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ZEROS OF OPERATORS ON FUNCTIONS AND THEIR ANALYTIC CHARACTER

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1. Introduction. This is a survey of recent work on the behavior of the zeros of iterated operators on analytic functions. It is motivated by the classical survey paper of G. Pólya **[50]**, the more recent survey due to Boas **[8]** and the collected papers by Pólya **[49]**. Some old and perhaps forgotten results coupled with new and interesting developments have helped to add a new dimension to some classical results as well as to some as yet unsolved problems. The classical problems include the determination of the behavior of the zeros of the successive derivatives of an analytic function and the influence of the behavior of the sign changes of the derivatives of a function on its analytic character.

The present paper is by no means exhaustive. The author's intention is merely to present recent results designed to indicate the flavor of the work being done on the topics under discussion. The reader should consult the original papers for more detailed information and perhaps will tackle some of the many interesting problems that remain.

2. The complex domain; the case of derivatives. The classical problem involves the determination of the distribution of the zeros of the successive derivatives $f(z)$, $f'(z)$, $f''(z)$, $\ldots$. To be more precise, following Pólya, given a function $f(z)$ analytic on a domain D, we say that a point z_0 lies in the final set S of f when every neighborhood of z_0 contains zeros of infinitely many of the derivatives of f. The final set thus determines the final location of the zeros of the successive derivatives.

3. Meromorphic Functions. For meromorphic functions, the final set for the zeros of the successive derivatives is easy to describe. Pólya's remarkable theorem **[52]** says that the poles behave like repellers for the zeros of the successive derivatives. Specifically, if a meromorphic function f has at least two distinct poles, then a point z lies in the final set S of f if and only if z is equidistant from the two poles that are nearest to it. If f has only one pole, the final set of f is empty. Proofs of this theorem can be found in Whittaker **[82]** and Hayman **[30]**. When f is meromorphic

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with a single pole, the principle is that the pole repels the zeros of the successive derivatives, thereby creating a zero free region. Information on the asymptotic size of this zero free region was recently obtained **[72]**.

Let $F(z)$ be of the form

(1) $F(z) = a_{-1}/z + A(z)$, where $a_{-1} \neq 0$ and $A(z)$ is analytic in a neighborhood of $z = 0$ and possibly entire.

(1a) When $A(z)$ is analytic in $|z| < R$ and $0 < \Gamma < 1/2$, then for all n sufficiently large, $F^{(n)}(z)$ has no zero in $|z| \leqq \Gamma R$. The constant $1/2$ is best possible.

(1b) If $A(z)$ is entire of exponential type T or less, C_0 denotes the unique positive root of the equation $xe^{x+1} = 1$, and $0 < C < C_0$, then for all n sufficiently large, $F^{(n)}(z)$ has no zero in $|z| \leqq CT^{-1}(n + 1)$. The constant C_0 is the best possible.

(1c) If $A(z)$ is entire of order ρ, type τ $(0 < \rho, \tau < \infty)$ and if $0 < \gamma < [2(e\rho\tau)^{1/\rho}]^{-1}$, then for all n sufficiently large, $F^{(n)}(z)$ has no zero in $|z| \leqq \gamma(n + 1)^{1/\rho}$.

With only slight modifications, one can handle an F with a pole of order N. It is not known whether the constant $[2(e\rho\tau)^{1/\rho}]^{-1}$ in (1c) is best possible.

Not specifically mentioned in Polya's paper **[50]**, but germane to the topic, is the characterization of meromorphic functions for which the derivatives fail to have zeros at all. Recent significant work stems from a well known conjecture of Hayman **[30]** that the only meromorphic functions f for which $f(z)$ and $f^{(\ell)}(z)$ have no zeros for some $\ell \geqq 2$ are those of the form $f(z) = \exp(az + b)$ or $f(z) = (Az + B)^{-n}$. This conjecture has been proven by G. Frank **[27]**. In addition, Frank, Hennekemper and Pollaczek **[28]** have shown that if f is meromorphic and if f and $f^{(\ell)}$ have only finitely many zeros, then $f = (P_1/P_2)\exp(P_3)$, where P_1, P_2 and P_3 are polynomials. The proof of the result appears for $\ell \geqq 3$. The authors promise a proof when $\ell = 2$ in a future paper.

4. Entire and analytic functions. The distribution of the zeros of the successive derivatives of an entire function is an active topic. Many of the classical references are listed in **[50]**. Whether the zeros of the successive derivatives scatter or condense is determined by whether the function has order less than or greater than one, respectively. Moreover, if Γ_n denotes the radius of the largest disk centered at the origin in which $f^{(n)}(z)$ is zero-free, then the behavior of Γ_n depends on whether the function has order less than or greater than one. If Γ_n, instead, is taken to be the radius of the largest disk somewhere in the plane in which $f^{(n)}$ is zero-free, then Boas and Reddy **[12]** have shown that the critical order is two instead of one. In this regard there is a conjecture which shall be referred to as the Boas conjecture: If $\rho > 2$, then there is an entire function f of order ρ such that for some positive A, every disk of radius A, anywhere in the plane, contains a zero of every $f^{(n)}(z)$. In a sequel to the paper of Boas and Reddy, Reddy [71] investigated zero free disks for so called D operators, which we now define. Let $\{d_p\}_{p=1}^{\infty}$ denote a nondecreasing sequence of positive numbers. Define the operator D on $f(z) = \sum_{k=0}^{\infty} a_k z^k$ by $Df(z) = \sum_{k=0}^{\infty} d_{k+1} a_{k+1} z^k$. The j-th iterate of D on f is given by

$$D^j f(z) = \textstyle\sum_{k=0}^{\infty} (e_k/e_{j+1}) a_{j+k} z^k,$$

where $e_0 = 1$ and $e_k = (d_1 d_2 \cdots d_k)^{-1}$. The author proves that many results for derivatives appear as special cases of a general theory for these operators.

