

OPERATOR ALGEBRAS RELATED TO MEASURE PRESERVING TRANSFORMATIONS OF FINITE ORDER

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1. Introduction. The study of both multiplication and composition operators has a well established and productive history. In both cases questions of norms and spectra are, in varying degrees of complexity answered. This paper is concerned with the study of operators of the form $Tf = \sum_{i=0}^N a_i f \circ \tau^i$, acting on f in $L^2(X, \Sigma, m)$ where each a_i is a measurable function and τ is a measure preserving transformation on X . Special attention is paid to the case where $\tau^N(x) = x$ a.e. for some positive integer N , and τ is invertible. In this case we characterize the spectrum of T . The set of all such operators is shown to be a von Neumann algebra, and each such operator is shown to have a representation with the coefficient functions a_0, \dots, a_N in $L^\infty(X)$. The question of uniqueness of representation is answered. Finally a technique is developed enabling one to exhibit the coefficient functions concretely in terms of the operator itself.

2. Preliminaries and notation. Let (X, Σ, m) be a complete finite measure space. For each set Y in Σ , l_Y represents both the characteristic function of Y and the act of restricting a σ -algebra, measure, or function to Y . A Σ -measurable mapping τ from X onto X is said to be measure preserving if $m(\tau^{-1}(A)) = m(A)$ for each set A in Σ (equivalently $dm \circ \tau^{-1}/dm = 1$ a.e. dm). Throughout this article we assume that τ is both measure preserving and invertible. For any integer k , τ^k represents the k -fold composition of τ with itself, with the obvious interpretation if $k = 0$. All statements about equality, inclusion and disjointness are to be understood to hold up to a set of m -measure 0. If V is a vector space and k is a positive integer then $V^{(k)}$ is the k -fold direct sum of V with itself. In case V is a Hilbert space we endow $V^{(k)}$ with the inner product

$$\langle \langle v_i \rangle, \langle u_i \rangle \rangle = \sum_{i=0}^{k-1} \langle v_i, u_i \rangle.$$

For H a Hilbert space $B(H)$ is the ring of all bounded linear operators

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on H . Each operator T in $B(H^{(k)})$ is uniquely represented by a $k \times k$ matrix $[T_{ij}]$ with entries in $B(H)$. This representation is an algebra isomorphism between $B(H^{(k)})$ and $M_k(B(H))$, the ring of all $k \times k$ matrices with entries in $B(H)$.

If $\Phi = [\phi_{ij}]$ is a member of $M_k(L^\infty(X))$, then Φ acts as a bounded operator on $L_k^2(X) = [L^2(X)]^{(k)}$. One easily verifies that the usual operator-theoretic operations may be performed pointwise a.e., e.g.,

$$\begin{aligned}(\Phi^*)(x) &= [\bar{\phi}_{ji}(x)] = [\Phi(x)]^* \text{ a.e. and} \\ \Phi^{-1}(x) &= [\Phi(x)]^{-1} \text{ a.e.}\end{aligned}$$

Of course $[\Phi(x)]^{-1}$ may exist almost everywhere without Φ being invertible on $[L^2(x)]^{(k)}$. However it is easily verified that Φ is invertible if and only if the function $(\det \Phi)^{-1}$ is in $L^\infty(X)$.

We will write $\Phi \circ \tau$ for the matrix $[\phi_{ij} \circ \tau]$. Since composition distributes over all the usual algebraic operations we see that

$$\begin{aligned}[\Phi \circ \tau]^{-1} &= \Phi^{-1} \circ \tau \text{ and} \\ \det(\Phi \circ \tau) &= (\det \Phi) \circ \tau\end{aligned}$$

We will be concerned with the study of those operators T on $L^2(X)$ of the form

$$(Tf)(x) = \sum_{k=-M}^M a_k(x) f(\tau^k(x))$$

where M is a nonnegative integers and each a_k is a measurable function. Of course not all such operators are bounded. Let A be the algebra of all such bounded operators. Clearly A is a vector space. If $Tf = a \cdot f \circ \tau^k$ and $Sf = b \cdot f \circ \tau^j$ are in A , then $TSf = [a b \circ \tau^k] f \circ \tau^{k+j}$.

