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ON CONTINUED FRACTIONS CORRESPONDING TO ASYMPTOTIC SERIES

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ABSTRACT. Let $\{f_n\}$ be a sequence of complex-valued functions of a complex variable, each holomorphic at a point z_0 and meromorphic in a domain D containing z_0. Let $\{f_n\}$ correspond to a formal power series or to a formal Laurent series at z_0. [3, 148] Let a set $S \subset D$ and let z_0 be a limit point of S. Conditions are given for the functions f_n which insure that the corresponding series is the asymptotic expansion as $z \to z_0$, $z \in S$, of the limit of a subsequence of $\{f_n\}$. Applications are made to regular C-fractions, to general T-fractions, and to J-fractions.

DEFINITION. Let $\{f_n\}$ be a sequence of complex-valued functions of a complex variable, each holomorphic at a point z_0. Let $L = \sum_{k=0}^{\infty} c_k(z - z_0)^k$ be a formal power series, and let $G_m(z) = \sum_{k=0}^{m} c_k(z - z_0)^k$. The sequence $\{f_n\}$ is said to correspond to L at z_0, with order of correspondence ν_n, if there exists a sequence $\{\nu_n\}$ of positive integers such that $\nu_n \to \infty$ and

$$f_n(z) - G_{\nu_n - 1}(z) = O((z - z_0)^{\nu_n}),$$

as $z \to z_0$.

DEFINITION. Let $\{f_n\}$ be a sequence of complex-valued functions of a complex variable, each holomorphic at ∞. Let $L = \sum_{k=0}^{\infty} c_k z^{-k}$ be a formal Laurent series, and let $G_m(z) = \sum_{k=0}^{m} c_k z^{-k}$. The sequence $\{f_n\}$ is said to correspond to L at ∞, with order of correspondence ν_n, if there exists a sequence $\{\nu_n\}$ of negative integers such that $\nu_n \to -\infty$ and

$$f_n(z) - G_{\nu_n + 1}(z) = O(z^{\nu_n})$$

as $z \to \infty$.

A continued fraction with n^{th} approximant $f_n(z)$ is said to correspond to a formal power series or to a formal Laurent series if $\{f_n\}$ corresponds to the series.

THEOREM 1. *Let $\{f_n\}$ be a sequence of functions, holomorphic at z_0 and meromorphic in a domain D, with $z_0 \in D$. Let z_0 be a limit point of a set $S \subset D$. Let $\{f_n\}$ correspond to a formal power series $L = \sum_{k=0}^{\infty} c_k(z - z_0)^k$*

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at $z = z_0$, with order of correspondence greater than or equal to $n + 1$. If there exists a function f, defined on S, and positive constants $k_n (n = 1, 2, \ldots)$ such that

$$(1) \qquad |f(z) - f_n(z)| \leqq k_n |f_n(z) - f_{n-1}(z)|$$

for $z \in S$ and $n = 1, 2, \ldots$, then L is the asymptotic expansion of f as $z \to z_0$, $z \in S$.

PROOF. Since the order of correspondence is not less than $n + 1$,

$$f_n(z) - f_{n-1}(z) = \gamma_n (z - z_0)^n + \cdots$$

for $|z - z_0|$ sufficiently small. Substituting in (1), we obtain

$$|f(z) - f_n(z)| \leqq k_n |\gamma_n (z - z_0)^n + \cdots| \leqq k_n |\gamma_n| \, |z - z_0|^n + \cdots,$$

for $|z - z_0|$ sufficiently small and $z \in S$. Let $G_n(z) = \sum_{k=0}^{n} c_k (z - z_0)^k$, and let the Taylor series expansion of $f_n(z) = \sum_{k=0}^{\infty} a_k^{(n)} (z - z_0)^k$. Then

$$\begin{aligned} &|f(z) - G_{n-1}(z)| - |c_n (z - z_0)^n| \\ &\leqq |f(z) - G_n(z)| \leqq |f(z) - f_n(z)| + |f_n(z) - G_n(z)| \\ &\leqq k_n |\gamma_n| |z - z_0|^n + \cdots + \textstyle\sum_{k=n+1}^{\infty} |a_k^{(n)}| \, |z - z_0|^k. \end{aligned}$$

Therefore,

$$f(z) - G_{n-1}(z) = O((z - z_0)^n) \text{ as } z \to z_0, \ z \in S, \quad \text{for } n = 1, 2, \ldots,$$

and the theorem is proved. [**1**, 355].