Another interesting result is a verification of a conjecture of Erdös by Barth and Schneider [4]. They prove that if $\{S_k\}_{k=1}^{\infty}$ is any sequence of sets in the complex plane each of which has no finite limit point, then there is a sequence of positive integers $\{n_k\}$ and a transcendental entire function $f(z)$ such that $f^{n_k}(z) = 0$ if $z \in S_k$.

In [74], I. J. Schoenberg, after 40 years, returned to an investigation about determining a lower bound for the type of an entire function of exponential type having the property that all its derivatives vanish at least k times in a real interval of length $(k - 1/2)$. He poses the following conjecture.

If $f(z)(\not\equiv 0)$ is entire of exponential type δ and is such that each $f^{(v)}(x)$ $(v = 0, 1, 2, \ldots)$ has at least k zeros in $I_k = [0, k - 1/2]$, then $\delta \geqq \pi$ and the function $f(z) = \cos(\pi z)$ would show that π is the best constant for this inequality.

This problem is related to the problem of determining the Whittaker constant. For the definition of the Whittaker constant and other progress towards determining the Whittaker constant due to Evgrafov and Buckholtz, see Buckholtz [14].

For results on analytic functions having univalent derivatives, an up-to-date survey has been written by Shah and Trimble [75]. The reader should consult this paper for information on this.

Also of interest is the recent investigation of Boas [9] on the zeros of successive derivatives of a function analytic at infinity. He proves the following.

Let $F(w) = \sum_{n=1}^{\infty} b_n w^{-n}$ be analytic at ∞, with F nonconstant. Then there is a constant $c > 0$ such that for all n sufficiently large, $F^{(n)}(w)$ has no finite zero outside the circle $|w| = nc$.

Among other reasons, its interest lies in the fact that the results are not obtainable from the results on zeros of derivatives of functions analytic in a neighborhood of $z = 0$ with the substitution $w = z^{-1}$. These results have been generalized in [77] to include behavior of zeros of successive derivatives of functions analytic in a half plane. Related results may be found in Widder [**83**].

5. Real entire functions. A real entire function is an entire function which is real valued on the real axis. As discussed in [**50**], the principle at work on the distribution of the zeros of the successive derivatives of such functions is that the non real zeros of $f^{(n)}(z)$ move toward the real axis if f has order less than 2 and away from it if f has order greater than 2.

Of primary importance and interest are the three as yet unsolved conjectures of Pólya [**50**], which he refers to as hypothetical theorems. They are stated as follows:

A. If the order of a real entire function f is less than two and $f(z)$ has only a finite number of complex zeros, then its derivatives, from a certain one onwards, will have no complex zeros at all.

B. If the order of the real entire function $f(z)$ is greater than two and f has only a finite number of complex zeros, then the number of complex zeros of $f^{(n)}(z)$ tend to infinity as $n \to \infty$.

C. If a real entire function of order greater than one remains bounded for real values of the variable, then its final set contains the whole real axis.

It seems that no progress has been made on conjecture A since the appearance of Pólya's paper [**50**]. Pólya proved that conjecture A holds for functions having order less than 4/3 in [**59**].

Some advances towards the verification of conjecture B have recently been made by Hellerstein, Shen, and Williamson, Edrei and Gethner.

In 1914 G. Pólya posed the problem of classifying all entire functions f which have, along with all of their derivatives, only real zeros. He conjectured that such a function f must have one of the following forms:

1. $f(z) = A \exp(Bz)$,
2. $f(z) = A \exp(\exp(icz) - \exp(id))$,
3. $f(z) = Az^m \exp(-az^2 + bz)\prod_n(1 - z/a_n)\exp(z/a_n)$,

where A and B are complex constants, $a \geqq 0$, b, c, d and the a_n are real, $\sum_n(a_n)^{-2} < \infty$ and m is a nonnegative integer.

Hellerstein, Shen and Williamson [**32**] prove a strong form of this conjecture and thus completely settle Pólya's problem. They prove the following theorem.

THEOREM. *Let f be an entire function which has, along with its first three derivatives, only real zeros. Then f has one of the three forms* 1., 2., *or* 3.

This result is a consequence of earlier work of the authors [**33**], [**34**], results of Levin and Ostrowski as well as of their main theorem.

THEOREM. *Let f be a strictly nonreal meromorphic function* (*i.e., not a constant multiple of a real meromorphic function*). *Suppose that f, f' and f''*

have only real zeros. If f is entire, then it has the form 1., 2., *or has one of the following two forms*:

4. $f(z) = A \exp(\exp(i(cz + d)))$,
5. $f(z) = A \exp\{K[i(cz + d) - \exp(i(cz + d))]\}$,

where A is a complex constant, c and d are real and $-\infty < K \leqq -1/4$.

If f has at least one pole, then f has one of the following two forms:

6. $f(z) = A \exp(-i(cz + d))/\sin(cz + d)$,
7. $f(z) = A \exp[-2i(cz + d) - 2\exp(2i(cz + d))]/\sin^2(cz + d)$,

where A, c and d are constants and c and d are real.

An immediate corollary of this and the results of **[35]** is the classification of all entire functions f such that $f, f', \ldots, f^{(n)}, \ldots$ have only non-positive zeros.