Thus A is an algebra. Also the identify operator I is a member of A . For any f and g in L^2

$$\begin{aligned}(Tf, g) &= \int (a)(f \circ \tau^k)(\bar{g}) \\ &= \int (a \circ \tau^{-k} \bar{g} \circ \tau^{-k}) f \\ &= (f, (\bar{a} \circ \tau^{-k})(\bar{g} \circ \tau^{-k})).\end{aligned}$$

That is, $T^*g = (\bar{a} \circ \tau^{-k})g \circ \tau^{-k}$, so that A is in fact a *-algebra.

We will on several occasions call into play certain properties of *-algebras. For conciseness we state here the relevant properties. Good references for this material are [2] and [4].

(i) A von Neumann algebra is a *-algebra of bounded operators on a Hilbert space which contains the identity operator and is closed in the weak operator topology.

(ii) [The von Neumann Double Commutant Theorem]. If B is a von Neumann algebra and B' is its commutant (i.e., the set of all operators commuting with all operators in B), then B' is von Neumann algebra and $B'' = B$.

(iii) [Kaplansky Density Theorem]. If B and B_0 are $*$ -algebras with $B \supseteq B_0$, then B_0 is weakly dense in B if and only if the unit ball of B_0 is weakly dense in the unit ball of B .

(iv) If B is a von Neumann algebra, then the unit ball of B is weakly and weakly sequentially compact.

(v) If T is in $B(H)$ and $\sigma(T)$ is its spectrum, and if B is any von Neumann algebra containing T , then $\sigma_B(T)$, the spectrum of T with respect to B , is $\sigma(T)$. Equivalently, T is invertible in $B(H)$ if and only if T is invertible in B .

Of special interest is the von Neumann algebra $M(X)$ of multiplication operators $M_\phi f = \phi f; f$ in $L^2(X)$ where ϕ is in $L^\infty(X)$. In this case it is well known that $M' = M$.

We may apply some of these concepts immediately to the algebra A above. Since A is a $*$ -algebra its weak closure is a von Neumann algebra. One sees that the weak closure of A is the smallest von Neumann algebra containing $M(X)$ and the composition operator $Cf = f \circ \tau; f$ in $L^2(X)$. Thus an operator B is in A' if and only if B is in $M' = M$ and $BC = CB$. Write $B = M_\phi$. Then for every f in L^2 $\phi f \circ \tau = (\phi \circ \tau)(f \circ \tau)$ hence $\phi = \phi \circ \tau$ a.e.

LEMMA 2.1. A is weakly dense in $B(L^2)$ if and only if τ is ergodic.

PROOF. From [5, p. 23] it follows that τ is ergodic if and only if the only bounded functions ϕ satisfying $\phi \circ \tau = \phi$ are the constant functions. From the preceding remarks we see that this is equivalent to having $A' = \{\lambda I: \lambda \text{ in } \mathbf{C}\}$. By the von Neumann double commutant Theorem this is equivalent to having A weakly dense in $B(L^2)$.

For the remainder of this paper we will concern ourselves only with τ of finite order (periodic). We assume then that N is the smallest positive integer for which $\tau^N(x) = x$ a.e. (and that such an N exists). The following lemma will be of considerable value in analysing the algebra A in this case.

LEMMA 2.2. Let $X_0 = \{x: \tau(x) = x\}$ and for $1 \leq k \leq N$ let $x_k = \{x: \tau^{k+1}(x) = x, \tau^j(x) \neq x \text{ for } 1 \leq j \leq k\}$ then

- (a) $\{X_0, \dots, X_N\}$ is a partition of X ; and
- (b) For $1 \leq k \leq N$, if $A \subseteq X_k$ and $A \neq \emptyset$, then A contains a subset $B \neq \emptyset$ such that $B, \tau(B), \dots, \tau^k(B)$ are mutually disjoint.

