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UPPER AND LOWER SOLUTIONS FOR SYSTEMS OF SECOND ORDER EQUATIONS WITH NONNEGATIVE CHARACTERISTIC FORM AND DISCONTINUOUS NONLINEARITIES

CHRIS COSNER AND FRANK SCHINDLER

1. Introduction. The method of upper and lower solutions has a long history and a wide application in the study of nonlinear elliptic and parabolic boundry value problems. It is related to the methods of monotone operators as discussed in [1], and has been applied to elliptic, parabolic, and first order equations and systems by various authors, see [1], [2], [3], [4], [8], [10], [11], [12], [13], among many others; for further references see [1], [12].

The present article gives a unified treatment for systems of second order equations with nonnegative characteristic form, as studied in [5], [6], [9]. Such equations include elliptic, parabolic, and first order equations, among others. The results obtained here extend existing ones in various ways. First, we consider the general setting of systems of equations with nonnegative characteristic form. Single equations of that type are considered in [4], but are required to be elliptic near the boundary of the domain where they are studied. We consider weak solutions; in fact, since we allow discontinuities in our nonlinearity, and equations with nonnegative characteristic form need not have any smoothing properties, we can do no better.

Specifically, we consider systems of the form

$$-L^r[u^r] = f^r(x, \vec{u}) \text{ in } \Omega, \qquad r = 1, \ldots, m \tag{1.1}$$

where $\Omega \subseteqq \mathbf{R}$ is a smooth bounded domain, $\vec{u} = (u^1, \ldots, u^m)$ and

$$L^r[u] \equiv \sum_{i,j=1}^{n} a^r_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b^r_i(x)u_{x_i} + c^r(x)u \tag{1.2}$$

with $\sum_{i,j,=1}^{n} a^r_{ij}(x)\, \xi_i \xi_j \geqq 0$. The functions $f^r(x, \vec{u})$ are required to be measurable in all variables and quasimonotone, that is, f^r must be nondecreasing in u^s for $s \neq r$; we also require that $f^r(x, \vec{u}) + Mu^r$ be increasing in u^r for some $M > 0$. We also require that if $\vec{u}$ is bounded and

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measurable, then so is $f^r(x, \vec{u})$ for each r. That will be true under various conditions discussed in §3, but in particular, it suffices that f^r be independent of x and be a Borel function in $\vec{u}$, which allows f^r to be discontinuous. In **[13]**, discontinuous nonlinearities are considered, but only for a single equation.

In the elliptic case, equations and systems have been studied where the nonlinearity depends on the first derivatives of the dependent variables; see **[2]**, **[12]**. However, the operators in (1.1) may be first or even zero order, and no compactness results are available, so we consider only the case where no derivatives occur in the nonlinearity.

We also discuss the requirement of quasimonotonicity for (1.1). This condition is a common one for results based on monotone operator theory. In the case $r = 2$ it is noted in **[3]**, **[10]**, **[11]** that the condition can be replaced by requiring f^i to be decreasing in u^j, $i \neq j$, with suitable redefinition of upper and lower solutions. However, such systems can be made quasimonotone by a change of variables, as noted in **[10]**. In fact, a similar situation occurs in some systems with $m \geqq 3$; we address the question of reducing systems to equivalent ones which are quasimonotone. We give a simple method of deciding if such a reduction can be performed by replacing u^r by $-u^r$ for some values of r.

2. Preliminaries. Let $\Omega \subseteqq \mathbf{R}^n$ be a bounded domain, with $\partial\Omega$ of class $C^{2+\alpha}$ for some $\alpha \in (0, 1)$. For $r = 1, \ldots, m$ let L^r denote the operator

$$L^r[u] \equiv \sum_{i,j=1}^{n} a_{ij}^r(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i^r(x)u_{x_i} + c^r(x)u, \tag{2.1}$$

and let L^{r*} denote the formal adjoint of L^r, so that

$$L^{r*}[u] \equiv \sum_{i,j=1}^{n} a_{ij}^r(x)u_{x_ix_j} + \sum_{i=1}^{n} b_i^{r*}(x)u_{x_i} + c^{r*}(x)u. \tag{2.2}$$