For functions that are reciprocals of real entire functions Hellerstein and Williamson **[36]** and Rossi **[72]** prove the following.

THEOREM. *Let F be real entire with only real zeros and not zero free. If $F = 1/f$ and F', F'' have only real zeros, then $F(z) = (Az + B)^{-n}$, A and B real constants, $A \neq 0$ and n is a positive integer.*

For real meromorphic functions, Hellerstein, Shen and Williamson **[91]** prove the following.

THEOREM. *Let F be a real meromorphic function with only real zeros and real poles (and at least one of each). If F' has no zeros and F'' has only real zeros, then F has one of the forms*:

$$F(z) = A\tan(az + b) + B,$$
$$F(z) = A(az + b)/(cz + d),$$
$$F(z) = A[(az + b)^2 - 1]/(az + b)^2$$

where A, B, a, b, c and d are real constants, A, a and c are non-zero and where, in the second case, $ad - bc \neq 0$.

Recently, Gethner **[90]** has verified Pólya's conjecture B for certain classes of functions. We cite certain examples to which his results apply.

EXAMPLE. Let $a \in \mathbf{C}$ and define $L_a(f) = \{z \in \mathbf{C}$: every neighborhood of z contains solutions w to the equation $f^{(k)}(w) = a$ for infinitely many values of $k\}$.

Let $c, d \in \mathbf{C}$ and write $c = |c|e^{i\gamma}$. Let $N \geqq 2$ be an integer. Suppose that f is an entire function satisfying the following two conditions.

a) There exists $\rho > 0$ such that $\log|f(z)\exp(-cz^N - dz^{N-1})| = o(r^{N-1})$ as $r \to \infty$ for

$$z \in S(\rho, \gamma) = \bigcup_{p=1}^{N} \{z: |\arg(z) + (\gamma + 2\pi p)/N| < \rho\},$$

and

b) Max $\{\log |f(z)|: |z| = r\} = o(r^N)$ $(r \to \infty)$. Then for each $a \in \mathbf{C}$, $L_a(f)$ consists of q rays with vertex at $-d/Nc$, passing through the points $-d/Nc + \exp\{i(\pi - \gamma + 2\pi p)/N\}, p = 1, \ldots N$.

In particular, one may take $f(z) = z^m P(z) \exp(Q(z))$, where $Q(z) = cz^N + dz^{N-1} + \cdots$ is a polynomial of degree N and P is a canonical product of genus at most $N - 2$, having finitely many zeros in $S(\rho, \gamma)$.

Let $n_z(\rho, a, f^{(k)})$ denote the number of zeros of $f^{(k)}(w) - a$ in $|w - z| \leqq \rho$, where $a \in \mathbf{C}$ and f is as above. If $z = -d/Nc$, then for some $\rho_0 > 0$,

$$n_z(\rho, a, f^{(k)}) \sim (\pi^{-1} N^{1+1/N} \sin(\pi/N)) k^{1-1/N} \rho$$

as $k \to \infty$ for each ρ, $0 < \rho < \rho_0$.

If $z \in L_a(f)$ but $z \neq -d/Nc$ then for some $\rho_0 > 0$,

$$n_z(\rho, a, f^{(k)}) \sim (2\pi^{-1} \sin(\pi/N)) k^{1-1/N} \rho$$

as $k \to \infty$ for each ρ, $0 < \rho < \rho_0$.

The entire functions described above have finite order. Gethner also has results which apply to functions of infinite order.

EXAMPLE. Let ϕ be an entire function satisfying

$$\limsup \log^+ \log^+ M(r, \phi)/r < 1.$$

Suppose that for each $q \in \mathbf{Z}$, there exists numbers n_q, $0 < n_q < \pi$ and $N_q > 0$ such that ϕ has no zero in the half strip

$$\{x + iy: x > N_q, |y - (2q - 1)\pi| < n_q\}.$$

Let

$$f(z) = \phi(z) \exp(-e^z).$$

Then, for each $a \in \mathbf{C}$, $L_a(f)$ is the union of the horizontal lines

$$\{y = 2\pi q\}. \quad (q \in \mathbf{Z})$$

Futhermore, there exists $\rho_0 > 0$ such that if $z \in L_a(f)$, then

$$n_z(\rho, a, f^{(k)}) \sim 2\rho k/(\log(k))^2 \quad (k \to \infty)$$

for each ρ, $0 < \rho < \rho_0$.

Edrei **[24]**, **[25]** had proved this result when $a = 0$. Gethner has a multitude of other results to which the reader should refer. Gethner considers entire functions whose directions of maximal growth are rays in the plane. These rays repel the zeros of the successive derivatives. In recent related work, Abi-Khuzam **[86]** obtains information on the local and global maxima of Lindelöf functions.

For Pólya's conjecture C, the best results remain those of Edrei **[23]**. His two main theorems are as follows.

THEOREM. 1. *Let $f(z)$ be a real entire function of finite order ρ and of mean type. Assume that,* for $x > 0$, $\limsup \log |f(\pm x)|/x^{\rho} < 0$ *as* $x \to \infty$. *Then $\rho > 1$ and the set of all the real zeros of the successive derivatives of f is everywhere dense on the real axis.*

THEOREM. 2. *Let $f(z)$ be a real entire transcendental function which satisfies a differential equation of the form $Q_0(z)f^{(m)}(z) + \cdots + Q_m(z)f(z) = P(z)$, where P and the Q's are polynomials and $Q_0(z) \not\equiv 0$. Putting $M(r) = \max|f(z)|$ on $|z| = r$, there exists a suitable value of ρ $(0 < \rho < \infty)$ such that* $\lim \log M(r)/r^{\rho} = \tau > 0$ *as $r \to \infty$. If $\rho > 1$ and* $\liminf \log |f(r)|/r^{\rho} < \tau$, $\liminf \log |f(-r)|/r^{\rho} < \tau$ *as $r \to \infty$, then every point of the real axis belongs to the final set of $f(z)$.*

A.C. Schaeffer **[73]** proved the following intriguing theorem, which, in Pólya's own words "... seems to open a new vista on the hypothetical theorem C ..." **[50**, p. 189].