Theorem 1 remains true if the right side of (1) is replaced by $k_n |f_{n+1}(z) - f_n(z)|$; in fact, the proof is easier. We have chosen the present form because it is more easily applied to specific continued fractions.

There is a somewhat similar theorem for $z_0 = \infty$.

THEOREM 2. *Let $\{f_n\}$ be a sequence of functions, holomorphic at ∞ and meromorphic in a domain D with $\infty \in D$. Let ∞ be a limit point of a set $S \subset D$. Let $\{f_n\}$ correspond to the formal Laurent series $L^* = \sum_{k=0}^{\infty} c_k z^{-k}$ at $z = \infty$ with order of correspondence less than or equal to $-n$. If there exist a function f, defined on S, and positive constants k_n $(n = 1, 2, \ldots)$ such that condition (1) holds for $z \in S$ and $n = 1, 2, \ldots$, then L^* is the asymptotic expansion of f as $z \to \infty$, $z \in S$.*

The proof is omitted, since it is similar to the proof of Theorem 1.

Theorems 1 and 2 provide a method of studying a problem about which surprisingly little is known. Given a formal power series in z^{-1} (or in z) and a continued fraction to which the series corresponds at $z = \infty$ ($z = 0$), suppose that the continued fraction converges to a holomorphic function $f(z)$ in a region D with $z = \infty$ ($z = 0$) on its boundary. Is the series the asymptotic expansion of $f(z)$ at $z = \infty$ ($z = 0$) with respect to D? Previous results were obtained using the theory of moments. Theorems 1 and 2 give a different approach to the problem and extend these previous results. Also our proofs are somewhat simpler. See [**3**, 331] for a more detailed discussion.

THEOREM 3. *Let $\{f_n\}$ be a sequence of functions, each holomorphic at z_0 and meromorphic in a domain containing z_0. Let*

$$|f_{m+n}(z) - f_n(z)| \leqq c|f_n(z) - f_{n-1}(z)|, \tag{2}$$

for $m, n = 1, 2, \ldots, z \in S$, where c is a positive constant independent of m, n, and $z \in S$. Let a subsequence $\{f_{n_k}(z)\}$ converge to $f(z)$ for $z \in S$. Then condition (1) *is satisfied.*

PROOF. Let $m \to \infty$ in (2) in such a way that $f_{n+m}(z)$ is always an element of the subsequence $\{f_{n_k}(z)\}$.

In 1971, Jones and Thron, in a paper [**2**] on truncation errors of continued fractions, showed that the approximants of a number of different kinds of continued fractions satisfy condition (2). Such sequences are called simple sequences. If the continued fraction converges, then the right side of (2) gives an upper bound to the truncation error.

We shall give three applications of Theorems 1–3. The notation $K_{n=1}^{\infty}(a_n/b_n)$ is used to denote the continued fraction $\frac{a_1}{b_1}+\frac{a_2}{b_2}+\frac{a_n}{b_n}+\cdots$. The even part of a continued fraction is said to converge if the sequence of approximants $\{f_{2n}\}$ converges. Convergence of the odd part is similarly defined.

Regular C-fractions. A continued fraction of the form $\mathbf{K}(a_n z/1)$, $a_n \neq 0$ for all n, is called a regular C-fraction.

By an equivalence transformation [**3**, 31] any regular C-fraction can be put in the form $\mathbf{K}(z/b_n)$, with $b_n \neq 0$ for all n. The b_n's may be obtained recursively by $b_1 = a_1^{-1}$, $b_n = a_n^{-1}b_{n-1}^{-1}$ for $n = 2, 3, \ldots$. In the following theorem, conditions are given in terms of the b_n, rather than the a_n, because of the greater simplicity.