Assume that $a_{ij}^r = a_{ji}^r$ for all r, that for all, i, j, r, $a_{ij}^r \in C^2(\bar{\Omega})$, $b_i^{r*} \in C^1(\bar{\Omega})$, b_i^r, c^r, $c^{r*} \in C^\alpha(\bar{\Omega})$, and that for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{R}^n$, $x \in \bar{\Omega}$, and r,

$$\sum_{i,j=1}^{n} a_{ij}^r(x)\, \xi_i\xi_j \geqq 0. \tag{2.3}$$

Following Fichera **[6]** (see also **[5]**, **[9]**), divide $\partial\Omega$ in regions as follows: let $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ be the *inward* unit normal to $\partial\Omega$. For each r, define Σ^{or} to be the subset of $\partial\Omega$ where $\sum_{i,j=1}^n a_{ij}^r\, \nu_i\nu_j = 0$. Define a function $B^r(x)$ on Σ^{or} by

$$B^r(x) = \sum_{i=1}^{n} (b_i^r - \sum_{j=1}^{n} a_{ijx_j}^r)\nu_i.$$

($B^r(x)$ is generally known as the Fichera Function for L^r; its sign at any point of Σ^{or} is invariant under nondegenerate changes of coordinates.) Define Σ_0^r, Σ_1^r, Σ_2^r to be the subsets of Σ^{or} where $B^r(x) = 0$, $B^r(x) > 0$, and $B^r(x) < 0$, respectively. Define Σ_3^r by $\Sigma_3^r = \partial\Omega \backslash \Sigma^{or}$. Define Γ^r to be the set boundary points of $\Sigma_2^r \cup \Sigma_0^r$ in $\partial\Omega$. Suppose that $\partial\Omega$ is given in some neighborhood of $x_0 \in \partial\Omega$ by $F(x) = 0$, with $F > 0$ inside Ω and $\nabla F \neq 0$. Define $\beta^r(x)$ at x_0 by $\beta^r(x_0) = L^r[F(x)]|_{x=x_0}$. (We will be concerned with only the sign of β^r, which is independent of the choice of F. In many cases, the sign of $\beta^r(x)$ will agree with that of $B^r(x)$, but that is not always true; see **[8]**, p. 31.)

We will be interested in weak solutions to problems involving the operators L^r. The simplest problem, which provides the basis for studying more complicated ones, is the following:

$$-L^r[u] = f(x) \text{ in } \Omega, \qquad u = g(x) \text{ on } \Sigma_2^r \cup \Sigma_3^r, \tag{2.4}$$

where f and g are bounded measurable functions on Ω and $\Sigma_2^r \cup \Sigma_3^r$ respectively. We follow **[5]**, **[6]** and **[9]**, and define a weak solution to (2.4) to be a bounded, measurable function u such that for any $v \in C^2(\bar{\Omega})$ with $v = 0$ on $\Sigma_1^r \cup \Sigma_2^r$, the equation

$$-\int_\Omega uL^{r*}[v]\, dx = \int_\Omega vf\, dx - \int_{\Sigma_3^r} g\frac{\partial v}{\partial \nu}\, ds + \int_{\Sigma_2^r} B^r gv\, ds \tag{2.5}$$

is satisfied, where ds denotes surface measure on $\partial\Omega$. Observe that although the notion of a weak solution includes having $u \in L^\infty(\Omega)$, equation (2.5) still makes sense for $u \in L_p(\Omega)$, $p \geqq 1$.

We will need the following existence-uniqueness theorem.

THEOREM 1. *Suppose that the following conditions are met*: (*E*) *the coefficient* $c^r(x)$ *satisfies* $c^r(x) \leqq c_0 < 0$; $f \geqq 0$ *is a bounded, measurable function on* Ω, *and* $g \geqq 0$ *is a bounded measurable function on* $\Sigma_2^r \cup \Sigma_3^r$ such that there exist functions $g_n \in C^{2+\alpha}(\bar{\Omega})$ *with* $g_n \geqq 0$ *and* $g_n \to g$ *in* $L^2(\Sigma_2^r \cup \Sigma_3^r)$; *and* (*U*) *the coefficient* c^{r*} *of* L^{r*} *satisfies* $c^{r*} < 0$ *in* Ω; $\beta^{r*} < 0$ *on* Σ_1^r, (*where* β^{r*} *is the function corresponding to* β^r *for* L^{r*}), *the n*–1 *measure of a* δ *neighborhood of* Γ^r *in* $\partial\Omega$ *has order* δ^q, $q > 0$, *and that the coefficients of* L^{r*} *can be extended into a* δ *neighborhood of* $\Sigma_0^r \cup \Sigma_2^r$ *so that the smoothness hypotheses given above, and* (2.3), *remain true.*

Then the problem (2.4) *has a unique bounded measurable weak solution, u, in the sense of* (2.5), *with*

$$0 \leqq u \leqq \max\{\text{ess sup } f/c_0, \text{ ess sup } g\} \tag{2.6}$$

holding almost anywhere.

