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SOME DISTORTION THEOREMS FOR A CLASS OF CONVEX FUNCTIONS

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1. Introduction. Let A denote the class of analytic functions f in the unit disc $E = \{z \,|\, |z| < 1\}$ with $f(0) = f'(0) - 1 = 0$. For a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in A, Ruscheweyh has defined [4] the δ-neighbourhood of f as

$$N_\delta(f) = \{g(z) = z + \sum_{k=2}^{\infty} b_k z^k \,\Big|\, \sum_{k=2}^{\infty} k\,|a_k - b_k| \leqq \delta\}.$$

This paper deals with the following subclasses of A.

$$T = \{f \in A \,|\, \left|\frac{zf'(z)}{f(z)} - 1\right| < 1,\ z \in E\}$$

$$\tilde{T} = \{f \in A \,|\, \left|\frac{zf''(z)}{f'(z)}\right| < 1,\ z \in E\}.$$

The functions in $\tilde{T}(T)$ are convex (starlike) univalent functions. The following result was proved in [1].

THEOREM A. *Let $g \in \tilde{T}$. Then $N_\delta(g) \subset T$ for $\delta = 1/e$. Moreover if for a function $g \in T$ we have* $\sup_{z \in E} |(zg'(z)/g(z)) - 1| = 1$, *then $N_\delta(g) \not\subset T$ for any $\delta > 0$.*

It follows clearly from Theorem A and the compacity of the class $\tilde{T}$ that

$$\sup_{\substack{|z|<1 \\ g \in \tilde{T}}} \left|\frac{zg'(z)}{g(z)} - 1\right| =: \rho < 1$$

and therefore we have $\tilde{T} \subset T$.

In this paper we will be mainly concerned with the precise determination of ρ. Some new distortion theorems for the classes T and $\tilde{T}$ will also be obtained.

2. An estimate for ρ. It is easily seen from the definitions that

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$$(1)\qquad \begin{aligned} g \in \tilde{T} &\Leftrightarrow g'(z) = e^{\int_0^z \frac{w(\xi)}{\xi} d\xi}, & w(z) &= \frac{zg''(z)}{g'(z)} \\ f \in T &\Leftrightarrow f(z) = z e^{\int_0^z \frac{w_1(\xi)}{\xi} d\xi}, & w_1(z) &= \frac{zf'(z)}{f(z)} - 1. \end{aligned}$$

Here $w(z)$, $w_1(z)$ are analytic functions in E of modulus bounded by 1. By differentiation and substitution we find that $\tilde{T} \subset T$ is equivalent to the fact that for any function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ in E the differential equation

$$(2)\qquad w(z) = \frac{zw_1'(z)}{1 + w_1(z)} + w_1(z)$$

admits a solution $w_1(z)$ again with $w_1(0) = 0$ and $|w_1(z)| < 1$ in E. Explicitly the solution of (2) is given by

$$(3)\qquad \frac{w_1(z)}{1 + w_1(z)} = \frac{1}{z} \int_0^z w(\xi) e^{-\int_\xi^z \frac{w(u)}{u} du} d\xi.$$

We first show that $|w_1(z)| < 1$ in E. If this was not the case there would exist, according to Jack's lemma [2], $z_1 \in E$ such that

$$1 = |w_1(z_1)| = \max_{|z|=|z_1|} |w_1(z)| \text{ and } \frac{z_1 w_1'(z_1)}{w_1(z_1)} =: k \geqq 1.$$

But then it follows from (2) that

$$(4)\qquad \begin{aligned} w(z_1) = \frac{z_1 w_1'(z_1)}{1 + w_1(z_1)} + w_1(z_1) &= \frac{z_1 w_1'(z_1)}{w_1(z_1)} \frac{w_1(z_1)}{1 + w_1(z_1)} + w_1(z_1) \\ &= w_1(z_1) \frac{w_1(z_1) + (k+1)}{w_1(z_1) + 1}. \end{aligned}$$

Since $\min_{|\xi|=1} |(\xi + (k+1))/(\xi + 1)| = (k+2)/2$ we then obtain from (4) that $|w(z_1)| \geqq (k+2)/2 \geqq 3/2$ which contradicts the assumption that $|w(z_1)| < 1$. Therefore $|w_1(z)| < 1$ in E.

