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MULTIPLIERS FOR SOME SPACES OF BANACH ALGEBRA VALUED FUNCTIONS

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ABSTRACT. Let G be a locally compact abelian group, and A be a commutative Banach algebra. Let $C_0(G, A)$ be the Banach algebra of A-valued continuous functions on G which vanish at infinity. It is the object of this paper to characterize the space of multipliers for the space $C_0(G, A)$ regarded as a Banach algebra and regarded as an $L^1(G, A)$-module, respectively, where $L^1(G, A)$ is the Banach algebra of A-valued Bochner integrable functions on G. We prove that the space of algebra multipliers of $C_0(G, A)$ is isometrically isomorphic to $C^b(G, \mathscr{M}(A))$, the bounded continuous $\mathscr{M}(A)$-valued functions on G where $\mathscr{M}(A)$ denotes the multiplier algebra of the Banach algebra A with a bounded approximate identity. It is proved also that the $L^1(G, A)$-module homomorphisms of $C_0(G, A)$ is identified with $M(G, A)$ when A has identity of norm 1 where $M(G, A)$ is the A-valued regular Borel measure of bounded variation on G.

1. Introduction and preliminaries. Let G be a locally compact abelian group with Haar measure dt, and A be a commutative Banach algebra with a bounded approximate identity. The space $C_0(G, A)$ of A-valued continuous functions on G vanishing at infinity forms a commutative Banach algebra under pointwise products. $M(G, A)$ is the space of A-valued regular Borel measures of bounded variation on G.

For any commutative Banach algebra A, a linear map $T: A \to A$ is called a multiplier for A if $T(ab) = a(Tb) = (Ta)b$. We denote by $\mathscr{M}(A)$ the space of all multipliers for A. Clearly $\mathscr{M}(A)$ is a Banach algebra as a subalgebra of bounded linear operators on A. For the general theory of multipliers we refer to Larsen [7], and some characterizations of multipliers of Banach algebras studied also in Lai [6]. For the theory of vector valued functions or vector measures, one can consult Dinculeanu [1], [2] and Johnson [4] for the spaces of Banach algebra valued functions on a locally compact group.

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Recently, in 1981 Tewari, Dutta the Vaidya **[10]** and Khalil **[5]** studied the multipliers for some spaces of vector-valued functions on a locally compact abelian group G. In **[10]**, they proved that the multiplier algebra for $L^1(G, A)$ is isometrically isomorphic to $M(G, A)$ where $L^1(G, A)$ is the Banach algebra of A-valued functions on G under convolution and A has identity of norm 1. If G is a compact abelian group, Khalil **[5]** showed that $\mathscr{M}(L^1(G, A))$ is isomorphic to $M(G, \mathscr{M}(A))$ and the multipliers of the Hilbert algebra $L^2(G, A)$ is isomorphic to $L^\infty(\hat{G}, \mathscr{M}(A))$.

We shall use the concept of module tensor products and their relations to multipliers (see Rieffel **[8]** and **[9]**). If V and W are A-modules, the A-module tensor product $V \otimes_A W$ is defined to be the quotient Banach space $V \hat{\otimes}_\gamma W/K$ where K is the closed linear subspace of the projective tensor product $V \hat{\otimes}_\gamma W$ spanned by the elements $av \otimes w\text{-}v \otimes aw$ with $a \in A$, $v \in V$ and $w \in W$. A continuous linear transformation T from V to W is called A-module homomorphism if

$$T(a \cdot v) = a \cdot Tv \qquad \text{for all } v \in V \text{ and } a \in A.$$

The space of all A-module homomorphisms from V to W is denoted by $\mathrm{Hom}_A(V, W)$ which is a Banach space under the operator norm. Evidently $\mathrm{Hom}_A(A, A) = \mathscr{M}(A)$ the multiplier algebra of A. In **[9]** Rieffel has shown that $\mathrm{Hom}_A(V, W^*) \cong (V \otimes_A W)^*$, where $\cong$ denotes the isometric isomorphism under which an operator $T \in \mathrm{Hom}_A(V, W^*)$ defines a linear functional on $V \otimes_A W$ with value $\langle Tv, w\rangle$ at $v \otimes w \in V \otimes_A W$.