THEOREM. *In an interval $a - L < x < a + L$, let $f(x) \in C^n$, $n \geqq 2$, where $f(x)$ is a real valued function. If $|f(x)| \leqq M$ and if $|f'(a)| \geqq (10n)^{2n}M/L$, then $f^{(n)}(x)$ changes sign at least $(n-1)$ times in the interval.*

Because of Bernstein's theorem (see, e.g., [7, Theorem 11.1.2]), entire functions of exponential type cannot satisfy the hypotheses of this theorem. Consequently, among the functions that might satisfy the hypotheses of Schaeffer's theorem are real entire functions (or infinitely many of their derivatives) of order > 1, precisely the functions given in conjecture C.

6. The complex domain; the case of operators. In light of the results described above on zeros of derivatives, a natural question arises. Do similar results hold when differentiation is replaced by other operators? As we shall see, this question has its source in a number of works by Pólya. When successive differentiation is replaced by successive iterates of other operators, the distribution of the zeros of the iterates of the operators on analytic or entire functions is as much influenced by the operator in question as it is by the order of the entire function or the growth of the analytic function.

The motivation for looking at the zeros of successive iterates of differential operators on analytic functions can be found in a number of papers of Pólya. Expanding on work of Laguerre, Pólya and Schur **[61]** characterized two types of multiplier sequences, that is, sequences $\{\gamma_k\}_{k=1}^{\infty}$ which transform a polynomial or entire function having real zeros or real zeros of the same sign into a function having only real zeros by termwise multiplication of the k-th coefficient of the original function by γ_k. See also Levin [**44**, Chapter 8]. In a sequel to this paper Pólya [**51**] determined what the effect is on the real zeros upon applying a differential operator (possibly of infinite order) to a real entire function having only real zeros.

Quite recently, Craven and Csordas have investigated various extensions of the above ideas in a number of fascinating papers [**16**], [**17**], [**18**], [**19**], [**87**], [**88**], [**89**]. For example, in [**16**], they present results on multiplier sequences for algebraic fields other than the real numbers and in [**17**] investigate those multiplier sequences which do not increase the number of non real zeros of real polynomials. Of particular interest is their result [**19**] on the characterization of zero diminishing linear transformations defined on the set of all polynomials with real coefficients. That is, a linear transformation T defined on polynomials $\sum_{k=0}^{n} A_k x^k$ by $T(\sum_{k=0}^{n} A_k x^k) = \sum_{k=0}^{n} A_k \lambda_k x^k$ is said to be zero diminishing when $\mathbf{Z}_{\mathbf{R}}[Tf] \leqq \mathbf{Z}_{\mathbf{R}}[f]$ for all polynomials, where $\mathbf{Z}_{\mathbf{R}}[h]$ denotes the number of real zeros of h. Craven and Csordas prove the following result.

THEOREM. *The following are equivalent.*

(i) *Either* (a) $\mathbf{Z}_+[Tf] \leqq \mathbf{Z}_+[f]$ *for all polynomials* f *or*
(b) $\mathbf{Z}_+[Tf] \leqq \mathbf{Z}_-[f]$ *for all polynomials* f.

(ii) T *is a zero diminishing linear transformation*; *i.e.*, $\mathbf{Z}_{\mathbf{R}}[Tf] \leqq \mathbf{Z}_{\mathbf{R}}[f]$ *for all polynomials* f.

(iii) *For each integer* n, *the polynomial* $\sum_{k=0}^{n} \binom{n}{k} \lambda_k^{-1} X^k$ *has only real zeros, all of the same sign.*

(iv) *The series* $\varphi(z) = \sum_{k=0}^{\infty} (k!\lambda_k)^{-1} z^k$ *converges in the whole plane and the entire function* $\varphi(z)$ *can be represented in the form* $\varphi(z) = ce^{\tau z} \prod_{n=1}^{\infty} (1 + z/z_n)$ *where* $\tau \geqq 0$, $z_n > 0$, c *is real and* $\sum_{n=1}^{\infty} z_n^{-1} < \infty$.

Furthermore, in case (i)(a) *all of the numbers* λ_k *have the same sign and in case* (i)(b) *they alternate in sign.*

In the papers [**87**], [**88**] Craven and Csordas develop a theory of n-sequences and prove many interesting results to which the reader should refer.

Related results were done by Verzhbinskii [**81**], who investigated conditions under which a differential operator applied to a transcendental real entire function of genus 1 has zeros close to the real axis. We state a typical result of Verzhbinskii [Theorem 6].

THEOREM. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be a transcendental entire function of genus* $g \leqq 1$ *and order* $\lambda < g + 1$ *with a finite number of non real zeros. Let* $P(w)$ *be a transcendental entire function of genus* $\leqq 1$ *with negative zeros* $-k_1, -k_2, -k_3, \ldots$. *If*

$$\min((1/(\tau) - \ell, (1/\lambda) - \ell) < 2^{-1}$$

where $\ell = \lim \sum_{j=1}^{n}, k_j^{-1}$ *as* $n \to \infty$ *and* τ *is the exponent of convergence of the zeros of* P, *then the non real zeros of the function* $P(zd/dz)f(z) = \sum_{k=0}^{\infty} a_k P(k) z^k$ *asymptotically approach the real axis as* $Re(z) \to \pm\infty$.