Using a similar technique we can obtain a better bound for $|w_1(z)|$. Let again $|w_1(z_1)| = \max_{|z|=|z_1|} |w_1(z)| < 1$. According to Jack's lemma $z_1 w_1'(z_1)/(w_1(z_1)) =: k \geqq 1$ and it follows from (2) that

$$(5)\qquad \frac{w(z_1)}{w_1(z_1)} = \frac{z_1 w_1'(z_1)}{w_1(z_1)} \frac{1}{1 + w_1(z_1)} + 1 = \frac{k}{1 + w_1(z_1)} + 1.$$

Taking into account that $\min_{|\xi| \leqq r < 1} \operatorname{Re}(1/(1 + \xi)) = 1/(1 + r)$ we obtain from (5) that

$$\operatorname{Re}\left(\frac{w_1(z_1)}{w_1(z_1)} \geq \frac{1}{1 + |w_1(z_1)|} + 1 = \frac{2 + |w_1(z_1)|}{1 + |w_1(z_1)|}\right.$$

and therefore

$$\left|\frac{w_1(z_1)}{w(z_1)} - \frac{1}{2}\frac{1+|w_1(z_1)|}{2+|w_1(z_1)|}\right| = \frac{1}{2}\frac{1+|w_1(z_1)|}{2+|w_1(z_1)|}$$

from which it follows easily that

$$|w_1(z_1)| \leqq \frac{1+|w_1(z_1)|}{2+|w_1(z_1)|} \text{ and } |w_1(z_1)| \leqq \frac{-1+\sqrt{5}}{2} \simeq .618.$$

Since $|z_1|$ is arbitrary it follows that $|w_1(z)| < (-1+\sqrt{5})/2$ for z in E and we have proved

THEOREM 2. *$\tilde{T} \subset T$. In fact $g \in \tilde{T} \Rightarrow |(zg'(z)/g(z)) - 1| < (-1+\sqrt{5})/2$, $z \in E$.*

It may also be of some interest to remark that the evaluation of the integral in (3) using the Schwarz lemma yields the estimates

$$g \in \tilde{T} \Rightarrow \left|1 - \frac{g(z)}{zg'(z)}\right| \leqq \frac{e^{|z|} - |z| - 1}{|z|} \text{ and } \frac{|z|}{e^{|z|}-1} \leqq \left|\frac{zg'(z)}{g(z)}\right|, \; z \in E.$$

These estimates are sharp as seen from $g(z) = e^z - 1$ for $z < 0$.

3. The exact value of ρ. In this section we prove some distortion theorems for the classes T and $\tilde{T}$ and determine the exact value of ρ. We first need

LEMMA 3.1. *Let $g \in \tilde{T}$. Then for any $z \in E$,* $\operatorname{Re}\left(\frac{g(z)}{zg'(z)}\right) \geqq \frac{e^{|z|}-1}{|z|e^{|z|}}$.

PROOF. It follows from (1) that

$$\frac{g(z)}{zg'(z)} = \frac{\int_0^z e^{\int_0^\xi \frac{w(u)}{u}du} d\xi}{z e^{\int_0^z \frac{w(u)}{u}du}} = \frac{1}{z}\int_0^z e^{-\int_\xi^z \frac{w(u)}{u}du} d\xi$$

where $w(0) = 0$ and $|w(z)| < 1$ for $z \in E$. We therefore have

$$\operatorname{Re}\left(\frac{g(z)}{zg'(z)}\right) = \int_0^1 \operatorname{Re}\left(e^{-\int_t^1 \frac{w(zr)}{r}dr}\right)dt.$$