PROOF. This theorem is essentially Theorems 1.5.1 and 1.6.2 of **[9]**; the hypotheses given under (E) are needed for existence, those under (U) for uniqueness. The only new assertion is the nonnegativity of u; that follows from the construction of the weak solution in theorem 1.5.1 of [9], which is done by approximating f and g by sequences of smooth functions $\{f_n\}$ and $\{g_n\}$ which converge to f and g in the L_2 sense on Ω and $\Sigma_2^r \cup \Sigma_3^r$ respectively, and then solving $-\varepsilon\Delta u - L^r u = f_n$ in Ω $u = g_n$ on $\partial\Omega$ to obtain $u_{\varepsilon,n}$, and passing to a limit as $\varepsilon \to 0$, $n \to \infty$. We can obtain $f_n \geqq 0$ by mollifying f, and $g_n \geqq 0$ by hypothesis. Then $u_{\varepsilon,n} \geqq 0$ so that $u \geqq 0$ a.e.

Another result that will be needed is a theorem of Krasnosel'skii on monotone operators. The setting will be the space $[L_p(\Omega)]^m$ of vector functions $\vec{u} = (u^1, \ldots, u^m)$ with $u^r \in L_p(\Omega)$ for $r = 1, \ldots, m$; the cone K used to define order monotonicity in $[L_p(\Omega)]^m$ will be the cone of functions $\vec{v}$ with $v^r \geqq 0$ a.e. for $r = 1, \ldots, m$. The cone K in $[L_p(\Omega)]^m$ is regular and strongly minihedral; that is, every bounded sequence monotone with respect to K has a limit, and every bounded set has a least upper bound. (For further discussion see [7].) The relation $\vec{u} \leqq \vec{v}$ then means $u^r \leqq v^r$ a.e. for $r = 1, \ldots, m$. An operator on all or part of $[L^p(\Omega)]^m$ is monotone (with respect to K) if $\vec{u} \leqq \vec{v}$ implies $A\vec{u} \leqq A\vec{v}$. The conical segment in $[L^p(\Omega)]^m$ defined by the elements $\vec{v}$, $\vec{w}$, is the set of all $u \in [L^p(\Omega)]^m$ with $\vec{v} \leqq \vec{u} \leqq \vec{w}$.

The following is a theorem of Krasnosel'skii ([7], Theorem 4.1).

THEOREM 2. *Let A be an operator, monotone on the conical segment $\vec{v} \leqq \vec{u} \leqq \vec{w}$ and transforming that segment to itself, that is, with $A\vec{v} \geqq \vec{v}$ and $A\vec{w} \leqq \vec{w}$.*

If the cone K defining the ordering is strongly minihedral, then A has a fixed point in the segment.

REMARK. Note that A need not be continuous. That fact is crucial for the results that follow.

3. The Existence Theorem. In this section we establish an existence theorem for weak solutions of the system.

$$
\begin{aligned}
-L^r[u^r] &= f^r(x, u^1, \ldots, u^m) \text{ in } \Omega,\ r = 1, \ldots, m, \\
u^r &= 0 \text{ on } \Sigma_2^r \cup \Sigma_3^r,
\end{aligned}
\tag{3.1}
$$

by using results on monotone maps. We will require that either $\vec{f} = (f^1, \ldots, f^m)$ is quasimonotone (that is, f^r is nondecreasing in u^s for $s \neq r$) for $x \in \Omega$ and $\vec{u} = (u^1, \ldots, u^m)$ in an appropriate set, or that system (3.1) can be transformed to one in which the nonlinearity is quasimonotone. We will also require that for some $M > 0$, for all r

$$
\begin{aligned}
&f^r(x, u^1, \ldots a, u^{r-1}, u^r, u^{r+1}, \ldots, u^m) \\
(3.2) \qquad &\quad - f^r(x, u^1, \ldots, u^{r-1}, v^r, u^{r+1}, \ldots, u^m) \\
&\geqq -M(u^r - v^r)
\end{aligned}
$$

for $u^r \geqq v^r$, $x \in \Omega$, and $\vec{u}$ in an appropriate set. (The "appropriate set" will typically be a set including any values of u^r between the components χ^r, ϕ^r of the upper and lower solutions.)

