result is a theorem of Hahn ([5], [11] 6.1) which asserts that an infinite scalar matrix sums all bounded sequences if and only if it sums all sequences of 0's and 1's. In this note we consider an analogue of the Hahn summability result for infinite matrices of continuous linear operators and compare this result with both Maddox's operator generalization of the Schur summability theorem and the classical scalar case. In investigating the similarities and differences between the operator and scalar cases, we are also lead to consider an operator version of another classical scalar result of Schur pertaining to weak and norm convergence of sequences in $\ell^1$.

In order to fully appreciate the difficulties encountered in the case of operator-valued matrices, we first present some results on summability when the infinite matrices have values in a metric linear space. These vector results are then compared with the corresponding classical scalar results and are used in the later sections to point out the differences which arise in the infinite dimensional case of operator-valued matrices.

We then proceed to consider operator-valued matrices. We first consider Maddox's generalization of the classical Schur summability result. In establishing this result, Maddox employed Baire Category methods. We give a proof of Maddox's generalization which relies only upon a very simple lemma concerning infinite matrices with vector entries, and actually our method gives a generalization of Maddox's result to barrelled spaces. This matrix method can be regarded as an "abstract sliding hump" type of result and is employed throughout the paper in all of the proofs. Thus, the methods employed here can be regarded as an interesting contrast to the Baire Category methods.

We next present a generalization of the classical Hahn summability result to operator-valued matrices and contrast this result with Maddox's

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generalization of the classical Schur summability result. In particular, an example is given which shows that the infinite dimensional case covered by operator-valued matrices is quite different from the scalar or vector case.

Motivated by the results in the vector summability case, we also consider a generalization to the operator case of a result of Schur on weak and norm convergence of sequences in $\ell^1$.

Before presenting the results on vector summability, we state our basic matrix theorem which will be employed throughout the sequel. This matrix theorem is quite simple and can be viewed as an "abstract sliding hump" type of result. It has been successfully employed to treat a wide variety of topics in measure theory and functional analysis ([1], [2], [3]).

Throughout the sequel $X$ will denote a (real) metric linear space, i.e., $X$ is a topological vector space whose topology is generated by a quasi-norm $|\ |$. (A quasi-norm is a function $|\ |: X \to \mathbf{R}$ satisfying $|x| \geqq 0$, $|-x| = |x|$, $|0| = 0$ and $|x + y| \leqq |x| + |y|$; such a function generates a metric topology on $X$ via the translation invariant metric $d(x, y) = |x - y|$ ([10]).)

THEOREM 1. *Let $x_{ij} \in X$ for $i, j \in \mathbf{N}$ satisfy*

(I) $\lim_i x_{ij} = x_j$ *exists for each $j$ and*

(II) *for each subsequence $\{m_j\}$ there is a subsequence $\{n_j\}$ of $\{m_j\}$ such that $\{\sum_{j=1}^{\infty} x_{in_j}\}$ is a Cauchy sequence. Then* $\lim_i x_{ij} = x_j$ *uniformly in $j$. In particular,* $\lim_i x_{ij} = 0$.

See [2], Theorem 3, [1], Theorem 2, or [13], Theorem 1 for proof.

**1. Vector summability.** In this section we present several summability results for infinite matrices with entries in a metric linear space. The results are generalizations of the classical summability results of Schur and Hahn to such matrices. These results are presented in order to point out the basic differences which occur later when operator-valued matrices are considered; the results are also used later in §3. Most of the results in this section are known but do not seem to have been presented in the summability terminology.

We first recall the classical summability results of Schur and Hahn. Let $A = [a_{ij}]$ be an infinite real matrix. The matrix $A$ is said to be in the class $(\ell^\infty, c)$ if for each bounded sequence $x = \{t_j\} \in \ell^\infty$, the sequence $\{\sum_{j=1}^{\infty} a_{ij} t_j\}$ is convergent, i.e., for each $x \in \ell^\infty$, the formal matrix product $Ax$ belongs to the space of convergent sequences, $c$. The classical Schur summability theorem gives necessary and sufficient conditions for a matrix $A$ to belong to $(\ell^\infty, c)$ (see [6], 7.6, or Theorem 2 below).

Let $m_0$ be the subspace of $\ell^\infty$ which consists of those real sequences with finite range. Note that $m_0$ is a dense subspace of $\ell^\infty$. The matrix $A$ is said to be of class $(m_o, c)$ if for each sequence $x = \{t_i\} \in m_o$, the formal matrix product $Ax \in c$. The classical summability result of Hahn asserts that a matrix $A$ belongs to the class $(\ell^\infty, c)$ if and only if $A$ belongs to $(m_o, c)$ ([11], 6.1, or Proposition 4 below).

We now consider the analogue of the classical results of Schur and Hahn for matrices with vector entries. Let $A = [x_{ij}]$ be an infinite matrix with elements $x_{ij} \in X$. Let $c(X)$ be the space of $X$-valued sequences which are convergent in $X$. The matrix $A$ is said to be of class $(\ell^\infty, c(X))$ $((m_o, c(X)))$ if for each bounded scalar sequence $x = \{t_j\} \in \ell^\infty$ $(x \in m_o)$, the sequence $\{\sum_{j=1}^\infty t_j x_{ij}\}$ belongs to $c(X)$, i.e., the formal matrix product $Ax \in c(X)$.