Using the estimate $\operatorname{Re}(e^u) \geqq e^{-|u|}$ valid for $|u| < 1$ and the Schwarz lemma we obtain

$$\operatorname{Re}\left(\frac{g(z)}{zg'(z)}\right) \geqq \int_0^1 e^{-\left|\int_t^1 \frac{w(zr)}{r}dr\right|}dt \geqq \int_0^1 e^{-(1-t)|z|}dt = \frac{e^{|z|}-1}{|z|e^{|z|}}.$$

This result is sharp and the inequality is strict unless $g(z) = (e^{\alpha z} - 1)/\alpha$ for some α with $|\alpha| = 1$. Note that it follows from this lemma that

$$\text{(6)} \qquad \left|\frac{zg'(z)}{g(z)}\right| \leqq \frac{|z|e^{|z|}}{e^{|z|}-1}, \qquad z \in E.$$

We next prove

LEMMA 3.2. *Let $w(z)$ be an analytic function with $w(0) = 0$ and $|zw'(z)| < 1$ for $z \in E$. Let also $r(0 < r < 1)$ and $\theta(0 \leqq \theta < 2\pi)$ be fixed and define $E_{r,\theta} = \{(r_1, \theta_1) | 0 < r_1 \leqq r$ and $\theta_1 + \mathrm{Im}(w(r_1e^{i\theta_1})) = \theta + \mathrm{Im}(w(re^{i\theta}))\}$. Then $r_1\text{-}\mathrm{Re}(w(r_1e^{i\theta_1})) \leqq r - \mathrm{Re}(w(re^{i\theta}))$ if $(r_1, \theta_1) \in E_{r,\theta}$.*

PROOF. First of all we remark that the set $E_{r,\theta}$ is not empty; define the function f as $f(z) = ze^{w(z)}$. It is clear from (1) that f belongs to T and is therefore a starlike univalent function. This means, given r_1 with $0 < r_1 < r$, there exists one and only one θ_1 such that $\arg(f(r_1e^{i\theta_1})) = \arg(f(re^{i\theta}))$, which gives that $(r_1, \theta_1) \in E_{r,\theta}$. In fact it is clear that $E_{r,\theta}$ is a Jordan arc joining the origin and $re^{i\theta}$ which intersects each circle with center at the origin and radius smaller than r exactly once.

Put then $r^*\text{-}\mathrm{Re}(w(r^*e^{i\theta^*})) = \max_{(r_1, \theta_1) \in E r, \theta} r_1\text{-}\mathrm{Re}(w(r_1e^{i\theta_1}))$. It follows from a theorem of Kuhn and Tucker [3] that there exist real numbers λ and μ such that

$$r^*\text{-}\mathrm{Re}(r^*e^{i\theta^*}w'(r^*e^{i\theta^*})) - \lambda\,\mathrm{Im}(r^*e^{i\theta^*}w'(r^*e^{i\theta^*})) - \mu r^* = 0 \tag{7}$$

$$\mathrm{Im}(r^*e^{i\theta^*}w'(r^*e^{i\theta^*})) - \lambda(1 + \mathrm{Re}(r^*e^{i\theta^*}w'(r^*e^{i\theta^*}))) = 0. \tag{8}$$

The theorem of Kuhn and Tucker says further that if $\mu \neq 0$ then $r^* = r$. From (8) we can isolate λ and if we substitute the value of λ in (7) we obtain that

$$\frac{r^* - (1 - r^*)\mathrm{Re}(r^*e^{i\theta^*}w'(r^*e^{i\theta^*})) - |r^*e^{i\theta^*}w'(r^*e^{i\theta^*})|^2}{1 + \mathrm{Re}(r^*e^{i\theta^*}w'(r^*e^{i\theta^*}))} - \mu r^* = 0. \tag{9}$$

Now suppose that $\mu = 0$. It then follows from (9) that

$$\left| r^*e^{i\theta^*}w'(r^*e^{i\theta^*}) + \frac{1 - r^*}{2} \right| = \frac{1 + r^*}{2}.$$

But from the hypothesis on w we have that

$$\left| r^*e^{i\theta^*}w'(r^*e^{i\theta^*}) + \frac{1 - r^*}{2} \right| < r^* + \frac{1 - r^*}{2} = \frac{1 + r^*}{2}$$

at least in the case where $zw'(z) \not\equiv e^{i\gamma}z$ for some real γ. In that case it must then follow that $\mu \neq 0$ and therefore $r^* = r$. Since θ^* is uniquely determined by r^* we must also have $\theta^* = \theta$ and the conclusion of Lemma 3.2 follows. The result for the case $zw'(z) \equiv e^{i\gamma}z$ follows by continuity.