We will assume that $\vec{f}(x, \vec{u})$ is a bounded measurable function if u^r is bounded and measurable for each r. (Again, this need only hold between the lower and upper solutions.) There are various possible sufficient conditions for such functions. One is the Carathéodory condition that $f^r(x, \vec{u})$ be bounded and measurable in x and continuous in $\vec{u}$. Another is that $f^r(x, \vec{u}) = f^r(\vec{u})$, with $f^r(u)$ a Borel function for each r. (If $f^r(u)$ is separately continuous in each component of $\vec{u}$, then it is Borel.) Borel functions can be rather discontinuous; for example, $\{U_i\}$, $i = 1, 2, \cdots$, is a disjoint collection of Borel sets in $\mathbf{R}^m$ with $\bigcup_{i=1}^{\infty} U_i = \mathbf{R}^m$, then $f(\vec{u}) = \sum_{i=1}^{\infty} c_i \chi_{U_i}$ is Borel, and bounded if the coefficients c_i are bounded. Other conditions on $\vec{f}(x, \vec{u})$ are also possible. In the case of a function of a single variable, $f(w)$, condition (3.2) implies f is Borel since $f(w) + Mw$ is monotone.

We define a weak solution of (3.1) to be a bounded measurable function $\vec{u}$ on Ω such that for any $\vec{v} \in C^2(\bar{\Omega})$ with $v^r = 0$ on $\Sigma_1^r \cup \Sigma_2^r$, the equations

$$
(3.3) \qquad -\int_\Omega u^r L^{r*}[v^r]\, dx = \int_\Omega v^r f^r(x, \vec{u})\, dx, \quad r = 1, \ldots, m
$$

are satisfied. (Any classical solution is also a weak solution.)

THEOREM 3. *Suppose that there exist bounded measurable vector functions $\vec{\phi}$, $\vec{\chi}$ on Ω such that for $r = 1, \ldots, m$, ϕ^r satisfies*

$$
(3.4) \qquad
\begin{aligned}
L^r[\phi^r] &= \bar{F}^r(x) \text{ in } \Omega, \\
\phi^r &= \bar{g}^r(x) \text{ on } \Sigma_2^r \cup \Sigma_3^r
\end{aligned}
$$

in the sense of (2.5), *with* $\bar{F}^r(x) \geqq f^r(x, \vec{\phi})$ a.e. *and χ^r satisfies in the sense of* (2.5),

$$
(3.5) \qquad
\begin{aligned}
L^r[\chi^r] &= \underline{F}^r(x) \text{ in } \Omega, \\
\chi^r &= \underline{g}^r(x) \text{ on } \Sigma_2^r \cup \Sigma_3^r
\end{aligned}
$$

with $\underline{F}^r(x) \leqq f^r(x, \vec{\chi})$ a.e., *where $\bar{F}^r$, $\underline{F}^r$, $\bar{g}^r$, $\underline{g}^r$ are bounded measurable functions, and there exist functions $\bar{g}_n^r$, $\underline{g}_n^r \in C^{2+\alpha}(\bar{\Omega})$ with $\bar{g}_n^r \geqq 0$, $\underline{g}_n^r \leqq 0$ and $\bar{g}_n^r \to \bar{g}^r$, $\underline{g}_n^r \to \underline{g}^r$ in $L^2(\Sigma_2^r \cup \Sigma_3^r)$.*

Suppose also

$$
(3.6) \qquad \phi^r(x) \geqq \chi^r(x) \quad \text{a.e.,} \quad r = 1, \ldots, m.
$$

Suppose that for $x \in \bar{\Omega}$ *and* $\vec{u}$ with $\chi^r \leqq u^r \leqq \phi^r$ a.e., *for all r*, $\vec{f}(x, \vec{u})$ *is quasimonotone and satisfies* (3.2) *and the measurability hypotheses following* (3.2), *and that for each r*, L^r *satisfies the hypotheses of Theorem* 1, *but with the condition of* (*E*) *that* $c^r(x) \leqq -c_0 < 0$ *replaced by the condition that* $c^r(x) \leqq c_1$.

Then (3.1) *has a weak solution* $\vec{u}$ *in the sense of* (3.2) *with* $\chi^r \leqq u^r \leqq \phi^r$ a.e., $r = 1, \ldots, m$.