It is known that $L^1(G, A) \cong L^1(G) \hat{\otimes}_\gamma A$, the completed projective tensor product of $L^1(G)$ with A, and $C_0(G, A) \cong C_0(G) \otimes_\varepsilon A$, the completed injective tensor product of $C_0(G)$ with A. In **[10]** Theorem 4, it is proved that

$$\begin{aligned}\mathscr{M}(L^1(G, A)) &= \mathrm{Hom}_{L^1(G,A)}(L^1(G, A), L^1(G, A)) \\ &\cong M(G, A)\end{aligned}$$

where A is a commutative Banach algebra with identity of norm 1. It is proved also in **[10]** that an invariant operator of $L^1(G, A)$ need not be a multiplier for $L^1(G, A)$ which is different from the multipliers for $L^1(G)$ since a bounded linear operator on $L^1(G)$ is a multiplier of $L^1(G)$ if and only if it is an invariant operator.

Since $C_0(G, A)$ is a Banach algebra under pointwise product and supremum norm defined by $\|f\|_\infty = \sup_{t\in G}|f(t)|_A$, where $|\cdot|_A$ is the norm of A, and since it is also a Banach $L^1(G, A)$-module under convolution, we study in this paper the multipliers for $C_0(G, A)$ of the following two types.

(a) T is a linear operator of $C_0(G, A)$ such that

$$T(f \cdot g) = f \cdot Tg = Tf \cdot g \qquad \text{for } f, g \in C_0(G, A)$$

Since $C_0(G, A)$ is a commutative Banach algebra with an approximate identity under pointwise product, it is without order. Then by the Closed Graph Theorem, it can be shown that the linear operator T satisfying the formula in (a) is continuous.

(b) T is a bounded linear operator of $C_0(G, A)$ such that

$$T(f*g) = f*Tg \text{ for all } f \in L^1(G, A) \text{ and } g \in C_0(G, A).$$

We say that the operators of type (a) are algebra multipliers and operators of type (b) are $L^1(G, A)$-module multipliers for $C_0(G, A)$. We shall establish in this paper that

$$\text{(1)} \qquad \begin{aligned} \mathrm{Hom}_{C_0(G,A)}(C_0(G, A), C_0(G, A)) &= \mathcal{M}(C_0(G, A)) \\ &\cong C^b(G, \mathcal{M}(A)) \end{aligned}$$

and

$$\text{(2)} \qquad \begin{aligned} \mathrm{Hom}_{L^1(G,A)}(C_0(G, A), C_0(G, A)) &= \mathcal{M}_{L^1}(C_0(G, A)) \\ &\cong M(G, A). \end{aligned}$$

Note that $C_0(G, A)$ is not a Banach algebra under convolution.

2. A characterization of the algebra multipliers for C_0(G, A). The following lemma is useful subsequently.

LEMMA 1. *If $T \in \mathcal{M}(C_0(G, A))$, then $T(af) = aTf$ for $f \in C_0(G, A)$ and $a \in A$.*

PROOF. Since $C_0(G)$ is a Banach algebra with a bounded approximate identity, $\{u_\alpha\}$, letting $f = f_1 \otimes b \in C_0(G) \check{\otimes}_\varepsilon A = C_0(G, A)$, one has

$$\begin{aligned} T(af) &= \lim_\alpha T((u_\alpha \otimes a) \cdot (f_1 \otimes b)) \\ &= \lim_\alpha (u_\alpha \otimes a)\, T(f_1 \otimes b) \\ &= aTf \end{aligned}$$

for all $a \in A$, where the limit is in $C_0(G, A)$.

Our first result is to characterize the multipliers of type (a). It is similar to a result of Lai [**6**, Corrollary 6.5] where the strong continuity argument is used.

THEOREM 2. *Let A be a Banach algebra with a bounded approximate identity $\{e_k\}$. Then*

$$\text{(3)} \qquad \mathcal{M}(C_0(G, A)) \cong C^b(G, \mathcal{M}(A)).$$

PROOF. Let $h \in C^b(G, \mathscr{M}(A))$ and $f \in C_0(G, A)$. Then $h \cdot f$ is a continuous function on G vanishing at infinity, that is, $hf \in C_0(G, A)$. Evidently h defines a multiplier, $T \in \mathscr{M}(C_0(G, A))$, by $h(t)(f(t)) = Tf(t)$ and $\|T\| = \|h\|_\infty$.