We then obtain as a result of Lemma 3.2.

COROLLARY 3.1. *Let $f \in T$ and $\arg(f(u)) = \arg(f(v))$ for $0 < |u| < |v| < 1$. Then $|f(v)|/|f(u)| \leqq (|v|e^{|v|})/(|u|e^{|u|})$. This result is sharp as seen for $f(z) = ze^z$ with $0 < u < v < 1$.*

PROOF. The proof is immediate from Lemma 3.2 since

$$\left|\frac{f(v)}{f(u)}\right| = \frac{|v|}{|u|} e^{\mathrm{Re}(w(v))-\mathrm{Re}(w(u))} \leqq \frac{|v|}{|u|} z^{|v|-|u|}.$$

We now proceed to show a lemma similar to Lemma 3.2. This lemma will have an interesting application to the class $\tilde{T}$.

LEMMA 3.3. *Let $w(z)$ be an analytic function with $w(0) = 0$ and $|zw'(z)| < 1$ for $z \in E$. Let also r $(0 < r < 1)$ and θ $(0 \leqq \theta < 2\pi)$ be fixed and define*

$$E_{r,\theta} = \left\{(r_1, \theta_1) \,|\, 0 < r_1 \leqq r \text{ and } \arg\left(\int_0^{r_1 e^{i\theta_1}} e^{w(\xi)} d\xi\right) = \arg\left(\int_0^{re^{i\theta}} e^{w(\xi)} d\xi\right)\right\}.$$

Then

$$\ln(e^{r_1} - 1) - \mathrm{Re}\left(\ln\left(\int_0^{r_1 e^{i\theta_1}} e^{w(\xi)} d\xi\right)\right) \leqq \ln(e^r - 1) - \mathrm{Re}\left(\ln\left(\int_0^{re^{\theta i}} e^{w(\xi)} d\xi\right)\right)$$

if $(r_1, \theta_1) \in E_{r,\theta}$.

PROOF. Define the function g as $g'(z) = e^{w(z)}$ and $g(0) = 0$. This function belongs to the class $\tilde{T}$ and in particular is a starlike univalent function. It follows therefore that $E_{r,\theta}$ is a Jordan arc joining the origin and $re^{i\theta}$ intersecting each circle with center at the origin and radius smaller than r exactly once. Put then

$$\ln(e^{r^*} - 1) - \mathrm{Re}\left(\ln\left(\int_0^{r^* e^{i\theta^*}} e^{w(\xi)} d\xi\right)\right)$$
$$= \max_{(r_1, \theta_1) \in E_{r,\theta}} \left[\ln(e^{r_1} - 1) - \mathrm{Re}\left(\ln\left(\int_0^{r_1 e^{i\theta_1}} e^{w(\xi)} d\xi\right)\right)\right].$$

As in the case of Lemma 3.2 there must exist real numbers λ and μ such that

$$\text{(10)} \qquad \frac{r^* e^{r^*}}{e^{r^*} - 1} - \mathrm{Re}(\xi) - \lambda\, \mathrm{Im}(\xi) - \mu r^* = 0,$$

$$\text{(11)} \qquad \mathrm{Im}(\xi) - \lambda \mathrm{Re}(\xi) = 0$$

and if $\mu \neq 0$, then $r^* = r$. Here $\xi = (r^* e^{i\theta^*} g'(r^* e^{i\theta^*}))/(g(r^* e^{i\theta^*}))$