There are several observations that should be made pertaining to matrices in the classes $(\ell^\infty, c(X))$ and $(m_o, c(X))$. First, in order for the matrix $A$ to belong to $(\ell^\infty, c(X))$, the series $\sum_{j=1}^\infty t_j x_{ij}$ must be convergent for each $\{t_j\} \in \ell^\infty$ and $i \in \mathbf{N}$. That is, the rows of $A$ must be bounded multiplier convergent (BMC). ($A$ series $\sum y_j$ in $X$ is said to be BMC if the series $\sum_{j=1}^\infty t_j y_j$ converges in $X$ for each bounded sequence $\{t_j\} \in \ell^\infty$.) A series $\sum y_j$ in $X$ is said to be subseries (s.s.) convergent if for each subsequence $\{y_{k_j}\}$ of $\{y_j\}$, the subseries $\sum y_{k_j}$ is convergent in $X$ ([10] III.6). A series which is BMC is clearly s.s. convergent (choose $t_j$ equal to 0 or 1). In locally convex spaces and even in certain non-locally convex spaces, the converse also holds, i.e., a series which is s.s. convergent is also BMC ([10] III.6.5). However, in general, a series in a non-locally convex space may be subseries convergent and fail to be BMC ([10] III.6.9).

We now give a characterization of elements of the class $(\ell^\infty, c(X))$. This result is a generalization of the classical Schur summability theorem to vector-valued matrices ([6] 7.6).

THEOREM 2. *Let A be a matrix whose rows are* BMC. *The following are equivalent*:

(a) $A \in (\ell^\infty, c(X))$;

(b) (i) $\lim_i x_{ij} = x_j$ *exists for each* $j$,

(ii) *the series* $\sum_{j=1}^\infty t_j x_{ij}$ *converge uniformly for* $i \in \mathbf{N}$ *and* $|t_j| \leqq 1$;

(c) $\lim_i \sum_{j=1}^\infty t_j x_{ij}$ *exists* (*and equals* $\sum_{j=1} t_j x_j$, *where* $x_j = \lim_i x_{ij}$) *uniformaly for* $|t_j| \leqq 1$.

PROOF. That (a) implies (c) follows from Theorem 3 of [13], and (a) implies (b) follows from Corollary 4 of [13]. That (c) implies (a) is clear. Finally, the proof that (b) implies (a) can be given exactly as the proof that (B) implies (A) of Theorem 8 below (in particular, see (4)), and, for that reason, we do not repeat the proof.

We next consider the analogue of the classical summability theorem of Hahn for vector matrices.

Recall a series $\sum y_i$ in $X$ is said to be subseries (s.s) convergent if for each subsequence $\{y_{k_i}\}$ of $\{y_i\}$ the subseries $\sum_i y_{k_i}$ converges in $X$. (See, [10] III.6, for properties of s.s. convergent series.) If the series $\sum y_i$ is s.s. convergent and $\sigma$ is an infinite subset of $\mathbf{N}$, we write $\sum_{i\in\sigma} y_i$ for the sum of the subseries $\sum_{i=1}^{\infty} y_{k_i}$, where the elements of $\sigma$ are arranged in the subsequence $\sigma = \{k_i : k_i < k_{i+1}\}$. If $\sigma \subseteq \mathbf{N}$ is finite, the meaning of $\sum_{i\in\sigma} y_i$ is clear.

If $\sigma \subseteq \mathbf{N}$, then the characteristic function of $\sigma$ belongs to $m_o$ so if the matrix $A$ belongs to $(m_o, c(X))$, the series $\sum_{j\in\sigma} x_{ij}$ converges in $X$ for each $i$. That is, if $A$ belongs to $(m_o, c(X))$, its rows must be s.s. convergent.

This necessary condition for a matrix to belong to the class $(m_o, c(X))$ should be contrasted with the corresponding necessary condition for a matrix to belong to the class $(\ell^\infty, c(X))$; that is, for a matrix to belong to $(m_o, c(X))$ its rows must be s.s. convergent whereas for a matrix to belong to $(\ell^\infty, c(X))$ it rows must actually be BMC. (See the remarks preceding Theorem 2.)

We now give a generalization of the classical summability result of Hahn to vector-valued matrices ([11] 6.1). For the statement of the theorem, we require the following definition. A family of series $\sum_{i=1}^{\infty} y_{\alpha i}$, $\alpha \in A$, is said to be uniformly unordered convergent if for each $\varepsilon > 0$ there exists $N$ such that $|\sum_{i\in\sigma} y_{\alpha i}| < \varepsilon$ whenever $\min \sigma \geqq N$ and $\alpha \in A$.

THEOREM 3. *Let $A$ be a matrix whose rows are* s.s. *convergent. The following are equivalent*:

(d) $A \in (m_o, c(X))$;

(e) (i) $\lim_i x_{ij} = x_j$ *exists for each* $j$;

(ii) $\sum_j x_{ij}$ *is uniformly unordered convergent for* $i \in \mathbf{N}$;

(f) $\lim_i \sum_{j\in\sigma} x_{ij}$ *exists* (*and equals* $\sum_{j\in\sigma} x_j$, *where* $x_j = \lim_i x_{ij}$) *uniformly for* $\sigma \subseteq \mathbf{N}$.