Conversely, for any $a \in A$ and $f \in C_0(G)$, it is obvious that $af \in C_0(G, A)$ and $\|af\|_\infty = |a|_A |f|_\infty$. Thus if $T \in \mathscr{M}(C_0(G, A))$ then $T(af) \in C_0(G, A)$. Now if $f \in C_0(G)$, the mapping $t \to T(f \otimes a)(t)/f(t) = h_T(t)(a)$, for $a \in A$, defines an A-valued function whenever $f(t) \neq 0$. The function $h_T(t)$ defined in this way is independent of the choice of $f \in C_0(G)$. Indeed let $\{e_\alpha\}$ be a bounded approximate identity for A and f, $g \in C_0(G)$ such that $f(t) \neq 0$, $g(t) \neq 0$, we have

$$\begin{aligned} T(af \cdot e_\alpha g)(t) &= e_\alpha g(t) \cdot T(af)(t) \\ &= e_\alpha f(t) \cdot T(ag)(t) \end{aligned}$$

or

$$e_\alpha \cdot \frac{T(af)(t)}{f(t)} = e_\alpha \cdot \frac{T(ag)(t)}{g(t)},$$

and then

$$\frac{T(af)(t)}{f(t)} = \frac{T(ag)(t)}{g(t)}.$$

Therefore $h_T(t)$ is a linear operator on A and we write

$$\begin{aligned} T(af)(t) &= f(t)h_T(t)(a) \\ &= h_T(t)(af)(t) \qquad \text{for all } a \in A, f \in C_0(G). \end{aligned}$$

Moreover h_T is bounded and $\|h_T(af)\|_\infty \leqq \|T\| \; \|af\|_\infty = \|T\| \; |a| \; |f|_\infty$. This shows that h_T is strongly continuous.

We need to show, with emphasis on the fact, that the function $h_T(\cdot)$ is continuous on G with respect to the norm topology of $\mathscr{M}(A)$.

Let $t_0 \in G$. Then there exists $f \in C_0(G)$ such that $f(t_0) \neq 0$ and $N = N(t_0) = \{t \in G, f(t) \neq 0\}$ is an open neighborhood of t_0. Thus $h_T(t)a = (T(af)(t)/f(t)$, for $t \in N$, is a strong continuous function of values in A. We let $\{t_\alpha\} \subset N$ with $t_\alpha \to t_0$ in G. Then we have to show that

$$\begin{aligned} \|h_T(t_\alpha) - h_T(t_0)\|_{\mathscr{M}(A)} &= \sup_{|a|_A \leqq 1} |h_T(t_\alpha)a - h_T(t)a|_A \\ &\to 0 \text{ as } t_\alpha \to t_0. \end{aligned}$$

Indeed,

$$\begin{aligned}|h_T(t_\alpha)a - h_T(t_0)a|_A &= \left|\frac{T(af)(t_\alpha)}{f(t_\alpha)} - \frac{T(af)(t_0)}{f(t_0)}\right|_A \\ &= \frac{1}{|f(t_\alpha)f(t_0)|}\,|f(t_0)T(af)(t_\alpha) - f(t_\alpha)T(af)(t_0)|_A \\ &\leqq \frac{1}{|f(t_\alpha)f(t_0)|}\,\{|f(t_0)[T(af)(t_\alpha) - T(af)(t_0)]|_A \\ &\quad + |[f(t_\alpha) - f(t_0)]T(af)(t_0)|_A\}.\end{aligned}$$

Since $f \in C_0(G)$, $f(t_\alpha) \to f(t_0)$ as $t_\alpha \to t_0$ in G, it follows that the second term of $\{\cdot\}$ in the last inequality tends to zero when $t_\alpha \to t_0$. It remains to show that the first term of $\{\cdot\}$ tends to zero uniformly on $\{a \in A;\ |a|_A \leqq 1\}$. Let $\{e_k\}$ be a bounded approximate identity of A. Then for any $\varepsilon > 0$ there exists $k_0 = k_0(\varepsilon)$ depending on ε only such that $\|e_{k_0}T(af) - T(af)\|_\infty < \varepsilon/4$. For this $\varepsilon > 0$ and any $a \in A$ with $|a|_A \leqq 1$, we have

$$\begin{aligned}|e_{k_0}T(af)(t_\alpha) - e_{k_0}T(af)(t_0)|_A &= |T(ae_{k_0}f)(t_\alpha) - T(ae_{k_0}f)(t_0)|_A \\ &= |aT(e_{k_0}f)(t_\alpha) - aT(e_{k_0}f)(t_0)|_A \\ &\leqq |T(e_{k_0}f)(t_\alpha) - T(e_{k_0}f)(t_0)|_A \\ &< \frac{\varepsilon}{2}, \text{ whenever } t_\alpha \text{ is near } t_0,\end{aligned}$$