Now suppose that $\mu = 0$; to substitute $\lambda = \mathrm{Im}(\xi)/\mathrm{Re}(\xi)$ in (10) will mean that $|\xi|^2 - (r^* e^{r^*})/(e^{r^*} - 1)\mathrm{Re}\{\xi\} = 0$, which is equivalent to $\mathrm{Re}\{1/\xi\} = (e^{r^*} - 1)/(r^* e^{r^*})$. In view of Lemma 3.1, this is impossible if $zw'(z) \not\equiv e^{i\gamma} z$ for some real γ. In that case it must follow that $\mu \neq 0$ and $r^* = r$. Also $\theta^* = \theta$ and the conclusion of Lemma 3.3 is then shown. The result for the case $w(z) \equiv e^{i\gamma} z$ follows by continuity.

From Lemma 3.3 it is now easy to show (the proof is omitted).

COROLLARY 3.2. *Let $g \in \tilde{T}$ and* $\arg(g(u)) = \arg(g(v))$ *for* $0 < |u| < |v| <$

1. *Then* $|g(v)/g(u)| \leqq (e^{|v|} - 1)/(e^{|u|} - 1)$. *This result is sharp as seen for* $g(z) = e^z - 1$ *with* $0 < u < v < 1$.

We are now ready to show the main result of this paper.

THEOREM 3. *Let* $g \in \tilde{T}$. *Then* $|(zg'(z)/g(z)) - 1| \leqq (1 - (1 - |z|)e^{|z|})/(e^{|z|} - 1)$, $z \in E$. *This result is sharp as seen for* $g(z) = e^z - 1$ *with* $z > 0$.

PROOF. It is readily seen from (1) that

$$\frac{zg'(z)}{g(z)} - 1 = \frac{zg'(z) - g(z)}{g(z)} = \frac{\int_0^z \xi d''(\xi)d\xi}{g(z)} = \frac{\int_0^z \frac{\xi g''(\xi)}{g'(\xi)} g'(\xi)d\xi}{g(z)} = \frac{\int_0^z w(\xi)g'(\xi)d\xi}{g(z)}.$$

where $w(0) = 0$ and $|w(z)| < 1$ if $z \in E$. In the last expression we perform the change of variable $v = g(\xi)$ to obtain

$$\frac{zg'(z)}{g(z)} - 1 = \frac{\int_0^{g(z)} w(g^{-1}(v))dv}{g(z)}.$$

And since we can integrate on the segment $[0, g(z)]$ (because the function g is starlike), we have

$$\left|\frac{zg'(z)}{g(z)} - 1\right| = \left|\int_0^1 w(g^{-1}(tg(z)))dt\right| \leqq \int_0^1 |g^{-1}(tg(z))|dt. \tag{12}$$

It follows from Corollary 3.2 that for $t > 0$, we have

$$\frac{1}{t} = \left|\frac{g(z)}{tg(z)}\right| \leqq \frac{e^{|z|} - 1}{e^{|g^{-1}(tg(z))|} - 1},$$

i.e., $|g^{-1}(tg(z))| \leqq \ln(1 + t(e^{|z|} - 1))$, and the substitution of the last estimate in (12) yields the conclusion of Theorem 3. It also follows that $\tilde{T} \subset T$ and $\rho = 1/(e - 1)$.

4. Some distortion theorems for $\tilde{T}$. The following inequalities were first obtained by Singh [7]:

$$g \in \tilde{T} \Rightarrow 1 - e^{-|z|} \leqq |g(z)| \leqq e^{|z|} - 1 \quad \text{and } e^{-|z|} \leqq |g'(z)| \leqq e^{|z|}, \qquad z \in E \tag{13}$$

$$f \in T \Rightarrow |z|e^{-|z|} \leqq |f(z)| \leqq |z|e^{|z|}, \qquad z \in E. \tag{14}$$

In this section we would like to point out some generalizations of these inequalities.

