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GORENSTEIN TORIC THREEFOLDS WITH ISOLATED SINGULARITIES AND CYCLIC DIVISOR CLASS GROUP

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I. Introduction. This article was motivated by the following question. Is any 19-dimensional family of $K3$ surfaces realizable as a family of divisors on a toric threefold, other than the quartics in $\mathbf{P}^3$ and the double covers of $\mathbf{P}^2$? Since toric threefolds are described so explicitly, the hope was to deduce and/or describe certain properties of these $K3$ surfaces, especially concerning their degenerations, from purely combinatorial considerations. A $K3$ surface would necessarily be an anticanonical divisor in the toric threefold, hence we require the threefold to be Gorenstein; moreover, since the general member of the family is smooth, the threefold should have only isolated singularities. Finally, by Lefschetz, the Weil Divisor Class group of the threefold injects into that of a smooth $K3$ surface, so that it should be cyclic, since the general $K3$ has cyclic Picard group.

Unfortunately this approach does not lead to 'new' descriptions of $K3$ surfaces. In this article, I will prove the following theorem.

THEOREM 1.1. *Let X be a complete Gorenstein toric threefold with isolated singularities and cyclic divisor class group. Then X is isomorphic to either*

(a) $\mathbf{P}^3$, *or*

(b) *the cone in* $\mathbf{P}^{10}$ *over the triple Veronese surface* $V \cong \mathbf{P}^2$ *embedded into* $\mathbf{P}^9$ *via cubics* ($= \mathbf{P}(1, 1, 1, 3)$.)

In the first case the anticanonical divisors are the quartic $K3$ surfaces, and in the second case they are the double covers.

COROLLARY 1.2. *Let X be a projective toric threefold such that* $|-K_X|$ *contains a nonsingular $K3$ surface S with* Pic $S \cong \mathbf{Z}$. *Then* $X \cong \mathbf{P}^3$ *or* $\mathbf{P}(1, 1, 1, 3)$.

PROOF. Note that, since $S \in |-K_X|$, every $p \in S$ must be a smooth point of X. Therefore S is a Cartier divisor, so X must be Gorenstein. Since

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Pic $S \cong \mathbf{Z}$, the divisor class group of X must be cyclic, and therefore S must be ample, and hence very ample ([3], p. 32). The singularities of X must then be isolated, and theorem (1.1) applies.

In the last section I have given the classification of Gorenstein toric surfaces with cyclic divisor class group, omitting the combinatorial analysis. There are five such surfaces, each of which is a (singular) Del Pezzo surface; they are of degrees 1, 2, 3, 4, and 6.

I will use the notation of [1] throughout this article concerning toric varieties. In particular, σ will denote a cone in $\mathbf{Z}^3$, σ' will be its dual, and $X_{\sigma'}$ will be the corresponding affine toric variety. If $a_1, \ldots, a_n$ are integers, $(a_1, a_2, \ldots, a_n)$ will denote their greatest common divisor.

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II. The local criteria. Let σ' be the cone generated by the three (dual) vectors $v_1 = [i_1, j_1, k_1]$, $v_2 = [i_2, j_2, k_2]$, and $v_3 = [i_3, j_3, k_3]$. Let us determine the conditions under which the affine toric variety $X_{\sigma'}$ is Gorenstein, with only isolated singularities. By [2, Theorem 7.5], $X_{\sigma'}$ is Gorenstein if and only if the three generating vectors v_1, v_2, v_3 for σ' lie in a plane in $\mathbf{R}^3$ of the form $ax + by + cz = 1$, where a, b, and c are integers. Let d be the determinant of the matrix whose rows are the v_i; then this condition is equivalent to d dividing the three quantities

$$\begin{aligned} \ell_1 &= j_2k_3 - j_3k_2 + j_3k_1 - j_1k_3 + j_1k_2 - j_2k_1, \\ \ell_2 &= i_3k_2 - i_2k_3 + i_1k_3 - i_3k_1 + i_2k_1 - i_1k_2, \text{ and} \\ \ell_3 &= i_2j_3 - i_3j_2 + i_3j_1 - i_1j_3 + i_1j_2 - i_2j_1. \end{aligned}$$