PROOF. That (d) implies (f) follows from Theorem 3 of [2], and (d) implies (e) follows from Corollary 4 of [2]. That (f) implies (d) is clear.

We show that (e) implies (d). First note that condition (ii) implies that the series $\sum x_j$ is s.s. convergent. Let $\varepsilon > 0$ and let $N$ be as in the definition of uniform unordered convergence given above. Let $\sigma \subseteq \mathbf{N}$ and put $\sigma(N) = \{i \in \sigma : i \geqq N\}$. Then we have

$$(1) \qquad \Big|\sum_{j\in\sigma}(x_{ij} - x_j)\Big| \leqq \sum_{j=1}^{N} |x_{ij} - x_j| + \Big|\sum_{j\in\sigma(N)} x_{ij}\Big| + \Big|\sum_{j\in\sigma(N)} x_j\Big|.$$

Now the second and third terms on the right hand side of (1) are less than or equal $\varepsilon$ by the uniform unordered convergence. With $N$ fixed, $i$ can be chosen large enough so that the first term on the right hand side of (1) can be made less than $\varepsilon$. Since the characteristic functions of subsets of $\mathbf{N}$ span $m_o$, this establishes (d).

The equivalence of (d) and (f) can be viewed as a generalization of a classical result of Schur on the equivalence of weak and norm convergence of sequences in $\ell^1$ ([**15**], 1.3.2; [**6**], 7.6 Corollary; and [**2**], Theorem 3 and Corollary 5). We consider an operator version of this result of Schur in §4.

In the case of scalar matrices, conditions (a) through (f) are equivalent ([**11**] 6.1). However, in the case of non-locally convex spaces, these conditions are not, in general, equivalent. For example, let $X$ be a metric linear space which contains a series $\sum x_i$ which is s.s. convergent but not BMC ([**10**] III.6.9). Set $x_{ij} = 0$ if $j > i$ and $x_{ij} = x_j$ if $i \leqq j$. Then the matrix $[x_{ij}]$ satisfies (d) but not (a).

If the space $X$ is locally convex, then a series is BMC iff it is s.s. convergent and conditions (a) through (f) are actually equivalent (Proposition 4 below) and the equivalence of (a) and (d) give a generalization of the classical summability result of Hahn ([**11**] 6.1).

Proposition 4. *If $X$ is a locally convex space, then* (a)–(f) *of Theorems* 2 *and* 3 *are equivalent.*

Proof. We show (e) implies (b). Let $U$ be a closed, absolutely convex neighborhood of 0 in $X$. By (ii) of (e), there exists $N$ such that $\sum_{j\in\sigma} x_{ij} \in U$ for all $i$ and min $\sigma \geqq N$. Thus, if $x' \in U^\circ$, the polar of $U$, $|\sum_{j\in\sigma} < x', x_{ij} > | \leqq |$. This implies that $\sum_{j=N}^{\infty} |<x', x_{ij}>| \leqq 2$ for all $i$ ([**9**] 1.1.2). Hence, if $|t_j| \leqq |$, we have $|<x', \sum_{j=N}^{\infty} t_j x_{ij} >| \leqq 2$, and $\sum_{j=N}^{\infty} t_j x_{ij} \in 2U$ by the Biploar Theorem. This establishes (ii) of (b) and the result follows.

The equivalence of (c) and (d) in the scalar case is just (a slight generalization) the classical Lemma of Schur ([**15**] 1.3.2; [**6**] 7.6 Corollary; [**13**]).

Before proceeding to the main section on operator-valued matrices, we want to point out that the proofs of the results cited in [**13**] and [**2**] were obtained by use of the Basic Matrix Theorem 1 so that Theorems 2 and 3 are actually consequences of the Basic Matrix Theorem 1.

**2. Schur summability result for operator matrices.** We now consider operator-valued matrices. For the remainder of the paper, $X$ and $Y$ will denote normed linear spaces with norms $|\ |$. The space of all continuous linear operators from $X$ into $Y$ will be denoted by $L(X, Y)$.

The space of all bounded $X$-valued sequences, $\phi = \{x_j\}$, will be denoted by $\ell^\infty(X)$ and will be equipped with the sup-norm, $\|\phi\| = \sup \|x_j\|$. The subspace of $\ell^\infty(X)$ consisting of all sequences with finite range will be denoted by $m_o(X)$; this subspace is just the vector analogue of the space $m_o$. One of the basic differences in the vector case is that $m_o(X)$ is not in general a dense subspace of $\ell^\infty(X)$ (Example 12 below).