PROOF. Clearly (a) holds (of course some of the X_k 's may be vacuous). Let A be a nonempty subset of $X_k, 1 \leq k \leq N$. There is a nonempty subset

A'_0 of A such that $A'_0 \neq \tau(A'_0)$. For otherwise $\tau(x) = x$ a.e. on A . Let $A_0 = A'_0 - \tau(A'_0)$. If $A_0 = \emptyset$, then $A'_0 \subseteq \tau(A'_0)$ and since τ is measure preserving $A'_0 = \tau(A'_0)$. Thus $A_0 \neq \emptyset$ and $A_0 \cap \tau(A_0) = \emptyset$. If $k = 1$ we are finished. If $k > 1$, choose $A'_1 \subset A_0$ with $A'_1 \neq \emptyset$ and $A'_1 \cap \tau^2(A'_1) = \emptyset$. Once again, this is possible since $\tau^2(x) \neq x$ a.e. on X_k . Let $A_1 = A'_1 - \tau^2(A'_1) \neq \emptyset$. Then $A_1 \cap \tau^2(A_1) = \emptyset$. Also $A_1 \cap \tau(A_1) = \emptyset$ since $A_1 \subseteq A'_1 \subseteq A_0$. We continue this process until we arrive at a set $B \neq \emptyset$ with $B, \tau(B), \dots, \tau^k(B)$ mutually disjoint.

REMARKS. (a) Since τ is invertible, the sets $A, \dots, \tau^h(A)$ are disjoint in X_k if and only if $A, \tau^{-1}(A), \dots, \tau^{-h}(A)$ are disjoint in X_k . (b) Each X_k is τ invariant: $\tau^{-1}X_k = X_k$. It follows that $L^2(X_k)$ (as a subspace of $L^2(X)$) reduces the algebra A .

For convenience, define $J = \{k: 0 \leq k \leq N \text{ and } X_k \neq \emptyset\}$.

3. Representations of operators in A. In this section we show that each operator T in A has a representation $Tf = \sum a_i f \circ \tau^i$ where the a_i 's are in L^∞ . In general this representation is not unique, however we will show that there is a canonical representation. We also show that A is a von Neumann algebra and characterize the spectra of the operators in A . Since each $L^2(X_k)$ reduces the algebra A , and $L^2(X) = \sum_{k \in J} \oplus L^2(X_k)$ we many analyse the individual restrictions of A to each $L^2(X_k)$. Note also that if $Tf = \sum_{i=0}^{N-1} a_i f \circ \tau^i$, then on $L^2(X_k)$, $Tf = \sum_{i=0}^k a_{ki} f \circ \tau^i$ where $a_{ki} = \sum_{j=i(k+1)} a_j$. For each k in J let

$$D_k = \{A \subseteq X_k: m(A) > 0 \text{ and } A, \tau^{-1}(A), \dots, \tau^{-k}(A) \text{ are disjoint}\}.$$

THEOREM 3.1. *If $Tf(x) = \sum_{i=0}^{k-1} a_i(x) f(\tau^i(x))$ defines a bounded operator on $L^2(X_k)$ then each $\|a_i\|_\infty \leq \|T\|$ and $A|_{L^2(X_k)}$ is a von Neumann algebra. In particular the representation of T on $L^2(X_k)$ is unique and A is a von Neumann algebra on $L^2(X)$.*

PROOF. Let A be in D_k . Then

$$m(A) \|T\|^2 \geq \|T|_A\|^2 = \|\sum a_j|_{\tau^{-j}(A)}\|^2 = \sum_{j=0}^k \int_{\tau^{-j}(A)} |a_j|^2 dm.$$

Fix $j \leq k$ and note that A and $\tau^{-j}(A)$ have precisely the same iterates under τ .