since $T(e_{k_0}f) \in C_0(G, A)$. Hence

$$\begin{aligned}|T(af)(t_\alpha) - T(af)(t_0)|_A &\leqq |T(af)(t_\alpha) - e_{k_0}T(af)(t_\alpha)|_A \\ &\quad + |e_{k_0}T(af)(t_\alpha) - e_{k_0}T(af)(t_0)|_A \\ &\quad + |e_{k_0}T(af)(t_0) - T(af)(t_0)|_A \\ &\leqq 2\|T(af) - e_{k_0}T(af)\|_\infty \\ &\quad + |T(e_{k_0}f)(t_\alpha) - T(e_{k_0}f)(t_0)|_A \\ &< \varepsilon\end{aligned}$$

when t_α near t_0. Therefore

$$\lim_{t_\alpha \to t_0} |T(af)(t_\alpha) - T(af)(t_0)|_A < \varepsilon;$$

since ε is arbitrary, it follows that

$$\lim_{t_\alpha \to t_0} |T(af)(t_\alpha) - T(af)(t_0)|_A = 0$$

uniformly on $\{a \in A\colon |a|_A \leqq 1\}$. Hence

$$\lim_{t_\alpha \to t_0} \|h_T(t_\alpha) - h_T(t_0)\|_{\mathscr{M}(A)} = 0.$$

Finally, we have

$$\begin{aligned}\|h_T(t)\|_{\mathscr{M}(A)} &= \sup_{|af(t)|_A=1} |h_T(t)(af(t))|_A \\ &= \sup_{|af(t)|_A=1} |T(af)(t)|_A \\ &\leqq \|T\|,\end{aligned}$$

and so $\sup_t \|h_T(t)\|_{\mathscr{M}(A)} = \|h_T\|_\infty \leqq \|T\|$. On the other hand

$$\begin{aligned}\|T(af)\|_\infty &= \sup_t |h_T(t)(af(t))|_A \\ &\leqq \sup_t \|h_T(t)\|_{\mathscr{M}(A)} \|af\|_\infty \\ &= \|h_T\|_\infty |a|_A |f|_\infty.\end{aligned}$$

Consequently, $\|T\| \leqq \|h_T\|_\infty$ proves $\|h_T\|_\infty = \|T\|$. Hence the proof is completed.

3. A-valued duality between $C_0(G, A)$ and M(G, A). The arguments in this section are similar to their counterparts in Larsen [7] for scalar function spaces. At first we give the following definition in the space of vector valued functions.

DEFINITION 1. We say that a space $F(G, A)$ is an A-valued dual of the space $E(G, A)$ if for each $f \in E(G, A)$, the pair $\langle f, g \rangle$ defines an element of A by

$$f \to \langle f, g \rangle = \int_G f(t)g(t)dt \text{ for } g \in F(G, A)$$

and $|\langle f, g\rangle|_A \leqq \|f\|_E \|g\|_F$.

That is, each $g \in F(G, A)$ defines a bounded linear A-valued functional which maps $f \in E(G, A) \to \langle f, g \rangle \in A$.

Here any $f \in E(G, A)$ and $g \in F(G, A)$ form a dual pair $\langle f, g \rangle$ of A-valued, and $F(G, A)$ is considered as the A-valued dual space of $E(G, A)$ with respect to the weak*-topology induced from $E(G, A)$, that is, each $g \in F(G, A)$ corresponds to an A-valued linear functional

$$f \to \langle f, g \rangle = \int_G f(t)g(t)dt$$

which is continuous in the weak*-topology induced from $F(G, A)$. We denote by $F_W(G, A)$ the A-valued continuous linear functional of $E(G, A)$ in weak*-topology. Then $E(G, A)$ is the A-valued dual of $F_W(G, A)$.

By Definition 1, $M(G, A)$ is the A-valued dual of $C_0(G, A)$ under which each $\mu \in M(G, A)$ is associated with the functional defined by

$$f \to \langle f, \mu \rangle = \int_G f(t)d\mu(t) \qquad f \in C_0(G, A) \tag{4}$$

(cf. Dinculeanu [1], [2]). Evidently, $|\langle f, \mu\rangle|_A \leqq \|f\|_\infty \|\mu\|$, and the integration in (4) is well defined since $C_c(G, A)$ is dense in $C_0(G, A)$ and for $f \in C_c(G, A)$, the integral in (4) is approximable by a finite sum of elements of A (see Johnson [4]).