We also recall some basic properties of series of operators. If $\sum T_i$ is a series in $L(X, Y)$, then $\sum T_i$ is said to be bounded multiplier convergent (BMC) if the series $\sum_{i=1}^{\infty} T_i x_i$ converges for each bounded sequence

$\{x_i\} \in \ell^\infty(X)$. If the series $\sum T_i$ is BMC, it is uniformly BMC in the sense that the series $\sum_{i=1}^\infty T_i x_i$ are uniformly convergent for $\|x_j\| \leqq 1$ ([3] Theorem 2). In particular, this implies that a BMC series $\sum T_i$ is always s.s. convergent with respect to the operator norm on $L(X, Y)$. The converse is, in general, false (see Example 10 below). We say that a family of series $\sum_i T_{\alpha i}$ is uniformly BMC for $\alpha \in A$ if the series $\sum_{i=1}^\infty T_{\alpha i} x_i$ converge uniformly for $\alpha \in A$ and $\|x_j\| \leqq 1$.

We now establish Maddox's generalization of the classical Schur summability theorem ([7], [8]). Let $A_{ij} \in L(X, Y)$ for $i, j \in \mathbf{N}$. The matrix $A = [A_{ij}]$ is said to be of class $(\ell^\infty(X), c(Y))((m_o(X), c(Y)))$ if the sequence $\{\sum_{j=1}^\infty A_{ij} x_j\}_{i=1}^\infty$ belongs to $c(Y)$, for each sequence $\phi = \{x_j\} \in \ell^\infty(X)\cdot$ $(\phi \in m_o(X))$, i.e., if the formal matrix product $A\phi \in c(Y)$ for each $\phi \in \ell^\infty(X)$ $(\phi \in m_o(X))$. Note that a necessary condition for $A$ to be of class $(\ell^\infty(X), c(Y))$ is that the rows of the matrix $A$ must be BMC.

We first establish a preliminary lemma.

LEMMA 5. *Let the matrix $A$ be such that its rows are* BMC. *If $\sum_j A_{ij}$ is not uniformly* BMC *for $i \in \mathbf{N}$, then there exists $\varepsilon > 0$, a disjoint sequence of finite sets $\{\sigma_j\}$ with* $\max \sigma_j < \min \sigma_{j+1}$, *$x_{ij} \in X$ for $j \in \sigma_i$ with $\|x_{ij}\| \leqq 1$, and a subsequence $\{k_i\}$ such that $\|\sum_{j \in \sigma_i} A_{k_i j} x_{ij}\| > \varepsilon$ for each $i$.*

PROOF. If $\sum A_{ij}$ is not uniformly BMC for $i \in \mathbf{N}$, then

$$\text{there exists } \varepsilon > 0 \tag{2}$$

such that for each $i$ there exist a $k_i$, a finite $\sigma_i \subseteq \mathbf{N}$ with $\min \sigma_i \geqq i$ and $x_{ij}$ for $j \in \sigma_i$ with $\|x_{ij}\| \leqq 1$ such that $\|\sum_{j \in \sigma_i} A_{k_i j} x_{ij}\| \geqq \varepsilon$.

Now set $i_1 = 1$ and apply (2) to $i_1$. Thus, there exist a $k_1$, a finite $\sigma_1$, $\{x_{1j}\}_{j \in \sigma_1}$, with $\|x_{1j}\| \leqq 1$ such that $\|\sum_{j \in \sigma_1} A_{k_1 j} x_{1j}\| \geqq \varepsilon$. By the uniform BMC of each series, there exists $j_1$ such that

$$\|\textstyle\sum_{j \in \sigma} A_{ij} x_j\| < \varepsilon/2 \tag{3}$$

for $1 \leqq i \leqq k_1$, $\min \sigma \geqq j_1$ and $\|x_j\| \leqq 1$. Put $i_2 = \max\{\max \sigma_1 + 1, j_1\}$ and apply (2) to $i_2$. Thus, there exist a $k_2$, a finite $\sigma_2$ with $\min \sigma_2 > i_2$, $\{x_{2j}\}_{j \in \sigma_2}$ with $\|x_{2j}\| \leqq 1$ such that $\|\sum_{j \in \sigma_2} A_{k_2 j} x_{2j}\| \geqq \varepsilon$. Note from (3) that $k_2 > k_1$.

Thus, this construction can be continued inductively to obtain the sequences in the conclusion of the lemma.

We next establish a necessary condition for a matrix $A$ to be of class $(\ell^\infty(X), c(Y))$.

PROPOSITION 6. *Let the matrix $A$ be such that its rows are* BMC. *Suppose* $\lim_i \sum_{j=1}^\infty A_{ij} x_j$ *exists for each $\{x_j\} \in \ell^\infty(X)$. Then*

(4) $\lim_i A_{ij} x = A_j x$ *exists for each $j$ and $x \in X$*

(5) $\sum_j A_{ij}$ *is uniformly* BMC

PROOF. For (i) let $x \in X$. Applying the hypothesis of the proposition to the sequence in $\ell^\infty(X)$ that has $x$ in the $j$-th coordinate and 0 in the other coordinates gives (4) immediately.