The behavior of zeros of iterated multiplier sequence operators on analytic functions has been studied in **[70]**. The operators used here include fractional integration and infinite order operators $G(d/dz)$, where $G(w)$ is entire of exponential type.

In **[53]**, Pólya presents Gauss-Lucas theorems for differential operators and classifies those operators for which Gauss-Lucas theorems hold. Proofs of these results are found in a paper of E. Benz **[5]**. A simple proof of the Gauss-Lucas results of Benz using the B and B^* operators of Levin **[44**, Chapter 9] is given in **[67]**. See also Marden **[45]**.

Recently, Craven and Csordas have characterized the sequences $\Gamma = \{\gamma_k\}_{k=1}^{\infty}$ of real numbers having the following Gauss-Lucas property. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k \in \mathbf{C}$ be an arbitrary complex polynomial. If K is a convex region containing the origin and all the zeros of $f(z)$, then the zeros of the polynomial $\Gamma f(z) = \sum_{k=0}^{n} a_k \gamma_k z^k$ also lie in K. Among the many results they prove is the following.

THEOREM. *Let* $\Phi(x) = \sum \gamma_k x^k/k!$ *be a transcendental entire function of type* I *in the Laguerre-Pólya class. For each* $n = 1, 2, 3, \ldots$, *let* $g_n(x) = \sum_{k=0}^{n} \gamma_k x^k$ *denote the Jensen polynomial associated with* $\Phi(x)$. *Then the zeros of* $g_n(x)$, $n = 1, 2, 3, \ldots$ *all lie in* $[-1, 0]$ *if and only if either* $0 \leqq \gamma_0 \leqq \gamma_1 \leqq \cdots$ *or* $0 \geqq \gamma_0 \geqq \gamma_1 \geqq \cdots$.

A sequence $\Gamma = \{\gamma_k\}$ *of non zero real numbers has the Gauss-Lucas property if and only if* Γ *is a multiplier sequence of the first kind and either* $0 \leqq \gamma_n \leqq \gamma_{n+1}$, *for* $n = 0, 1, 2, \ldots$ *or* $0 \geqq \gamma_n \geqq \gamma_{n+1}$, *for* $n = 0, 1, 2, \ldots$

The reader is advised to examine the paper of Craven and Csordas for many related results.

As mentioned earlier, the distribution of the zeros of operators on analytic or entire functions can be drastically different from that of differentiation depending on the operator in question. To be more precise, we say that a point z_0 lies in the final set $S(L, f)$ of f with respect to the operator L when every neighborhood of z_0 contains zeros of infinitely many iterates of L. The following theorem demonstrates this point **[91]**.

THEOREM. *Let* $f(z)$ *be a real entire function of order one, normal type* λ, *satisfying* $|f(x)| \leqq 1$ *for* $x \in (-\infty, \infty)$. *Let* $L = \varphi(D)$, *with* $D = (d/dz)$ *and* $\varphi(w)$ *is a Laguerre-Pólya function satisfying* $\varphi(0) = 0$. *Then the final set of* f *with respect to the operator* L *is either a discrete set on the real axis or the whole axis depending on whether the number* β, $\beta = \arg(\varphi(i\lambda)/\varphi(-i\lambda))$, *is or is not a rational multiple of* π.

This result generalizes previous special cases **[11]**, **[64]**, **[65]**. The reader should also note that Pólya's conjectures change when differentiation is replaced by more general differential operators.

7. The real domain; the case of derivatives and operators. Wide use in various places has been made of regularly monotonic, absolutely monotonic, completely monotonic and completely convex functions. These are functions for which each derivative is of fixed sign on $[a, b]$, $f^{(k)}(x) \geqq 0$, $(-1)^k f^{(k)}(x) \geqq 0$, and $(-1)^k f^{(2k)}(x) \geqq 0$, respectively. Various characterizations by Bernstein and Widder of these functions and their properties are well known and can be found in Boas' survey **[8]** and in two of Widder's books **[84]**, **[85]**. All of these notions have been generalized to differential operators of various types. Generalized absolutely monotonic functions (denoted G.A.M. functions) have been characterized by Karlin and Zeigler **[41]** and Amir and Zeigler **[2]**. The author will define G.A.M. functions later. Džrbašjan and Saakjan **[21**, Theorem 1.1**]** have studied the notion of $\langle \rho \rangle$-absolutely monotonic functions which are defined by the non-negativity of the iterates of a fractional derivative of order $1/\rho$. They generalize a theorem of S. N. Bernstein (see Widder **[84]**, p. 144–47) by showing that $\langle \rho \rangle$-absolutely monotonic functions can be expressed in a generalized Taylor-Maclaurin series in powers of $x^{1/\rho}$. In addition, they partially generalize this result to the case of polynomial operators in $D^{1/\rho}$, $D = (d/dx)$, relative to the sequence $\{\lambda_k\}$ of the roots of a polynomial (Theorems 4.4 and 4.5). They give further results in **[22]**. A comprehensive study of the fractional derivative as an operator can be found in the paper of Gaer and Rubel **[29]**.