The convolution of $\mu, \nu \in M(G, A)$ is defined as an A-valued measure by the following formula.

$$\begin{aligned}\langle f, \mu * \nu\rangle &= \int_G f(t) d(\mu * \nu)(t) \\ &= \int_G \int_G f(ts) d\mu(s) d\nu(t) \text{ for any } f \in C_0(G, A).\end{aligned}$$

This is well defined by the same reason given above.

LEMMA 3. *If $T \in \mathcal{M}_{L^1}(C_0(G, A))$, then T commutes with the translation operator, ρ_s, that is, $T\rho_s = \rho_s T$ for every $s \in G$. Here $\rho_s f(t) = f(ts)$.*

PROOF. Let $f \in L^1(G, A)$, $g \in C_0(G, A)$ and $T \in \mathrm{Hom}_{L^1(G,A)}(C_0(G, A))$. Then $f * g$ and $T(f * g) = f * Tg$ are in $C_0(G, A)$. Thus for $s \in G$,

$$\begin{aligned}\rho_s T(f * g)(0) &= T(f * g)(s) \\ &= f * Tg(s) \\ &= \rho_s f * Tg(0) \\ &= T(\rho_s f * g)(0) \\ &= T\rho_s(f * g)(0).\end{aligned}$$

Hence $\rho_s T = T\rho_s$.

In [10. Theorem 3], it has shown that there exists an invariant operator T of $L^1(G, A)$, that is, T is a bounded linear operator of $L^1(G, A)$ commuting with transiation, such that T is not a multiplier of $L^1(G, A)$. This means that T does not commute with convolution in $L^1(G, A)$. It follows that an invariant operator of $C_0(G, A)$ need not be an $L^1(G, A)$-module homomorphism. We establish the following results which are those for the scalar valued functions in Larsen [7]. The only difference is that commuting with translation is modified by commuting with convolution.

The following theorem is essential in the characterization of $L^1(G, A)$-module multipliers for $C_0(G, A)$.

THEOREM 4. *Let A be a commutative Banach algebra with an identity e of norm 1. Then a continuous linear operator T on $M_W(G, A)$ commutes with convolution in $M(G, A)$ if and only if there exists a unique $\xi \in M(G, A)$ such that $T\mu = \xi * \mu$, for all $\mu \in M(G, A)$.*

PROOF. If T is a continuous linear operator on $M_W(G, A)$ commuting with convolution in $M(G, A)$, then for any $\mu, \nu \in M(G, A)$, $T(\nu * \mu) = T\nu * \mu$. Let $\nu = \delta$ be the Dirac measure with point mass at the origin of G. Then δ is an identity of $M(G, A)$. It follows that $T\mu = (T\delta) * \mu$ for each $\mu \in M(G, A)$. That is, $\xi = T\delta$ is a fixed unique element in $M(G, A)$ such that $T\mu = \xi * \mu$. Conversely, if $\mu \to T\mu = \xi * \mu$ for all $\mu \in M(G, A)$, then it is obvious that T is a linear operator on $M_W(G, A)$ commuting with convolution. We have only to show that T is continuous. In fact, let $\{\mu_\alpha\} \subset M_W(G, A)$ converge to $\mu \in M_W(G, A)$, that is, for any $h \in C_0(G, A)$, $\lim_\alpha \langle h, \mu_\alpha\rangle = \langle h, \mu\rangle$ in A-norm topology. Since $\langle h, T\mu_\alpha\rangle = \langle h, \xi * \mu_\alpha\rangle = \langle(\tilde{h} * \xi), \mu_\alpha\rangle$, where $\tilde{h}(t) = h(t^{-1})$ and $\tilde{h} \in C_0(G, A)$ if $h \in C_0(G, A)$, the convolution $\tilde{h} * \xi$ of $\tilde{h} \in C_0(G, A)$ and $\xi \in M(G, A)$ is given by $\tilde{h} * \xi(t) = \int_Q \tilde{h}(ts^{-1})d\xi(s) \in A$. This is an element of $C_0(G, A)$. It follows that

$$\begin{aligned}\langle h, T\mu_\alpha\rangle &= \langle(\tilde{h} * \xi)^{\sim}, \mu_\alpha\rangle \\ \to \langle(\tilde{h} * \xi)^{\sim}, \mu\rangle &= \langle h, \xi * \mu\rangle = \langle h, T\mu\rangle\end{aligned}$$

in the topology of A. Hence $\{T\mu_\alpha\rangle$ converges to $T\mu$ in $M_W(G, A)$, and the proof is completed.