If (5) fails to hold, then let the notation be as in Lemma 5 and set $z_{ij} = \sum_{n \in \sigma_j} A_{k_i n} x_{jn}$. Note $\|z_{ii}\| > \varepsilon$. Let $\sigma \subseteq \mathbf{N}$ and define a sequence $\{x_n\} \in \ell^\infty(X)$ by $x_n = x_{jn}$ if $n \in \sigma_j$ and $j \in \sigma$ and $x_n = 0$ otherwise. Then $\sum_{n=1}^\infty A_{k_i n} x_n = \sum_{j \in \sigma} \sum_{n \in \sigma_j} A_{k_i n} x_{jn} = \sum_{j \in \sigma} z_{ij}$ converges as $i \to \infty$ by hypothesis. The matrix $\{z_{ij}\}$ satisfies the hypothesis of the basic matrix Theorem 1 so that $\lim_i z_{ii} = 0$. This contradiction establishes (5).

Note that in Proposition 6 we do not assert that the $A_j$ are elements of $L(X, Y)$. If $X$ is complete (or barrelled), then this follows from the Banach-Steinhaus Theorem. In Maddox's treatment he assumes that the space $X$ is a $B$-space, and then (4) can be phrased as:

(4′) $\lim_i A_{ij} = A_j$ exists in the strong operator topology for each $j$ (condition (i) of [7]). Maddox derives condition (5) (actually another equivalent form of (5)) by utilizing the assumption that $X$ is complete and employing the Baire Category Theorem. Note that the Baire Category Theorem cannot be directly applied above since $X$ is not assumed to be complete.

Under the assumptions that the $A_j$ in condition (4) are continuous and $Y$ is complete, Proposition 6 has the following corollary.

COROLLARY 7. *Assume the $A_j$ in* (4) *above are continuous and that $Y$ is complete. Then* (4) *and* (5) *above imply that the series $\sum A_j$ is* BMC.

PROOF. Let $\{x_j\} \in \ell^\infty(X)$ and for convenience assume that $\|x_j\| \leqq 1$. Let $\varepsilon > 0$. By (5) there exists $N$ such that $\| \sum_{j=m}^n A_{ij} x_j \| < \varepsilon$ for $n > m \geqq N$. By (4), $\| \sum_{j=m}^n A_j x_j \| \leqq \varepsilon$ for $n > m \geqq N$, and the series $\sum A_j x_j$ converges in $Y$.

Finally we give the Maddox generalization of the classical Schur summability theorem ([7], [8] 4.6).

THEOREM 8. *Let the matrix $A$ be such that its rows are* BMC. *If $X$ is barrelled and $Y$ is complete, the following are equivalent*:

(A) $A \in (\ell^\infty(X), c(Y))$; *and*

(B) *conditions* (4) *and* (5) *of Proposition* 6.

PROOF. That (A) implies (B) follows from Proposition 6 and the Banach-Steinhaus Theorem for barrelled spaces.

Assume that (B) holds. Let $\varepsilon > 0$. If $\{x_j\} \in \ell^\infty(X)$, then by Corollary 7 we have

$$(6) \quad \Big\| \sum_{j=1}^{\infty} (A_{ij} - A_j) x_j \Big\| \leqq \sum_{j=1}^{N} \|(A_{ij} - A_j) x_j\| + \Big\| \sum_{j=N}^{\infty} A_{ij} x_j \Big\| + \Big\| \sum_{j=N}^{\infty} A_j x_j \Big\|.$$

By (5) and Corollary 7 there exists $N$ such that the last two terms on the right hand side of (6) are less than $\varepsilon$. Then by (4) there exists $i_0$ such that $i \geqq i_0$ implies that the first term on the right hand side of (6) is less than $\varepsilon$. Thus, $\lim_i \sum_{j=1}^{\infty} A_{ij}x_j = \sum_{j=1}^{\infty} A_j x_j$ and (A) holds.

Note that the essential tool used in deriving Theorem 8 was the Basic Matrix Theorem 1. This should be contrasted with the Baire Category techniques employed by Maddox in [7] and [8]. It might also be worth noting that in the proof of (A) implies (B) we make no completeness assumption on the normed space $X$ so that the Baire Category methods employed by Maddox are not directly applicable to the case under consideration in Theorem 8 or Proposition 6.

**3. Hahn summability result for operator matrices.** We next consider matrices in the class $(m_o(X), c(Y))$. We give a characterization of matrices in this class; this characterization can be considered to be a generalization of the classical summability result of Hahn ([5], [11] 6.1).

Note that functions of the form $C_\sigma x$ for $\sigma \subseteq N$ and $x \in X$ (here $C_\sigma$ is the characteristic function of $\sigma$) span the space $m_o(X)$. Hence, if the matrix $A = [A_{ij}]$ belongs to $(m_o(X), c(Y))$, then the series $\sum_{j \in \sigma} A_{ij}x$ is convergent for $\sigma \subseteq N$ and $x \in X$. That is, a necessary condition for $A$ to belong to $(m_o(X), c(Y))$ is that the rows of $A$ must be s.s. convergent in the strong operator topology of $L(X, Y)$.

We now give our characterization of matrices of class $(m_o(X), c(Y))$. This result should be compared with Theorem 3 for vector matrices.

