ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

SEMI-INTEGRAL EXTENSIONS AND PROPER MORPHISMS

M. EMILIA ALONSO

Dedicated to the memory of Gus Efroymson.

Semi-integral extensions were introduced by Brumfiel in [1] from an algebraic point of view, and applied to study algebraic varieties over real closed fields. In this note we develop the connection between semi-integral extensions and proper morphisms.

Let R be a real closed field. If X is an irreducible algebraic variety over R, we denote by X_c the maximum dimension locus of X, by $R[X]$ the coordinate ring of X and by $R(X)$ its field of rational functions; we denote by β_d the ordering in $R[X]$ induced by the weakest ordering in $R(X)$. A dominant morphism $\varphi: X \to Y$ of irreducible algebraic varieties over R is called semi-integral if $(R[X], \beta_d)$ is a semi-integral extension of $(R[Y], \beta_d \cap R[Y])$ via φ. Then we prove the following proposition.

PROPOSITION 1. *The following are equivalent.*

(a) φ *is semi-integral*

(b) *For every bounded semialgebraic set* $S \subset Y$, $[\varphi^{-1}(S)] \cap X_c$ *is bounded.*

(c) (i) *For every closed semialgebraic set* $S \subset X_c$, $\varphi(S)$ *is closed, and,* (ii) *For every* $y \in Y$, $[\varphi^{-1}(y)] \cap X_c$ *is bounded.*

From this proposition we can deduce a result of Brumfiel, with no need of Hironaka's desingularization theorem [2].

COROLLARY 2. *Let* $f, g \in R[X]$, *Then* f/g *is semi-integral over* $R[X]$ *if and only if* f/g *is bounded on bounded semialgebraic sets of* $X_c - V(g)$.

Condition (c) in proposition 1 can be taken as a definition of real properness. Then, in terms of real spectra [3], we obtain a valuative Criterion for properness analogous to the classical one. Note that $\tilde{X} = \mathrm{Spec}_R R[X]$, $\tilde{Y} = \mathrm{Spec}_R R[Y]$, and $\tilde{\varphi}: \tilde{X} \to \tilde{Y}$ the morphism induced by φ. We have

PROPOSITION 2. *The following are equivalent.*

(a) (i) *For every closed constructible set* $S \subset \tilde{X}$, $\tilde{\varphi}(S)$ *is closed and*

(ii) *For every rational point* $y \in Y$, $\tilde{\varphi}^{-1}(y)$ *is real complete.*

(b) (Valuative Criterion) *For every real closed field F, every real valuation* *ring C_F of F and all morphisms u, v like those in* $(*)$, *there is a unique morphism w which makes the diagram commutative.*

(c) $\tilde{\varphi}$ *is universally closed.*

$$(*)\qquad \begin{tikzcd} \mathrm{Spec}_R F \arrow[r, "u"] \arrow[d] & \tilde{X} \arrow[d, "\tilde{\varphi}"] \\ \mathrm{Spec}_R C_F \arrow[r, "v"] \arrow[ur, "w"] & \tilde{Y} \end{tikzcd}$$

References

1. G. W. Brumfiel, *Partially ordered rings and semialgebraic geometry*, London Math. Soc. Lect. Not. **37** (1979).

2. ———, *Real valuations and ideals*. Springer Lectures Notes, **959** (1983).

3. M. Coste and Coste-Roy, M. F. *La topologie du spectre reel*. Contemporary Math. **8** (1982) 27–59.

Departmento de Algebra y Fund, Universidad Complutense. Madrid, Spain