$X_{\sigma'}$ will have at worst an isolated singularity if the three faces of σ are generated by vectors which can be (separately) extended to three bases of $\mathbf{Z}^3$. Two vectors v and w can be so extended if and only if their cross-product $v \times w$ is a primitive vector, i.e., the g.c.d. of its coordinates is 1 (see [1], 3). Therefore $X_{\sigma'}$ will have at worst an isolated singularity if and only if

$$\begin{aligned} (j_2k_3 - j_3k_2, j_3k_1 - j_1k_3, j_1k_2 - j_2k_1) &= 1, \\ (i_3k_2 - i_2k_3, i_1k_3 - i_3k_1, i_2k_1 - i_1k_2) &= 1, \quad \text{and} \\ (i_2j_3 - i_3j_2, i_3j_1 - i_1j_3, i_1j_2 - i_2j_1) &= 1. \end{aligned}$$

III. The global criteria. Let X be a complete toric threefold with $Cl(X) = \mathbf{Z}$. Then the dual fan in $\mathbf{R}^3$ associated to X will have 4 cones (see [3], p. 27), and therefore it will be generated by 4 vectors v_1, v_2, v_3 and v_4, so that the four cones are each generated by three of the v_i. If X is to be Gorenstein, each of these triples must lie in a plane of the form $ax +$

$by + cz = 1$; we are free to change coordinates and assume that v_1, v_2, and v_3 lie in the plane $z = 1$. A further change of coordinates allows us to assume that $v_1 = [0, 0, 1]$, $v_2 = [1, 0, 1]$, and $v_3 = [A, B, 1]$, with

$$0 < A \leqq B. \tag{3.1}$$

Let $v_4 = [-C, -D, -E]$. The five integers A, B, C, D, and E determine X up to isomorphism. Let σ_i' be the cone generated by $\{v_j | j \neq i\}$, and let d_i be the determinant of the matrix whose rows are the three generators of σ_i'. Then $d_4 = B$, $d_3 = D$, $d_2 = BC - AD$, $d_1 = BE - D - (BC - AD)$, and

$$d_i > 0 \text{ for each } i. \tag{3.2}$$

X will have only isolated singularities if and only if the six vectors $v_1 \times v_2 = [0, 1, 0]$, $v_1 \times v_3 = [-B, A, 0]$, $v_1 \times v_4 = [D, -C, 0]$, $v_2 \times v_3 = [-B, A - 1, B]$, $v_2 \times v_4 = [D, E - C, -D]$, and $v_3 \times v_4 = [D - BE, AE - C, BC - AD]$ are primitive; this is equivalent to

$$\begin{aligned}(A, B) &= (A - 1, B) = (C, D) = (E - C, D)\\ &= (D - BE, AE - C, BC - AD) = 1.\end{aligned} \tag{3.3}$$

X will be Gorenstein if and only if each $X\sigma_i'$ is Gorenstein; for $i = 4$, this is automatic by our normalization. For the other three cases, the divisibility conditions are that

$$\begin{aligned}&d_3 = D \text{ must divide } E + 1\\ &d_2 = BC - AD \text{ must divide } B(E + 1) \text{ and } A(E + 1), \text{ and}\\ &d_1 = BE - D - (BC - AD) \text{ must divide } B(E + 1),\\ &\qquad (A - 1)(E + 1), \text{ and } BC - AD + B + D.\end{aligned} \tag{3.4}$$

Thus complete Gorenstein toric threefolds X with isolated singularities and cyclic divisor class group are classified by quintuples of integers $(A\ B\ C\ D\ E)$ satisfying (3.1)–(3.4). Certain quintuples yield isomorphic toric threefolds, however, and I will first find all quintuple solutions and then analyze the isomorphism types.

IV. The quintuple solutions.

PROPOSITION 4.1. *The only quintuples* $(A\ B\ C\ D\ E)$ *satisfying* (3.1)–(3.4) *are* (1 1 2 1 3), (1 1 2 1 5), (1 1 4 1 5), (1 1 4 3 5), *and* (2 3 1 1 1).

PROOF. By (3.4) we may write $E + 1 = FD$ for some $F > 0$, so that $E = FD - 1$. Since $(A, B) = (A - 1, B) = 1$, the rest of (3.4) reduces to