THEOREM 9. *Let $A = [A_{ij}]$ be such that its rows are* s.s. *convergent in the strong operator topology. The following are equivalent*:

(D) $A \in (m_o(X), c(Y))$,

(E) (4) $\lim_i A_{ij}x = A_j x$ *exists for each* $j \in \mathbf{N}$ *and* $x \in X$ *and*

(7) *for each* $x \in X$

*the series* $\sum_j A_{ij}x$ *are uniformly unordered convergent for* $i \in \mathbf{N}$,

(F) *for each* $x$, $\lim_i \sum_{j \in \sigma} A_{ij}x$ *exists*(*and equals* $\sum_{j \in \sigma} A_j x$, *where* $A_j x = \lim_i A_{ij}x$) *uniformly for* $\sigma \in \mathbf{N}$.

PROOF. Assume that (D) holds. Then (4) is clear by employing previous arguments. For (7), consider the matrix $A_x = [A_{ij}x]$. By (D), $A_x \in (m_o, c(Y))$ so (7) follows from Theorem 3.

Assume that (E) holds. For fixed $x$ the matrix $A_x$ satisfies condition (e) of Theorem 3. Hence, the equivalence of (e) and (f) in Theorem 3 gives condition (F).

That (F) implies (D) is clear.

Conditions (D), (E) and (F) should be compared with the corresponding conditions in Theorem 3.

Recall that in the locally convex case the analogue of conditions (A) and (D) are equivalent (i.e., (a) and (d) are equivalent, Proposition 4). However, in the operator case this is not the case as will be pointed out below in Example 11. To present an example showing that (A) and (D) are not in general equivalent we first show that there are series in $L(X, Y)$ which are s.s. convergent in norm (and, therefore, in the strong operator topology) but not BMC.

EXAMPLE 10. Let $X = Y = \ell^1$ and let $e_i$ be the sequence in $\ell^1$ which has a 1 in the $i$-th coordinate and 0 in the other coordinates. Define $T_i: \ell^1 \to \ell^1$ by $T_i\{t_j\} = (t_i/i)e_i$. If $\sigma \subseteq \mathbf{N}$, then $\|\sum_{i\in\sigma} T_i\| \leqq \sup_{i\in\sigma} 1/i$ so the series is s.s. convergent in the norm of $L(X, Y)$. However, the series $\sum T_i e_i$ is not convergent in $\ell^1$ so the series $\sum T_i$ is not BMC.

We now give a matrix which is in class $(m_o(X), c(Y))$ but not in $(\ell^\infty(X), c(Y))$. From Theorems 8 and 9 it follows that one can construct an example of such a matrix by constructing an example of a matrix in $(m_o(X), c(Y))$ whose rows are not BMC. However, the example which we give is such that its rows are actually BMC, and the matrix still fails to be in the class $(\ell^\infty(X), c(Y))$.

EXAMPLE 11. Let $\sum T_j$ be a series in $L(X, Y)$ which is norm subseries convergent but not BMC (see Example 10). Define $A_{ij} = 0$ if $j > i$ and $A_{ij} = T_j$ if $i \leqq j$. Since $\sum_{j=1}^{\infty} A_{ij} = \sum_{j=1}^{i} T_j$, each row of the matrix $A = [A_{ij}]$ is BMC. Since the $j$-th column is eventually $T_j$, the columns are clearly convergent in norm (i.e., (4) is satisfied). Also, the matrix $A \in (m_o(X), c(Y))$ since for $x \in X, \sigma \subseteq \mathbf{N}$, we have $\lim_i \sum_{j\in\sigma} A_{ij}x = \sum_{j\in\sigma} T_j x$.

Let $\{x_j\} \in \ell^\infty(X)$ be such that $\|x_j\| \leqq 1$ and $\sum_{j=1}^{\infty} T_j x_j$ does not converge. Then $\lim_i \sum_{j=1}^{\infty} A_{ij}x_j = \lim_i \sum_{j=1}^{i} T_j x_j$ does not exist so that the matrix $A$ does not belong to $(\ell^\infty(X), c(Y))$.

This phenomena should be compared to the result in Proposition 4 for vector summability (the equivalence of (a) and (d)).

One reason that the operator situation is so different from the vector (or scalar) case given in Corollary 4 might be that in the vector case the subspace $m_o(X)$ is not dense in $\ell^\infty(X)$ when $X$ is infinite dimensional. This is established in Example 12 below.

EXAMPLE 12. Let $X$ be an infinite dimensional $B$-space. Pick a sequence $\{x_i\}$ in $X$ such that $\|x_i\| = 1$ and $\|x_i - x_j\| \geqq 1/2$ for $i \neq j$ (Riesz Lemma, **[14]** 3.12 E). If $m_o(X)$ is dense in $\ell^\infty(X)$, then for $\varepsilon > 0$ there exists $y = \{y_i\} \in m_o(X)$ such that $\|y - x\| < \varepsilon$, where $x = \{x_i\} \in \ell^\infty(X)$. Since the range of $y$ is finite, this means that the set $S = \{x_i: i \in \mathbf{N}\}$ has a finite $\varepsilon$-net for each $\varepsilon > 0$ and, hence, $S$ is relatively norm compact ([**12**] 25 B). Since this is impossible by the choice of $\{x_i\}$, it follows that $m_o(X)$ is not dense in $\ell^\infty(X)$.