Let S_0 designate the subset of A consisting of the starlike univalent functions in E and let M be defined as

$$M = \{f \in A \mid \frac{f * g(z)}{z} \neq 0 \text{ for any } g \in S_0 \text{ and } z \in E\}.$$

Here "*" denotes the Hadamard product of two functions in A. A very important subset of M is

$$X = \{f(z) = \frac{1}{1+it}\left(\frac{z}{(1-xz)^2} + it\frac{z}{1-zx}\right) \mid t \in \mathbf{R} \text{ and } |x| \leqq 1\}.$$

Ruscheweyh and Singh have shown [5]

THEOREM B. *Let* $f \in M$ *and* $g \in \tilde{T}$. *Then* $f * g \in S_0$ *and*

$$\max_{|z|<1} |f*g(z)| \leqq \sqrt{2}\, \max_{|z|<1} |zg'(z)|. \tag{15}$$

Furthermore they conjectured that factor $\sqrt{2}$ in (15) may be lowered to 1.

First we disprove the conjecture. Define the function g_0 as $g_0(z) = z(1+cz)^i$. It is easy to check that for positive and small enough c the function g_0 is analytic in $\bar{E}$ and belongs to the class $\tilde{T}$. Moreover if $|g_0'(u)| = \max_{|z|=1} |g_0'(z)|$ and $|u| = 1$ some calculations will show that $(ug_0'(u))/(g_0(u))$ is not a real number. Therefore it follows that there must exist $t_0 \in \mathbf{R}$ such that

$$\left|\frac{ug_0'(u)}{g_0(u)}\right| < \left|\frac{\frac{ug_0'(u)}{g_0(u)} + it_0}{1+it_0}\right|$$

and this implies that

$$\max_{|z|<1} |zg_0'(z)| = |ug_0'(u)| < \left|\frac{ug_0'(u)+it_0g_0(u)}{1+it_0}\right| \leqq \operatorname{mas}_{|z|<1}\left|\frac{zg'(z)+it_0g_0(z)}{1+it_0}\right|.$$

Since $(zg_0'(z) + it_0g_0(z))/(1+it_0) = (1/(1+it_0))((z/(1-z)^2) + it_0(z/(1-z))) * g_0(z)$ it follows that the conjecture cannot hold. Next we prove

THEOREM 4. *Let* $f \in M$ *and* $g \in \tilde{T}$. *Then*

$$|z|e^{-|z|} \leqq |f*g(z)| \leqq |z|e^{|z|}, \qquad z \in E. \tag{16}$$

PROOF. In view of Ruscheweyh's Duality Theorem [6] it is enough to prove Theorem 4 for $f \in X \subset M$. Let $g \in \tilde{T}$. Since, for any real t and fixed $z \in E$, $(zg'(z) + itg(z))/(1+it)$ belongs to the disc of radius $|(zg'(z) - g(z))/2|$ and center $(zg'(z) + g(z))/2$, we obtain

$$|f*g(z)| \leqq \frac{|zg'(z)+g(z)| + |zg'(z)-g(z)|}{2}$$

$$= |g(z)|\left(\frac{\left|\left(\frac{zg'(z)}{g(z)} - 1\right) + 2\right| + \left|\frac{zg'(z)}{g(z)} - 1\right|}{2}\right)$$

$$\leqq |g(z)|\left|\left(1 + \left|\frac{zg'(z)}{g(z)} - 1\right|\right)\right|.$$

Using then the result of Theorem 3 we obtain, for $0 \leqq r < 1$,

$$\max_{|z|\leqq r} |f*g(z)| \leqq \frac{re^r}{e^r - 1} \max_{|z|\leqq r} |g(z)|,$$

and the right hand side of (16) follows from (13).

To prove the left-hand side of (16), we remark that, for $f(z) = (1/(1+it))\,((z/(1-z)^2) + it\,(z/(1-z)))$,

$$(17)\qquad |f*g(z)'| = |g'(z)|\left|\frac{1}{1+it}\frac{zg''(z)}{g'(z)} + 1\right| \geqq |g'(z)|\left(1 - \left|\frac{zg''(z)}{g'(z)}\right|\right) \geqq e^{-|z|}(1-|z|), \qquad z \in E.$$

Here we have used (13) and the definition of $\tilde{T}$. Since we know from Theorem B that $f*g$ is univalent we can integrate (17) to obtain

$$|f*g(z)| \geqq \int_0^{|z|} e^{-\xi}(1-\xi)d\xi = |z|e^{-|z|}$$

and this complete the proof of Theorem 4.