PROOF. Choose $p \in (1, \infty)$. (We want bounded measurable solutions, but to apply Theorem 2 we need the cone K of §2 to be minihedral, which is not true in $[L^\infty(\Omega)]^m$. Thus we must work in a subset of $[L_p(\Omega)]^m$. However, the necessary operators are well defined and need not be continuous, so that is not a problem.) Using the cone K in $[L_p(\Omega)]^m$ defined in §2, the relation $\chi^r \leqq w^r \leqq \phi^r$ a.e., $r = 1, \ldots, m$, defines a conical segment S. By adding the term Mu^r to both sides of the *rth* equation in (3.1), we can simultaneously make f^r monotone increasing in u^r and make the coefficient $-c^r + M$ of the undifferentiated term in $-L^r + M$ satisfy $-c^r + M \geqq c_0 > 0$, which is equivalent to what is needed in hypothesis (*E*) of Theorem 1. Also, (3.3) is unaffected. Thus we shall assume, without loss of generality, that f^r is monotone increasing in u^r and the hypotheses of Theorem 1 are satisfied.

For $\vec{w} \in F$, define $Z = A\vec{w}$ by taking Z^r to be the unique solution in the sense of (2.5) to

$$-L^r Z^r = f^r(x, \vec{w}) \text{ in } \Omega, \qquad Z^r = 0 \text{ on } \Sigma_2^r \cup \Sigma_3^r. \tag{3.7}$$

Theorem 1 insures that $\vec{Z}$ is well defined, and since $\vec{Z}$ is bounded and measurable, $\vec{Z} \in [L_p(\Omega)]^m$.

We now show that A maps S to itself and is monotone on S. Suppose $\vec{w} \in S$. Then for any admissible test function v^r, we have

$$-\int_\Omega \phi^r L^{r*}[v^r]dx = \int_\Omega v^r \bar{F}^r(x)dx - \int_{\Sigma_3^r} \bar{g}^r \frac{\partial v^r}{\partial \vec{\nu}} ds + \int_{\Sigma_2^r} B^r \bar{g}^r v^r ds$$

and for $\vec{Z} = A\vec{w}$, $-\int_\Omega Z^r L^{r*}[v^r] = \int_\Omega v^r f^r(x, \vec{w})\, dx$. Thus,

$$-\int_\Omega (\phi^r - Z^r)\, L^{r*}[v^r]dx$$

$$= \int_\Omega v^r \bar{F}^r(x) - f^r(x, \vec{w})dx - \int_{\Sigma_3^r} \bar{g}^r \frac{\partial v}{\partial \nu} ds + \int_{\Sigma_2^r} B^r\, \bar{g}^r v^r ds.$$

By 3.8, $\phi^r - Z^r$ is the unique bounded measurable weak solution to $-L^r[u] = \bar{F}^r(x) - f^r(x, \vec{w})$ in Ω, $u = \bar{g}^r$ on $\Sigma_2^r \cup \Sigma_3^r$. Since $\bar{F}^r(x) \geqq f^r(x,$

$\vec{\phi}$) and $\phi^s \geqq w^s$ a.e. for $s = 1, \ldots, m$, it follows from Theorem 1 that $\phi^r - Z^r \geqq 0$ a.e; thus $\vec{Z} = A\vec{w} \leqq \vec{\phi}$. A similar argument with inequalities reversed shows that $A\vec{w} \geqq \vec{\chi}$, so A maps S into itself. Suppose now that $\vec{w}_1, \vec{w}_2 \in S$ with $\vec{w}_1 \geqq \vec{w}_2$, and let $\vec{Z}_i = A\vec{w}_i$ $i = 1, 2$. Then it follows as (3.8) that for each r, for admissible $\vec{v}$,

$$-\int_\Omega (Z_1^r - Z_2^r)\, L^{r*}[v^r] = \int_\Omega v^r[f^r(x, \vec{w}_1) - f^r(x, \vec{w}_2)]dx. \tag{3.9}$$

But by the monotonicity properties of $\vec{f}, f^r(x, \vec{w}_1) - f^r(s, \vec{w}_2) \geqq 0$ a.e. By (3.9), $Z_1^r - Z_2^r$ is the unique weak solution of $L^r[u] = f^r(x, \vec{w}_1) - f^r(x, \vec{w}_2)$ in Ω, $u = 0$ on $\Sigma_2^r \cup \Sigma_3^r$; so by Theorem 1, $Z_1^r - Z_2^r \geqq 0$ a.e. Thus $A\vec{w}_1 \geqq A\vec{w}_2$, so A is monotone on S.

Since A is a monotone map of the conical segment S into itself and the cone K defining S is minihedral, it follows by Theorem 2 that A has a fixed point $u \in F$, which is equivalent to the conclusion of the theorem.