4. The L^1(G, A)-module multipliers for C_0(G, A). The space $C_0(G, A)$ is an $L^1(G, A)$-module under convolution, its multiplier is defined to be the space of module homomorphisms mentioned in §1. We characterize this type of multiplies as follows.

THEOREM 5. *Let A be a Banach algebra with identity of norm* 1, *and T be a bounded linear operator on $C_0(G, A)$. Then the following two statements are equivalent*:

i) $T \in \mathscr{M}_{L^1}(C_0(G, A))$

ii) *There exists a unique $\mu \in M(G, A)$ such that $Tf = \mu * f$ for all $f \in C_0(G, A)$.*

Moreover, in the correspondence of T and μ, we have the isometric isomorphic relation

$$\mathscr{M}_{L^1}(C_0(G, A)) \cong M(G, A). \tag{2}$$

PROOF. ii) implies i) is easy. In fact, let $\mu \in M(G, A)$, we define a mapping T by

$$f \to Tf = \mu * f \text{ for all } f \in C_0(G, A).$$

The convolution of $M(G, A)$ and $C_0(G, A)$ determines an element in $C_0(G, A)$ and this T is a bounded linear operator on $C_0(G, A)$. Evidently, T is an $L^1(G, A)$-module homomorphism since for $g \in L^1(G, A)$, $T(g * f) = \mu * g * f = g * \mu * f = g * Tf$ for all f in $C_0(G, A)$. Moreover,

$$\begin{aligned}\|Tf\|_\infty &= \|\mu * f\|_\infty \\ &= \sup_t |\mu * f(t)|_A \\ &= \sup_t \int_G |f(ts^{-1})|_A |d\mu(s)|_A \\ &\leqq \|f\|_\infty \|\mu\|.\end{aligned}$$

This implies $\|T\| \leqq \|\mu\|$. On the other hand, since $\|\mu * f\|_\infty = \|Tf\|_\infty \leqq \|T\| \, \|f\|_\infty$, we have $\|\mu\| \leqq \|T\|$, so that $\|T\| = \|\mu\|$.

i) implies ii). Let $T \in \mathcal{M}_{L^1}(C_0(G, A))$. Since $M(G, A)$ is the A-valued dual of $C_0(G, A)$, we can consider a mapping

$$T^*: M_W(G, A) \to M_W(G, A)$$

defined by $\langle Tf, \mu\rangle = \langle f, T^*\mu\rangle$ in A, for any $f \in C_0(G, A)$ and $\mu \in M(G, A)$. Then for any $\mu, \nu \in M(G, A)$,

$$\begin{aligned}\langle f, T^*(\mu * \nu)\rangle &= \langle Tf, \mu * \nu\rangle = \langle Tf * \tilde{\mu}, \nu\rangle \\ &= \langle T(f * \tilde{\mu}), \nu\rangle = \langle f * \tilde{\mu}, T^*\nu\rangle \\ &= \langle f, \mu * T^*\nu\rangle\end{aligned}$$

for all $f \in C_0(G, A)$, where

$$\begin{aligned}f * \tilde{\mu}(t) &= (\tilde{f} * \mu)^{\sim}(t) = (\tilde{f} * \mu)(t^{-1}) \\ &= \int \tilde{f}(t^{-1}s^{-1}) d\mu(s) = \int f(st) d\mu(s).\end{aligned}$$

Therefore $T^*(\mu * \nu) = \mu * T^*\nu$ in $M_W(G, A)$. That is, T^* commutes with convolution in $M_W(G, A)$. Applying Theorem 4, there is a unique $\xi \in M(G, A)$ such that $T^*\mu = \xi * \mu$. Hence $\langle Tf, \mu\rangle = \langle f, T^*\mu\rangle = \langle f, \xi * \mu\rangle = \langle f * \xi, \mu\rangle$ for all $\mu \in M(G, A)$. This implies $Tf = f * \xi$, for $\xi \in M(G, A)$. It is easy to verify that $\|T\| = \|\xi\|$. Therefore

$$\mathcal{M}_{L^1}(C_0(G, A)) \cong M(G, A).$$

The proof is completed.

REMARK. In Theorem 5, the condition on A having identity of norm 1 is necessary. Because if G is a trivial group consisting of the identity element only, then it reduces to $L^1(G, A) = A = C_0(G, A) = M(G, A)$, and the isometric isomorphic relation reduces to $\mathcal{M}(A) = A$. This equality holds if and only if A has identity.

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