THEOREM 4. *Let $E_1 = \{z\colon |\arg z| \leqq \alpha\}$, $E_2 = \{z\colon |\arg z| \leqq \beta\}$, and $S = \{z\colon |\arg z| \leqq \gamma\}$, where α, β and γ are non-negative numbers satisfying the following conditions: $\alpha + \gamma < \pi/2$, $\beta < \pi/2$, and $2\beta + \gamma < \pi$. Let $\mathbf{K}(z/b_n)$, $b_n \neq 0$ for all n, be a continued fraction with $b_{2n} \in E_1$, and $b_{2n-1} \in E_2$ for $n = 1, 2, \ldots$. The even and odd parts of the continued fraction converge in S to functions f and g respectively. Then the formal power series to which the continued fraction corresponds at the origin is the asymptotic expansion of f, and also of g, as $z \to 0$, $z \in S$.*

PROOF. The continued fraction $\mathbf{K}(z/b_n)$ is equivalent to the continued fraction

$$\frac{1}{b_1 z^{-1}} + \frac{1}{b_2} + \frac{1}{b_3 z^{-1}} + \frac{1}{b_4} + \cdots .$$

By the Van Vleck criterion [**3**, 88], the even and odd parts converge to finite values, since $b_{2n} \in E_1$, $b_{2n-1} \in E_2$ and $z \in S$. The n^{th} approximant of $\mathbf{K}(z/b_n)$ is equal to the n^{th} approximant of an equivalent regular C-fraction. Therefore, $\mathbf{K}(z/b_n)$ corresponds to a formal power series at the origin, with order of correspondence $n + 1$ [**3**, 222].

It remains to show that the approximants from a simple sequence in S To do this we use a special case of Corollary 2.2 [**2**, 699]

Let $\mathbf{K}(a_n/b_n)$ be a continued fraction with n^{th} approximant f_n If arg $a_n = \gamma$ for all n and if there is a constant θ such that $0 < \theta < \pi$ and such that for all $n \geqq 1$,

$$0 \leqq \arg\{b_n \exp[i(\theta - \gamma)/2]\} \leqq \theta, \tag{3}$$

then for all positive integers m and n

$$\begin{aligned} |f_{n+m} - f_n| &\leqq |f_n - f_{n-1}|, \text{ if } 0 < \theta \leqq \pi/2, \text{ and} \\ &\leqq \sec(\theta - \pi/2)|f_n - f_{n-1}|, \text{ if } \pi/2 < \theta < \pi. \end{aligned}$$

For $\mathbf{K}(z/b_n)$, condition (3) is equivalent to $|2\arg b_n - \arg z| \leqq \theta$. But, if $z \in S$, $0 \leqq |2 \arg b_n - \arg z| \leqq 2\alpha + \gamma$ or $2\beta + \gamma$ according as n is even or odd. We may take $\theta = \max(2\alpha + \gamma, 2\beta + \gamma)$. Thus, condition (2) is satisfied.

A regular C-fraction is called on S-fraction if $a_n > 0$ for all n. Theorem 4 applies to S-fractions, but it is possible, using similar methods, to obtain this result, with a "larger" set $S = \{z\colon |\arg z| \leqq \pi\}$. Of course, one can replace z by $1/z$ and obtain as asymptotic expansion as $z \to \infty$. This form of the theorem is given in [**1**, 565] with the set S as the positive real axis. The method of proof in [**1**] is different from that given here.

General T-fractions. A continued fraction of the form $\mathbf{K}(z/(e_n + d_n z))$, $e_n \neq 0$, $d_n \neq 0$, is called a general T-fraction.

THEOREM 5. *Let* $E_1 = \{z\colon |\arg z| \leqq \alpha\}$, $E_2 = \{z\colon |\arg z| \leqq \beta\}$, *and* $S = \{z\colon |\arg z| \leqq \gamma\}$, *where* α, β, *and* γ *are non-negative numbers satisfying the following conditions*: $\alpha + \gamma < \pi/2$, $2\alpha + 3\gamma < \pi$, *and* $\beta \leqq \alpha + \gamma$. *Given a general T-fraction with* $d_n \in E_1$ *and* $e_n \in E_2$ *for all n, the even and odd parts of the T-fraction converge to functions f and g, respectively, which are holomorphic in S. The formal power series to which the T-fraction corresponds at the origin is the asymptotic expansion of f, and also of g, as*

$z \to 0$, $z \in S$. *The formal Laurent series to which the T-fraction corresponds at* ∞ *is the asymptotic expansion of f, and also of g, as* $z \to \infty$, $z \in S$.