**4. Schur condition for operators.** Note that in Maddox's generalization of the Schur summability theorem for operator-valued matrices given in Theorem 8, we considered the analogues of conditions (a) and (b) of Theorem 2 (conditions (A) and (B) of Theorem 8) but not the analogue of condition (c). The operator analogue of (c) is given by:

$$\text{(C)} \qquad \lim_i \sum_{j=1}^{\infty} A_{ij}x_j \text{ exists (and equals } \sum_{j=1}^{\infty} A_j x_j, \text{ where } \lim_i A_{ij} = A_j) \text{ uniformly for } \|x_j\| \leqq 1.$$

Since the equivalence of (a) and (c) (or (d) and (c)) implies the classical result of Schur on the equivalence of weak and norm convergence of sequences in $\ell^1$ ([6] 7.6, Corollary; [15] 1.3.2), we refer to (C) as the generalized Schur condition. A natural question that arises is whether condition (C) is equivalent to conditions (A) or (B) as it is in the scalar (or vector) case, i.e., whether the operator analogue of the Schur result is valid. We show in Example 16 that this is not the case, i.e., that in general conditions (A), (B) and (C) are not equivalent. However, we show in Theorem 15 that there are slight variants of conditions (A) and (B) (conditions (A′) and (B′) below) which are equivalent to the generalized Schur condition (C).

We first establish a preliminary lemma.

LEMMA 13. *Let* $[A_{ij}]$ *be such that its rows are* BMC. *If*

$$\text{(8)} \qquad \lim_i \|A_{ij}\| = 0 \textit{ for each } j$$

*and*

$$\text{(9)} \qquad \lim_i \sum_{j=1}^{\infty} A_{ij}x_j = 0 \textit{ for each } \{x_j\} \in \ell^\infty(X),$$

*then* $\lim_i \sum_{j=1}^{\infty} A_{ij}x_j = 0$ *uniformly for* $\|x_j\| \leqq 1$.

PROOF. If the conclusion fails, we may assume (by passing to a subsequence if necessary) that there exists $\delta > 0$ such that $\sup\{\|\sum_{j=1}^{\infty} A_{ij}x_j\| : \|x_j\| \leqq 1\} \geqq \delta$ for each $i$. Put $k_1 = 1$. Then there exists $\{x_{1j}\} \in \ell^\infty(X)$ with $\|x_{1j}\| \leqq 1$ such that $\|\sum_{j=1}^{\infty} A_{k_1 j}x_{1j}\| > \delta$. There exists $M_1$ such that $\|\sum_{j=1}^{M_1} A_{k_1 j}x_{1j}\| > \delta$. By (8) there exists $k_2 > k_1$ such that $\sum_{j=1}^{M_1} \|A_{ij}\| < \delta/2$ for $i \geqq k_2$. Now there exists $\{x_{2j}\} \in \ell^\infty(X)$ with $\|x_{2j}\| \leqq$ and $\|\sum_{j=1}^{\infty} A_{k_2 j}x_{2j}\| > \delta$. There exists $M_2 > M_1$ such that $\|\sum_{j=1}^{M_2} A_{k_2 j}x_{2j}\| > \delta$. Note $\|\sum_{j=M_1+1}^{M_2} A_{k_2 j}x_{2j}\| \geqq \|\sum_{j=1}^{M_2} A_{k_2 j}x_{2j}\| - \sum_{j=1}^{M_1} \|A_{k_2 j}x_{2j}\| > \delta/2$. Continuing this construction produces increasing sequence $\{k_i\}$ and $\{M_i\}$ and a sequence $\{x_{ij}\}_{j=1}^{\infty} \in \ell^\infty(X)$ with $\|x_{ij}\| \leqq 1$ such that $\|\sum_{j=M_{i-1}+1}^{M_i} A_{k_i j}x_{ij}\| > \delta/2$, where $M_o = 0$.

Now consider the matrix $z_{ij} = \sum_{n=M_{j-1}+1}^{M_j} A_{k_i n}x_{jn}$ and note $\|z_{ii}\| >$

$\delta/2$. By (8), we have $\lim_i z_{ij} = 0$. Let $\sigma \subseteq \mathbf{N}$ and define a sequence $\{x_n\} \in \ell^\infty(X)$ by $x_n = x_{jn}$ if $M_{j-1} + 1 \leqq n \leqq M_j$ and $j \in \sigma$ and $x_n = 0$ otherwise. Then $\sum_{n=1}^\infty A_{k_i n} x_n = \sum_{j \in \sigma} \sum_{n=M_{j-1}+1}^{M_{j+1}} A_{k_i n} x_{jn} = \sum_{j \in \sigma} z_{ij}$ converges to 0 by (9). The matrix $[z_{ij}]$ satisfies the hypothesis of the matrix Theorem 1 and, hence, $\lim z_{ii} = 0$. This contradiction establishes the lemma.

We next consider a slight variant of condition (A) and its relation to (C). We strengthen (A) by making an additional assumption on the columns of $A$.

PROPOSITION 14. *Let $[A_{ij}]$ be such that its rows are* BMC. *If condition* (A′): (A) *and*

(10) $\lim_i A_{ij} = A_j$ *exists in operator norm for each $j$, then* (C) *holds and the series $\sum A_j$ is* BMC.