In view of (16) and (14) it is interesting to remark that, for $f \in M$ and $g \in \tilde{T}$ it does not follow necessarily that $f*g$ belongs to T. Ruscheweyh and Singh [5] have shown that for $h \in \tilde{T}$ it is true that $h'(z)$ is subordinated to e^z; this result is clearly equivalent to the fact that for $h \in T$ we have that $(h(z))/z$ is subordinated to e^z.

Choose $f(z) = (1/(1+it))\,((z/(1-z)^2) + it(z/(1-z)))$ in M and $g(z) = e^z - 1$ in $\tilde{T}$. Then for the function $h(z) = f(z)*g(z)$ we have

$$(18)\qquad \frac{h(z)}{z} = e^z \frac{1 + it\dfrac{1-e^{-z}}{z}}{1+it}.$$

Choose also $\xi = i$. It follows from elementary geometric consideration that, for any real number c different from 1, ce^{ξ} does not belong to $\bar{D}$, where $D = \{e^z | z \in E\}$. Also, since $(1 - e^{-\xi})/\xi$ is not a real number and $\mathrm{Re}((1-e^{-\xi})/\xi) \neq 1$, there must exist a real number t_0 such that

$$(19)\qquad \frac{1 + it_0\dfrac{1-e^{-\xi}}{\xi}}{1+it_0} \text{ is real and different from 1.}$$

Clearly it follows from (18) and (19) that $h(\xi)/\xi \notin \bar{D}$ for our choice of t_0, i.e., $(h(z))/z$ is not subordinated to e^z and $f*g$ does not belong to T.

Finally we would like to point out the following for the class $\tilde{T}$:

$$(20)\qquad g \in \tilde{T} \Rightarrow e^{-|u|} - e^{-|v|} \leqq |g(u) - g(v)| \text{ if } u, v \in E,\ |u| < |v|;$$

$$g \in \tilde{T} \Rightarrow |g(u) - g(v)| \leqq e^{|v|} - e^{|u|} \text{ if } u, v \in E, |u| < |v| \text{ and } \arg(g(u)) = \arg(g(v)); \tag{21}$$

$$g \in \tilde{T} \Rightarrow |g'(u) - g'(v)| \leqq |g(u) - g(v)| \text{ if } u, v \in E; \text{ and} \tag{22}$$

$$g \in \tilde{T} \Rightarrow |z|e^{-|z|} + |zg'(z) - g(z)| \leqq |g(z)| \leqq |z|e^{|z|} - |zg'(z) - g(z)|, \qquad z \in E. \tag{23}$$

The estimate (20) can be obtained as an application of a standard technique and (21) is a consequence of Corollary 3.2. It is also easy to show from (6) that (21) holds if we assume $\arg(u) = \arg(v)$ instead of $\arg(g(u)) = \arg(g(v))$. To obtain (22) it is enough to remark that, for $g \in \tilde{T}$,

$$\left|\frac{(g^{-1})''(\xi)}{(g^{-1})'(\xi)^2}\right| \leqq 1 \text{ if } \xi \text{ belongs to the range of } g. \tag{24}$$

The integration of (24) will then give (22). The left-hand side of (23) was proved in [1] and the right-hand side is a consequence of the fact that, for $g \in \tilde{T}$, we have

$$|zg'(z) - g(z)| = \left|\int_0^z \xi g''(\xi)d\xi\right| \leqq \int_0^1 |z|^2 te^{t|z|}dt = 1 - (1 - |z|)e^{|z|}, \qquad z \in E.$$

As a conclusion we would like to mention that many results from this paper are also consequences of more general results established by Ruscheweyh and Singh [5]. For example they were able to show that, for $g \in \tilde{T}$, $(zg'(z))/(g(z))$ is subordinate to $(ze^z)/(e^z - 1)$, but we were unable to use that result to establish Theorem 3 directly.

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