PROOF. The fact that the even and odd parts of the T-fraction converge to holomorphic functions in S follows from Theorem 4.64 [3, 143]. By Theorem 7.17 [3, 259] the T-fraction corresponds to a formal power series at the origin with order of correspondence $n + 1$ and also to a formal Laurent series at ∞ with order of correspondence $-n$.

For the general T-fraction, condition (3) is equivalent to

$$|2 \arg(e_n + d_n z) - \arg z| \leqq \theta.$$

We have $|\arg d_n z| \leqq \alpha + \gamma$. Since the sum of two complex numbers which lie in a convex angular opening also lies in that angular opening, $|\arg(e_n + d_n z)| \leqq \alpha + \gamma$. Thus,

$$|2 \arg(e_n + d_n z) - \arg z| \leqq 2\alpha + 3\gamma.$$

We may take $\theta = 2\alpha + 3\gamma$, and condition (2) is satisfied.

In 1980 Jones, Thron, and Waadeland [4, 519] proved the above result for T-fractions with $d_n > 0$ and $e_n > 0$, the set S being $\{z\colon |\arg z| \leqq \alpha < \pi\}$. This proof used integral representations of the approximants. Theorems 1–3 can be used to give a simpler proof.

J-fractions. A continued fraction of the form

$$\frac{1}{d_1 + z} - \frac{c_2^2}{d_2 + z} - \frac{c_3^2}{d_3 + z} - \cdots,$$

where the c_n's and d_n's are complex numbers, is called a J-fraction.

THEOREM 6. *A J-fraction, with* c_n *real*, $\mathrm{Im}(d_n) \geqq 0$, *and* $|\mathrm{Re}(d_n)| \leqq M$ *for some* $M > 0$, *converges to a holomorphic function f in* $\{z\colon \mathrm{Im}(z) > 0\}$, *provided* $|c_n| \leqq N$ *for some* $N > 0$ *and* $\mathrm{Im}(d_1) > 0$. *The formal Laurent series* $\sum_{k=1}^{\infty} a_k z^{-k}$ *to which such a continued fraction corresponds at* $z = \infty$ *is the asymptotic expansion of f as* $z \to \infty$, $z \in S$, *where* $S = \{z\colon \pi/4 \leqq \arg(z - Mi) \leqq 3\pi/4\}$.

PROOF. The continued fraction is a positive definite J-fraction; hence, it converges to a holomorphic function in $\{z\colon \mathrm{Im}(z) > 0\}$. [3, 138]. The J-fraction corresponds to a formal Laurent series at $z = \infty$, with order of correspondence $-2n - 1$. [3, 250].

It remains to show that the approximants from a simple sequence in S. Let $S_1 = \{z\colon 0 < \arg z \leqq \pi/2\}$, $S_2 = \{z\colon \pi/4 \leqq \arg(z + M) \leqq \pi/2\}$, $S_3 = \{z\colon \pi/2 \leqq \arg z < \pi\}$, and $S_4 = \{z\colon \pi/2 \leqq \arg(z - M) \leqq 3\pi/4\}$. By [2, 704–705], relation (2) holds with $c = 1$ for all $z \in S_1 \cap S_2$ and also for all $z \in S_3 \cap S_4$. Thus, (2) holds with $c = 1$ for all $z \in S = (S_1 \cap S_2) \cup (S_3 \cap S_4)$. But S is the set $\{z: \pi/4 \leqq \arg(z - Mi) \leqq 3\pi/4\}$.

Previous theorems concerning J-fractions corresponding to asymptotic series were proved using integral representations and were for real J-fractions (i.e., for J-fractions with c_n real and d_n real). See [**3**, 342].

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