Generalized completely monotonic functions (G.C.M.) have been characterized by Studden **[79]** for weighted iterates of differential operators. Specifically, let $\{W_i(t)\}_{i=0}^{\infty}$ be an infinite sequence of functions such that $W_i(t) > 0$ for $t \in [0, \infty)$, $W_i(t) \in C^{\infty}[0, \infty)$, and $D_i f = (d/dt)[f(t)/W_i(t)]$, $i = 0, 1, 2, \ldots$. A function $\phi(t)$ on $[0, \infty)$ is called a generalized completely monotonic function on $[0, \infty)$ with respect to $W_i(t)$ if $\phi \in C^{\infty}(0, \infty)$, $\phi \geqq 0$ and $(-1)^{n+1} D_n D_{n-1} \cdots D_0 \phi \geqq 0$ for all $t \in (0, \infty)$, $n = 0, 1, 2, \ldots$. Generalized absolutely monotonic functions (G.A.M.) are defined similarly, except that the multiplier $(-1)^{n+1}$ before the iterated operators is absent. Studden characterized G.C.M. functions in terms of an integral representation which generalizes the classical integral representation of completely monotonic functions due to Bernstein.

A representation for generalized completely convex functions that extends results of Widder **[84]** have recently been extended to a rather wide class of operators by Amir and Ziegler **[3]**.

Using the methods of total positivity, they consider self adjoint differential operators of the form

(a) $(Mu)(x) = (-1)^n (D_0^* \cdots D_{n-1}^* D_{n-1} \cdots D_0 u)(x)$ where $D_i u = (d/dx)$

$\cdot[u/W_i(x)]$, $D_i^* u = (du/dx)/W_i(x)$, and the $\{W_i(x)\}$ are positive functions with W_i being of class C^{n-i} on [0, 1]. Attach to (a) separable boundary conditions of the form

$$\text{(b)}\qquad \begin{gathered} D_1^* \cdots D_{n-1}^* D_{n-1} \cdots D_0 u(0) + (-1)^n C_1 u(0) = 0, \\ D_2^* \cdots D_{n-1}^* D_{n-1} \cdots D_0 u(0) + (-1)^{n+1} C_2 D_0 u(0) = 0, \\ \vdots \\ D_{n-1}^* \cdots D_0 u(0) + (-1)^{2n-1} C_n D_{n-2} \cdots D_0 u(0) = 0 \end{gathered}$$

and

$$\begin{gathered} D_1^* \ldots D_{n-1}^* D_{n-1} \ldots D_0 u(1) + (-1)^n d_1 u(1) = 0, \\ \vdots \\ D_{n-1}^* D_{n-1} \cdots D_0 u(1) + (-1)^{2n-1} d_n D_{n-2} \cdots D_0 u(1) = 0 \\ \text{where } 0 \leqq C_k \leqq d_k,\ 0 < d_k < \infty. \end{gathered}$$

For Amir and Ziegler, a function φ is said to be a generalized completely convex function with respect to the operator M and the boundary conditions (b) if $M^k\varphi \geqq 0$ for all k and φ satisfies (b). They prove that such a function φ has a uniformly convergent Taylor-Lidstone expansion. They are also able to handle non self-adjoint operators with regular boundary conditions and non regular separable boundary conditions.

Various specializations lead to the results of Buckholtz and Shaw [**15**], Pethe and Sharma [**48**] and Leeming and Sharma [**43**]. These results are a consequence of Amir's and Ziegler's investigation of a generalized Taylor-Lidstone series with respect to a linear operator L defined on a linear space E having m dimensional kernel and associated linear functionals $B_1, \ldots B_m$ on E. They generalize complete convexity in terms of convex cones and extreme rays and obtain representation theorems generalizing the classical Lidstone series. Related results can be found in [**6**].

Boas and Pólya [**10**] generalized classical results of Bernstein and Widder in their study of the influence of the sign changes of derivatives of functions on their analytic character. Sharma and Tzimbalario [**76**] generalized their results in their study of the influence of the sign changes of a sequence of linear differential operators on the analytic character of a function. To do so, they generalize the results of Kloosterman [**42**] and prove the following theorem.

THEOREM. *Let $\{\gamma_k\}_{k=1}^{\infty}$ be an infinite sequence of real numbers and let $\{n_k\}^{\infty}$, $\{q_k\}_{k=1}^{\infty}$ be two sequences of positive integers, with $\{n_k\}$ strictly increasing. Let $L_n(D) = \prod_{j=1}^{n}(D - \gamma_j)$, $n = 1, 2, \ldots$; $L_0(D) = I$. Let $f(x)$ be a real valued function $\in C^\infty[-1, 1]$ and assume that $L_{n_k}(D)f$, $L_{n_k+2q_k}(D)f$ do not change sign in $[-1, 1]$ with $L_{n_k}(D)fL_{n_k+2q_k}(D)f \leqq 0$, $k = 1, 2, \ldots$ and $x \in [-1, 1]$.*

(I) *If* $n_k - n_{k-1} = 0(1)$, $q_k = 0(1)$, *and* $\gamma_k = 0(1)$, *then* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function of growth not exceeding order one and finite type.*

(II) *If* $n_k - n_{k-1} = o(n_k^\delta)$, $q_k = o(n_k^\delta)$, $0 < \delta < 1$, $\sum_{j=1}^k q_j = o(n_k)$ *and if* $|\gamma_k| \leqq A + \eta \log(k)$ *for all* k *for some positive constants* A *and* $\eta < (1-\delta)/2$, *then* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function of finite order not exceeding* $1/(1 - (\delta + 2\eta))$.

