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RESEARCH ANNOUNCEMENT ON EXTENDING NASH FUNCTIONS OFF SINGULAR CURVES

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The general extension problem is: given sets $W \subset V$ and a function $f\colon W \to \mathbf{R}$, to find a function $g\colon V \to \mathbf{R}$ with $g = f$ on V. We say g extends f. To make the problem interesting, we need restrictions on f and g. In our case we want f and g to be Nash. The most general extension theorem about Nash functions doesn't quite fit the above description.

THEOREM [2] *Let V be a Nash variety in $\mathbf{R}^n$ (i.e., $V = h^{-1}(0)$ for a Nash function $h\colon \mathbf{R}^n \to \mathbf{R}$). Suppose U is an open neighborhood of V and f a Nash function $f\colon U \to \mathbf{R}$. Then there exists a Nash function g defined on $\mathbf{R}^n$ with $g = f$ on V.*

Note we have to assume f is defined on a neighborhood of V to extend it. But from the above theorem it easily follows that

THEOREM. [2] *If V is a non-singular variety in $\mathbf{R}^n$ and $f\colon V \to \mathbf{R}$ is Nash then there exists $g\colon \mathbf{R}^n \to \mathbf{R}$ extending f.*

At this point, it would be a good idea to say what we mean by a Nash function on a possibly singular variety. Recall first that if V is nonsingular, every point of V has a neighborhood which is essentially euclidean, and so the usual definition applies, i.e., f on V is Nash if f is analytic on V and f is algebraic. For a singular point of V, I don't know what an analytic function is, so I will define a globally algebraic function on V to be Nash at a point p of V if f has an analytic extension to some neighborhood of P in $\mathbf{R}^n$.

QUESTION. Can you always extend a Nash function $f\colon V \to \mathbf{R}$ to $g\colon \mathbf{R}^n \to \mathbf{R}$ where V is a possibly singular variety?

ANSWER. No. For example there is a Nash function on the Whitney umbrella $(x^2 + y^2)z = x^3$, which can't be extended.

I think the problem with the Whitney umbrella is that it is not coherent [4]. Since curves are always coherent, they are a good starting place for proving an extension result.

THEOREM 1. *Let V be a Nash curve, $V \subset \mathbf{R}^n$, and f a Nash function $V \to \mathbf{R}$. Then f can be extended to $\mathbf{R}^n$.*

We will sketch the proof in the planar case; the general case is similar. So let $q(x, y) = 0$ be the equation of the Nash curve and $p(x, y) = 0$ be the equation of the polynomial so that $V_q \subset V_p$. Let f: $V \to \mathbf{R}$ be a Nash function as defined above. By [1], there exists a Nash transformation of $\mathbf{R}^n$ so that in the transformed plane, $p = \prod(z - \alpha_i(x))$ with the following properties:

1) All singular points (a, b) of $p = 0$, with a real, lie on the y axis.
2) If $p(a, b) = \partial p/\partial y(a, b) = 0$, a, b real, $a \neq 0$, then $\partial p/\partial y(a, b) \neq 0$.
3) If $\partial p/\partial y(a, b) = 0, p(a, b) = 0, a \neq 0$, a real, then b is real also.

LEMMA. *The function f has an extension to that part of $V_{\mathbf{C}}$ which lies along the $\alpha_i(x)$.*

PROOF. This follows from above by analytic continuation, since the only points where the $\alpha_i(x)$ meet away from $x = 0$ are at real points.

We now wish to construct a function, $r(x, y) = \sum_{i=0}^{d-1} \beta_i(x)y^i$, $d =$ degree of q. We want $r(x, y) = f(x, y)$ along $y = \alpha_i(x)$, and the $\beta_i(x)$ should be Nash functions. Then it will be clear that $r(x, y)$ will extend f to $\mathbf{R}^2$. We take x real, but $\alpha_i(x)$ can be complex.

The idea is to write $f(x, y) = \sum_{i=0}^{d-1} \beta_i(x)y^i$. Then $f(x, \alpha_j(x)) = \sum_{i=0}^{d-1}\beta_i(x)\alpha_j(x)^i$, $j = 1, \ldots, d$ gives d linear equations in d unknowns, $\beta_0, \ldots, \beta_{d-i}$, which we can solve, but we must show that the $\beta_i(x)$ are Nash functions. Using Cramer's rule, we set as denominator for the β_i, $\prod_{i<j}(\alpha_i - \alpha_j)$ which is 0 only at multiple points of q and so otherwise is analytic. So we must look at each point above $x = 0$ where we have a singularity. Let $P = (0, y_0)$ be a singular point of V. Then let $\alpha_1(x), \ldots, \alpha_e(x)$ be the α_i for which $\alpha_i(0) = y_0$. Then, by essentially the residue theorem of complex variables, it follows as in [3, p. 167] that all symmetric functions $s_k(\alpha_1, \ldots, \alpha_e)$ are analytic and so Nash. We use the local power series for $f(x, y)$ to represent f in terms of symmetric functions of the α_i, which we use to get analyticity of the part of Cramer's formula which comes from $\alpha_1, \ldots, \alpha_e$. Putting together the various factorizations at points $(0, y_i)$, will give in a somewhat messy way a proof that all the $\beta_i(x)$ are Nash.

REFERENCES

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