PROOF. We first claim that the sequence $\{\sum_{j=1}^\infty A_{ij} x_j\}_{i=1}^\infty$ satisfies a Cauchy condition uniformly for $\|x_j\| \leqq 1$. If this is not the case, there exist $\delta > 0$ and a subsequence $\{n_i\}$ such that

$$(11) \qquad \sup \{\| \textstyle\sum_{j=1}^\infty (A_{n_{i+1}j} - A_{n_i j}) x_j : \|x_j\| \leqq 1\} > \delta \qquad \text{for all } i.$$

Now consider the matrix $[A_{n_{i+1}j} - A_{n_i j}]$. By (A) and (10), this matrix satisfies the conditions of Lemma 13. Hence, by Lemma 13, $\lim_i \sum_{j=1}^\infty (A_{n_{i+1}j} - A_{n_i j})\, x_j = 0$ uniformly for $\|x_j\| \leqq 1$. But, this contradicts (11).

Thus, given $\varepsilon > 0$, there exists $N$ such that $\| \sum_{j=1}^\infty (A_{mj} - A_{nj}) x_j \| < \varepsilon$ for $n, m \geqq N$ and $\|x_j\| \leqq 1$. Letting $m \to \infty$ in this inequality gives (C) and also the fact that $\sum A_i$ is BMC.

Note that (A′) is obtained from (A) by adding a strengthened form of (4), i.e., we require $\lim_i A_{ij}$ exist in norm instead of in the strong operator topology. In general (A) implies (4) but not (10) as is pointed out in Example 16.

Finally we show that (A′) and a slight variant of condition (B) are equivalent to (C). We first state the strengthened form of (B).

(B′): conditions (10) and (5).

Note that (B′) is obtained from (B) by strengthening the requirement that $\lim_i A_{ij}$ converges pointwise on $X$ to the condition that the limit exists in the operator norm.

THEOREM 15. *Let $[A_{ij}]$ be such that its rows are* BMC. *Then conditions* (A′) *and* (C) *are equivalent. Condition* (A′) *implies* (B′), *and if $Y$ is complete.* (A′) *and* (B′) *are equivalent.*

PROOF. That (A′) implies (C) is Proposition 14.

If (C) holds, then condition (A) clearly holds and (10) holds by taking the sequence $\{x_j\}$ in (C) to have a 0 in each coordinate but the $j$-th and an arbitrary $x$ with $\|x\| \leqq 1$ in the $j$-th. Thus, (C) implies (A′). That (A′) implies (B′) follows from Proposition 6. If $Y$ is complete, Corollary 7 and the proof of (B) implies (A) shows that (B′) implies (A′).

Note that the difference in conditions (B) and (B′) is that in condition (B) the columns of the matrix $A$ are convergent in the strong operator topology whereas in (B′) the columns are convergent in the norm topology. We show in Example 16 below that a matrix can satisfy (B) but not (B′) so that conditions (A), (B) and (C) are not in general equivalent. Thus, Theorem 15 can be viewed as a natural generalization of the classical result of Schur on the convergence of sequences in $\ell^1$(**[6]** 7.6 Corollary; **[15]** 1.3.2; **[13]**, Theorem 3).

For Example 16, we need the following notation. For $S$ a non-void set, we let $c_o(S)$ be the space of all real-valued functions on $S$, $f\colon S \to \mathbf{R}$, which vanish at $\infty$, i.e., for each $\varepsilon > 0$ the set $\{t \in S\colon |f(t)| \geqq \varepsilon\}$ is finite (**[4]** II. 2.(1)). We equip $c_o(S)$ with the sup-norm and then recall that the dual of $c_o(S)$ is $\ell^1(S)$ (**[4]** Theorem II.2.1).

EXAMPLE 16. Let $X = Y = c_o(\mathbf{N} \times \mathbf{N})$ and let $e_{ij} \in c_o(\mathbf{N} \times \mathbf{N})$ be the canonical unit vector defined by $e_{ij}(m, n) = 1$ if $m = i$, $n = j$ and 0 otherwise. Define $A_{ij}\colon X \to X$ by $A_{ij}(x) = (1/j) < e_{ij}, x > e_{ij}$, and consider the matrix $A = [A_{ij}]$. For fixed $j$, $\lim_i A_{ij}x = 0$ since $\lim_i < e_{ij}, x > = 0$ $(x \in c_o(\mathbf{N} \times \mathbf{N}))$; i.e., (B) (4) is satisfied. However, for each $j$, $\|A_{ij}\| = 1/j$ so (B′) (10) is not satisfied. If $x_j \in X$ and $\|x_j\| \leqq 1$, then for each $i$ and $N$, $\|\sum_{j=N}^{\infty} A_{ij}x_j\| \leqq \sup\{1/j\colon j \geqq N\}$, so that (5) is satisfied; i.e., the series $\sum_{j=1}^{\infty} A_{ij}x_j$ are uniformly BMC for $\|x_j\| \leqq 1$, $i \in \mathbf{N}$. From Theorem 8 it follows that $A \in \ell^\infty((X), c(X)$ but $A$ does not satisfy the general Schur condition (C) of Theorem 15.

The phenomena illustrated in this example should be compared with the vector summability result given in Theorem 2 (the equivalence of (a) and (c)).

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