(III) *If* $n_k - n_{k-1} = o(n_k)$, $q_k = o(n_k)$, $\sum_{j=1}^k q_j = o(n_k)$ *and if* $|\gamma_k| = o(\log k)$, *then* $f(x)$ *is the restriction to* $[-1, 1]$ *of an entire function.*

In the process of proving this theorem, Sharma and Tzimbalario prove an extension to linear differential operators $\prod_{i=1}^n (D-\lambda_i)$, λ_i real, of the Landau-Kolmogorov inequalities for a finite interval. Sharp Landau-Kolmogorov inequalities for such operators on the infinite intervals $(-\infty, \infty)$ and $[0, \infty)$ have been given by Karlin [**40**].

Pólya and Wiener [**63**] and Szego [**80**] investigated the analytic character of periodic functions as a function of the sign changes of derivatives of large order within a period. In a little known paper, V. E. Kacnelson [**38**] has investigated almost periodic functions in the simplest case as suggested by Pólya and Wiener. Due to the lack of easy access to the results of this paper we give an account of them.

As suggested by Pólya and Wiener, Kacnelson replaces the notion of the number of sign changes within a period by the density of the sign changes on the real axis. If f is uniformly almost periodic and is of class $C^\infty(-\infty, \infty)$, let $\eta_k(\alpha, \beta)$ be the number of sign changes of $f^{(k)}(x)$ in (α, β). Denote the upper sign density of $f^{(k)}(x)$ by the expression $\bar{N}_k =$ lim sup $n_k(-t, t)/2t$ as $t \to \infty$. If this limit exists, we denote it by N_k and call it the sign change density. Let f be an almost periodic function, with $[-\delta, \delta]$ being the smallest interval containing the spectrum of $f(0 \leqq \delta \leqq \infty)$. We say that f belongs to the class $[\delta]$, if $\delta < \infty$ and $+\delta$ and $-\delta$ both belong to the spectrum of f. The zeros of functions in the class $[\delta]$ are discussed in Levin's book [**44**, Chapter 6]. Kacnelson proves the following results.

THEOREM. *Let f be an infinitely differentiable real valued almost periodic function in the sense of Bohr* [**13**]. *Let* $[-\delta, \delta]$ *be the smallest segment containing the spectrum of f. Let $\bar{N}_k$ be defined as above. Then* lim $\bar{N}_k = \delta/\pi$ *as* $k \to \infty$.

If f has bounded spectrum, a necessary and sufficient condition that there exists an m such that $\bar{N}_k = \delta/\pi$ for $k \geqq m$ is that f belongs to the class $[\delta]$. *In this case $\bar{N}_k = N_k$ for $k \geqq m$.*

Kacnelson constructs an example of a function f of class $[\delta]$ having an absolutely convergent Fourier series which fails to have a sign change density N_0, although by the result stated above, from some k on, there exists densities N_k.

Pólya [**50**, p. 188] remarked that one can generalize the results of Pólya and Wiener [**63**] to Fourier integrals by looking at the density of the sign changes of $f^{(n)}(x)$ in $(-\infty, \infty)$. A result in this direction has recently been done for entire characteristic functions.

THEOREM. *Let*

$$f(x) = \int_{-\infty}^{\infty} e^{ixt}\phi(t)dt,$$

where $\phi(t) \geqq 0$ *is a real valued function that is absolutely continuous on* $(-\infty, \infty)$, *satisfying* $\int_{-\infty}^{\infty} \phi(t)dt = 1$, $\int_{-\infty}^{\infty} |t^k\phi(t)|dt < \infty$ *for every* $k = 1, 2, \ldots$ *and* $\phi(-t)=\phi(t)$. *Let* $\bar{N}_k$ *be the upper sign change density of* $f^{(k)}(x)$ *in* $(-\infty, \infty)$. *Then*

A) *Assume that* ϕ *is nonincreasing in* $(0, \infty)$ *and that there exists a* $\tau > 0$ *such that* $\phi(t)$ *is strictly decreasing in* $[M - \tau, M]$ *and* $\phi(M) \neq 0$. *If* $\lim\bar{N}_k = \lim N_k = M/\pi$ *as* $k \to \infty$, *then*

$$f(x) = \int_{-M}^{M} e^{ixt}\phi(t)dt.$$

Conversely, if $f(x) = \int_{-M}^{M} e^{ixt}\phi(t)dt$, *where* $\phi \in L_1[-M, M]$, $\phi(-t) = \phi(t)$, $\phi(M) \neq 0$, *and the convex hull of support* $[\phi]$ *is* $[-M, M]$, *then* $\lim \bar{N}_k = \lim N_k = M/\pi$ *as* $k \to \infty$.

B) Assume that ϕ *satisfies the hypotheses of the theorem. Furthermore, for the infinitely many values of s for which*

$$\phi(t)/\phi(s) \leqq \exp(s^p - t^p), \qquad 0 \leqq t < s \qquad (p > \rho > 1),$$

and $\phi(t) \leqq \phi(s)$, $t \geqq s$, *hold, assume that there exists a* $\tau > 0$, *independent of* s, *such that* $\phi(t)$ *is strictly decreasing in* $[s-\tau, s]$. *Then* $\bar{N}_k = O(k^{1/(\rho+1)})$, *for* $\rho > 1$, *implies that f is the restriction to the real axis of an entire function of finite order* $\leqq \rho/(\rho - 1)$.

C) *Assume that* ϕ *satisfies the hypotheses of the theorem. Furthermore, for the infinitely many values of s for which*

$$\phi(t)/\phi(s) \leqq \exp(\theta(s - t)), \qquad 0 \leqq t < s \qquad (\theta \text{ a constant}),$$

and $\phi(t)/\phi(s) \leqq 1$, $t \geqq s$ *hold, assume that there exists a* $\tau > 0$, *independent of* s, *such that* $\phi(t)$ *is strictly decreasing in* $[s - \tau, s]$. *Then* $\bar{N}_k = o(k^{1/2})$. *implies that f is the restriction to the real axis of an entire function.*

It is known that the result in B) is not best possible, whereas in *C*) it is not known.