$$BC - AD \text{ must divide } FD,$$

$$(4.2)\quad BFD - B - D - (BC - AD) \text{ must divide } FD, \text{ and}$$

$$BFD - B - D - (BC - AD) \text{ must divide } BC - AD + B + D.$$

However, $BC - AD + B + D = B(FD) - (BFD - B - D - (BC - AD))$, so that this last condition is implied by the previous one. Let $H = (B, D)$. Since $(C, D) = 1$, $(BC - AD, D) = H$ so that $BC - AD \mid FD \Leftrightarrow (BC - AD)/H \mid F$. Write $F = G(BC - AD)/H$. The only remaining divisibility condition is now that $I = BGD(BC - AD)/H - B - D - (BC - AD)$ must divide $GD(BC - AD)/H$. But $(D/H, I) = (D/H, B + BC) = 1$ since $(D, C + 1) = (D, C + 1 - FD) = (D, C - E) = 1$, so that the above reduces to $I \mid G(BC - AD)$. Now

$$\begin{aligned}
(I, BC - AD) &= (B + D, BC - AD) \\
&= (B + D, (A + C)D, BC - AD) \text{ since} \\
&\quad\ (A + C)D = (B + D)C - (BC - AD) \\
&= (B + D - BFD, (A + C - AFD)D, BC - AD \\
&\quad\ \text{since } BC - AD|FD \\
&= (D - BE, (AE - C)D, BC - AD) \\
&= (D - BE, D, BC - AD) \text{ since} \\
&\quad\ (D - BE, AE - C, BC - AD) = 1 \\
&= (BE, D, BC) \\
&= (BFD - B, D, BC) \\
&= (B, D) = H,
\end{aligned}$$

so $I|G(BC - AD)$ if and only if $I/H|G$.

Let $x = B/H$, $y = D/H$, and $z = (BC - AD)/H$. Then x, y, and z are all strictly positive, pairwise relatively prime, and $GHxyz - x - y - z$ must divide G. Hence $GHxyz - x - y - z \leqq G$, which implies $xyz \leqq x + y + z + 1$. The only solution to this last inequality are the triples $\{1, 1, n \geqq 1\}$, $\{1, 2, 2\}$, $\{1, 2, 3\}$, and $\{1, 2, 4\}$; since they must be pairwise relatively prime, we are left with $\{x, y, z\} = \{1, 1, n \geqq 1\}$ or $\{1, 2, 3\}$.

The case $\{1, 1, n\}$. We now seek solutions to $0 < GHn - 2 - n|G$. These are $(G\ H\ n) = (4\ 1\ 1)$, $(6\ 1\ 1)$, $(1\ 4\ 1)$, $(2\ 2\ 1)$, $(1\ 2\ 3)$, $(2\ 1\ 3)$, $(3\ 2\ 1)$, $(4\ 1\ 2)$ and $(2\ 1\ 4)$. There are three subcases to consider.

The case $x = y = 1$, $z = n$. In this case $B = D = H$, and since D is relatively prime to C and $C + 1$, H must be odd. This leaves the five solutions $(G\ H\ n) = (4\ 1\ 1)$, $(6\ 1\ 1)$, $(2\ 1\ 3)$, $(4\ 1\ 2)$, and $(2\ 1\ 4)$. Hence $H = B = D = 1$, forcing $A = 1$. (3.3) forces $(E - 1, C - 1) = 1$;

however $E - 1 = G(C - 1) - 2$ so C must be even, and $n = z = C - 1$ must be odd, leaving the three solutions $(G\ H\ n) = (4\ 1\ 1)$, $(6\ 1\ 1)$, and $(2\ 1\ 3)$. These give the quintuple $(A\ B\ C\ D\ E) = (1\ 1\ 2\ 1\ 3)$, $(1\ 1\ 2\ 1\ 5)$, and $(1\ 1\ 4\ 1\ 5)$ respectively.

The case $x = z = 1,\ y = n \geqq 2$. Here $B = H = BC - AD$ and $D = nH$. Again D must be odd, leaving only the solution $(G\ H\ n) = (2\ 1\ 3)$, giving the quintuple $(A\ B\ C\ D\ E) = (1\ 1\ 4\ 3\ 5)$.

The case $y = z = 1,\ x = n \geqq 2$. Here $D = H = BC - AD$ and $B = nH$. Since $(A, B) = (A - 1, B) = 1$, B is odd, leaving only the solution $(G\ H\ n) = (2\ 1\ 3)$, giving the quintuple $(A\ B\ C\ D\ E) = (2\ 3\ 1\ 1\ 1)$.

The case $\{x, y, z\} = \{1, 2, 3\}$. We now seek solutions to $0 < 6GH - 6 | G$ or $6(GH - 1) | G$; since $(G, GH - 1) = 1$, this implies $6 | G$. Write $G = 6K$; then the above is equivalent to $6K - 1 | K$, which has no solutions.