REMARKS. We require that $\vec{f}(x, \vec{u})$ be quasimonotone. Counter examples show that even for elliptic equations, some conditions of that type are needed; see for example **[12]**. However, in the case of two elliptic or parabolic equations, it has been noted that if f^i is decreasing in u^j for $j \neq i$, then the method of upper and lower solutions can still be used; see **[3]**, **[10]**, **[11]**. In **[10]**, such a system is reduced to a quasimonotone system by replacing u^2 with $m - u^2$ for some constant m. In fact, such a transformation in always possible in that case, and in many others.

Suppose that $f^r(x, \vec{u})$ is increasing in u^r for each r (if (3.2) is satisfied, simply assume that we have added Mu^r to both sides of (3.1) as in the proof of Theorem 3. If we replace u^r by $u^r = -\tilde{u}^r$, we obtain for the *rth* equation

$$\begin{aligned} -L^r\tilde{u}^r &= -f^r(x, u^1, \ldots, u^{r-1}, -\tilde{u}^r, u^{r+1}, \ldots, u^m). \\ &= \tilde{f}^r(x, u^1, \ldots, u^{r-1}, \tilde{u}^r, u^{r+1}, \ldots, u^m), \end{aligned} \tag{3.10}$$

where (because of the two minus signs on the right of the first equation in (3.10) $\tilde{f}^r$ is increasing in $\tilde{u}^r$. Also, if f^r was increasing in u^s for $s \neq r$, then $\tilde{f}^r$ is decreasing in u^s, and vice versa. For $s \neq r$, $-L^s u^s = f^s(x, u^1, \ldots, u^{r-1}, -\tilde{u}^r, u^{r+1}, \ldots, u^m)$ so if f^s was decreasing in u^r, it is increasing in $\tilde{u}^r$, and vice-versa.

A method of accounting for the above considerations is as follows: for a given $\vec{f}(x, \vec{u})$ with f^r monotone in u^r, construct a matrix $\underline{P} = ((P_{ij}))$, $i, j = 1, \ldots, m$ with P_{ij} equal to -1, 0, or 1 as f^i is decreasing in u^j, independent of u^j, or increasing in u^j respectively. Replacing u^r with $\tilde{u}^r$ leads to a new nonlinearity $f(x, u)$ with matrix $\underline{P}$ obtained from $\underline{P}$ by multiplication on the left and right by a matrix with zeros off the diagonal, ones on the diagonal except for the *rth* row and column, and a negative one in the rth row and column. If by repeated multiplications by such matrices P can be transformed to a matrix with all entries nonnegative, then f can be transformed to a quasimonotone function.

EXAMPLES. If $m = 2$ and f^i is decreasing in u^i, $i \neq j$, then we have

$$\underline{P} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{3.11}$$

so f can be made quasimonotone. For the reduction to work, $\underline{P}$ must be symmetric, but that is not sufficient. If $m = 3$ and

$$\underline{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \tag{3.12}$$

then the reduction is possible; if

$$\underline{P} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.13}$$

then it is not.

A note on sub- and supersolutions: In the case (3.11), the appropriate requirement is that there exist (ϕ^1, χ^2), (χ^1, ϕ^2) with $\phi^i \geqq \chi^i$ and

$$\begin{aligned} -L^1[\phi^1] &\geqq f^i(x, \phi^1, \chi^2), \\ -L^2[\chi^2] &\leqq f^2(x, \phi^1, \chi^2), \\ -L^1[\chi^1] &\leqq f^1(x, \phi^1, \chi^2), \\ -L^2[\phi^2] &\geqq f^2(x, \phi^1, \chi^2); \end{aligned}$$

see **[3]**. **[10]**. **[11]**. (Such pairs are called upper-lower and lower-upper solutions.) If the transformation $\tilde{u}^2 = -u^2$ is performed on such a system, then $(\phi^1, -\chi^2)$ and $(\chi^1, -\phi^2)$ give upper and lower solutions in the sense of Theorem 3 for the system with f replaced by f. In the case of (3.12) a similar situation occurs; upper-upper-lower and lower-lower-upper solutions lead to upper and lower solutions when the system is transformed by $\tilde{u}^3 = -u^3$. A similar analysis is possible in more general situations.

We have the following result.

COROLLARY 4. *If the original system of equations* (3.1) *can be made quasimonotone by transformations as above, and appropriate "upper" and "lower" solutions can be found (and the remaining hypotheses of Theorem* 3 *holds, then there exists a solution to* (3.1) *in the sense of* (3.3).

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UNIVERSITY OF MIAMI, CORAL GABLES, FL 33124