As mentioned by Pólya [**50**, p. 188], a generalization in another direction of the results of Pólya and Wiener [**63**] was done by Hille [**37**], who studies the problem in the context of characteristic series of boundary value problems. We briefly describe his work. The reader should examine the original paper for details. He considers a general second order differential operator L such that $Ly = P_0(x)y'' + P_1(x)y' + P_2(x)y$, with analytic coefficients. Let (a, b) be a finite or infinite interval and associate to L a boundary value problem $(L + \mu)u = 0$ with appropriate boundary conditions B.C. $(u, a, b) = 0$. Assume that there exists an infinite number of eigenvalues $\{\mu_n\}$ with associated eigenfunctions $\{u_n(x)\}$, normalized in weighted mean square. Consider the characteristic series

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x),$$

where $\sum_{n=1}^{\infty} |f_n \mu_n^m| < \infty$ for every $m = 0, 1, 2, \ldots$. Computation shows that

$$L^m f(x) = \sum_{n=1}^{\infty} (-\mu_n)^m f_n u_n(x).$$

The Pólya and Wiener problem in this setting is to determine the relationship between the oscillations of the transforms $L^m f(x)$ in (a, b) and the analytic character of $f(x)$. For admissible functions $f(x)$ (see **[37]** for the somewhat lengthy definition), Hille proves the following results.

THEOREM. *If $f(x)$ is admissible and* $\liminf N_k = N$, *as $k \to \infty$, where N_k is the number of sign changes of $L^k f(x)$ in (a, b), then there exists an integer $M = M(N)$ such that $f(x) = \sum_{n=1}^{M} f_n u_n(x)$.*

In the case of eigenvalues of multiplicity one, $M(N)$ is the largest integer such that $u_n(x)$ has at most N sign changes in (a, b). In the case of double eigenvalues $M(N)$ can exceed this number by at most one unit.

Extensions of the other results of Pólya and Wiener in this setting have recently been done where N_k is allowed to grow as a function of k. The following results hold **[70]**.

THEOREM. *Let $f(x)$ be admissible and satisfy*

$$\limsup_{n\to\infty} |f_n| \exp(\tau\, \mu_n^{1/2}) = \infty$$

for some $\tau > 0$. Let $\{C_m\}_{m=1}^{\infty}$ be a sequence such that $k_m = C_m \mu_m^2$ is an integer for each m, and for all m sufficiently large,

$$C_m \mu_m \geqq 1, \qquad (\mu_{m+1} - \mu_m)/\mu_m \leqq 1,$$

$$0 < \mu_2 - \mu_1 < \mu_3 - \mu_2 < \ldots,$$

$$C_m(\mu_m - \mu_{m-1}) \geqq \tau(3 + 2\tau), \qquad C_m \mu_m^2 \geqq 2(\beta+\delta)\mu_{m+1}/(\mu_2 - \mu_1),$$

$$m \mu_{m+1}^{\beta+\delta+1} \exp[(\mu_m - \mu_{m-1})(\tau - C_m(\mu_m - \mu_{m-1})/9] \to 0$$

as $m \to \infty$, where β and δ are appropriately chosen constants. Then for infinitely many m, $N_{k_m} \geqq m$.

THEOREM. *Let $f(x) = \sum_{n=1}^{\infty} f_n u_n(x)$ be admissible with the eigenfunctions $u_n(z)$ analytic in a region Ω containing (a, b). Suppose that*

$$\limsup |f_n| \exp(\tau \mu_n^{1/2}) < \infty$$

for every $\tau > 0$. Then this eigenfunction expansion converges uniformly on compact subsets of Ω. Thus $f(x)$ is analytically continuable into Ω.

Various other results are also given.

8. Some open questions.

1. What form do Pólya's conjectures A, B, C take for operators more general than differentiation?

2. Can the results of Schaeffer [**73**] be generalized to operators using the methods of Sharma and Tzimbalario [**76**]?

3. Edrei and MacLane [**26**] have shown that if K is a compact set on the Riemann sphere such that $\infty \in K$, then there exists an entire function f of any order ρ, $0 \leqq \rho \leqq \infty$, whose final set with respect to differentiation is K. Does something similar hold for other differential operators?

3. Let $L = \varphi(zd/dz)$, where $\varphi(w)$ is a real entire function of genus one having negative real zeros. Let f be a transcendental real entire function of genus one having real zeros. It is well known that $(Lf)(z) = \varphi(zd/dz)f(z)$ has only real zeros (see Pólya [**58**], Obreschkoff [**47**], Craven and Csordas [**17**] and Marden [**45**]). Under what conditions is the final set of f with respect to L the real axis?

5. If $F(z)$ and $w(z)$ are Laguerre-Pólya, $F(z) = \exp(-\gamma_1 z^2)F_1(z)$, $w(z) = \exp(-\gamma_2 z^2)w_1(z)$ where γ_1, $\gamma_2 \geqq 0$, and F_1, w_1 are real entire functions of genus one having only real zeros, then a theorem of Pólya [**51**] (see also Levin [**44**, Chapter 9]) gives that $F(D)w(z)$ is a Laguerre Pólya function provided $0 < \gamma_1\gamma_2 < 4^{-1}$. Under what conditions is the final set of F with respect to $L = F(D)$, $D = d/dz$, the real axis?

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