V. The proof of theorem 1.1. I will collect the solutions obtained in the previous section in Table (5.1).

Quintuple solutions (ABCDE) to (3.1)–(3.4)

A	B	C	D	E	d_1	d_2	d_3	d_4
1	1	2	1	3	1	1	1	1
1	1	2	1	5	3	1	1	1
1	1	4	1	5	1	3	1	1
1	1	4	3	5	1	1	3	1
2	3	1	1	1	1	1	1	3

Table 5.1

The solution (1 1 2 1 3) gives the toric threefold $\mathbf{P}^3$, since $v_4 = -(v_1 + v_2 + v_3)$ in this case (see [1], §5.3). The other solutions each have exactly one cone with determinant 3, and the reader can check that they are in the same orbit under the action of $GL(3, \mathbf{Z})$; hence they give one other threefold. Since the cone over the triple veronese surface is a Gorenstein toric threefold with one isolated singularity (at the vertex), and with cyclic divisor class group, this solution must lead to it. (This may of course be checked directly, also.)

VI. Gorenstein toric surfaces with cyclic Picard groups. In this section I will state the classification of proper Gorenstein toric surfaces with cyclic Picard group. The analysis is similar to that for three folds, only easier, and will be omitted. Let X be such a surface. The fan for X is generated by three vectors in $\mathbf{Z}^2$, which can be normalized to be $v_1 = [1, 0]$, $v_2 = [A, B]$, $v_3 = [-C, -D]$, with

(6.1) $$0 < A \leqq B,\ D > 0, \text{ and } BC > AD.$$

The Gorenstein condition is again a divisibility one: it is

$$d_3 = B \text{ must divide } A - 1,$$
(6.2) $$d_2 = BC - AD \text{ must divide } B + D \text{ and } A + C, \text{ and}$$
$$d_1 = D \text{ must divide } C + 1.$$

(The d_i are the determinants of the three cones.)

By switching the cones, if necessary, we may also assume that

(6.3) $$1 \leqq d_2 \leqq d_1 \leqq d_3, \text{ i.e., } 1 \leqq BC\text{–}AD \leqq D \leqq B.$$

There are exactly five solutions to (6.1)–(6.3), given in Table (6.4).

Solutions to (6.1) – (6.3)

A	B	C	D	d_1	d_2	d_3
1	1	2	1	1	1	1
1	2	1	1	1	1	2
1	3	1	2	2	1	3
1	4	1	2	2	2	4
1	3	2	3	3	3	3

Table 6.4

These five numerical solutions give five non-isomorphic surfaces. The solution (1 1 2 1) gives the plane $\mathbf{P}^2$. The solution (1 2 1 1) gives the quadric cone in $\mathbf{P}^3$. The solution (1 3 1 2) gives a surface whose resolution is a three-fold blowup of $\mathbf{P}^2$ with an anticanonical divisor a cycle of smooth rational curves with self-intersections 0, 1, -2, -1, -2, -2; the anticanonical map collapses the -2 curves to realize the surface as a sextic surface in $\mathbf{P}^6$ with one A_1 singularity and one A_2 singularity. The solution (1 4 1 2) gives a surface whose resolution is a five-fold blowup of $\mathbf{P}^2$ with an anticanonical divisor a cycle of smooth rational curves with self-intersections 0, -2, -1, -2, -2, -2, -1, -2; the anticanonical map realizes the toric surface as a quartic in $\mathbf{P}^4$, with two A_1 singularities and one A_3 singularity. The final solution (1 3 2 3) gives a surface whose resolution is a six-fold blowup of $\mathbf{P}^2$ with an anticanonical divisor a cycle of smooth rational curves with self-intersections -1, -2, -2, -1, -2, -2, -1, -2, -2; the anti-canonical map realizes this toric surface as the cubic surface $w^3 = xyz$ in $\mathbf{P}^3$ which has three A_2 singularities.

In the surface case, the isolated singularity condition is automatic, since every toric surface is normal. As seen in the threefold analysis, this condition (which forces variables to be relatively prime) when coupled with the Gorenstein condition (which forces certain variables to divide others) is very strong. I conjecture that in dimensions n greater than three, the only proper Gorenstein toric varieties with isolated singularities and cyclic divisor class group are the projective n-space $\mathbf{P}^n$, and the cone in $\mathbf{P}^{\binom{2n-1}{n}}$ over the n-fold veronese embedding of $\mathbf{P}^{n-1}$ into $\mathbf{P}^{\binom{2n-1}{n}-1}$.

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