Moreover, $m(\tau^{-j}(A)) = m(A)$. Thus we have that

$$\|T\|^2 \geq (1/m(A)) \int_A |a_j|^2 dm.$$

In fact this holds for any nonempty subset of A since this subset must also belong to D_k . It follows that $|a_j| \leq \|T\|$ a.e. on every subset of D_k . Let $S = \{x \text{ in } X_k: |a_j(x)| > \|T\|\}$. If $m(S) \neq \emptyset$ then by Lemma 2.2, S would

contain a subset S_0 of positive measure such that $\|T\| < |a_i(x)| \leq \|T\|$ a.e. on S_0 . Since this is impossible, $m(S) = 0$, i.e., $|a_j| \leq \|T\|$ a.e., $0 \leq j \leq k$.

In order to verify the assertion that $A|_{L^2(X_k)}$ is a von Neumann algebra, we need only show it is weakly closed. By the Kaplansky density Theorem we need only show that the unit ball of $A|_{L^2(X_k)}$ is weakly closed. Let $\{T_\lambda\}$ be a net in the unit ball of $A|_{L^2(X_k)}$ converging weakly to an operator T , and suppose T_λ has the representation $T_\lambda f = \sum_{j=0}^k a_{\lambda j} f \circ \tau^j$. The norm of the multiplication operator $M_{a_{\lambda j}}$ is precisely $\|a_{\lambda j}\|_\infty$ and so each $\|M_{a_{\lambda j}}\| \leq \|T\|$. We may also express T_λ as $T_\lambda = \sum_{j=0}^k M_{a_{\lambda j}} C^j$ (where $Cf = f \circ \tau$). Now, for each $j \leq k$, $\{M_{a_{\lambda j}}\}$ is a net in the unit ball of $M(X^k)$, and so, by passing through $k + 1$ subnets, we may assume that

$$\text{weak limit}_\lambda M_{a_{\lambda j}} = M_{a_j}, \quad 0 \leq j \leq k,$$

where each a_j is in $L^\infty(X_k)$. It follows immediately that (this subnet of) T_λ converges weakly to $\sum M_{a_j} C^j$, which is in $A|_{L^2(X_k)}$. Thus $A|_{L^2(X_k)}$ is weakly closed.

It is important to note that the above result does not imply uniqueness of representation of operators in A , but only on each direct summand $A|_{L^2(X_k)}$ k in J . One may easily construct examples with $N = 1$ (i.e., $\tau^2(x) = x$ a.e.) where uniqueness fails.

The verification of the next result is easy, straightforward, and omitted.

LEMMA 3.2. *Let T be in $A|_{L^2(X_k)}$ be given by $Tf = \sum_{i=0}^k a_i f \circ \tau^i$, and let E_T be the $(k + 1) \times (k + 1)$ $L^\infty(X_k)$ -valued matrix*

$$E_T = \begin{bmatrix} a_0 & a_1 & \cdots & a_k \\ a_k \tau & a_0 \tau & \cdots & a_{k-1} \tau \\ \vdots & & & \\ a_1 \circ \tau^k & a_2 \circ \tau^k & \cdots & a_0 \circ \tau^k \end{bmatrix}$$

(that is, each row is the cyclic permutation of the preceding row, composed with τ), Let $E_k = \{E_T : T \text{ in } A|_{L^2(X_k)}\}$. Then the map $T \rightarrow E_T$ is a *-algebra isomorphism. It follows that T is invertible in $A|_{L^2(X_k)}$ if and only if E_T is invertible in E_k .

THEOREM 3.3. *Let T be an operator in $A|_{L^2(L_k)}$ given by $Tf = \sum_{j=0}^k a_j f \circ \tau^j$ and let E_T be the matrix defined in the statement of Lemma 3.2. Then the spectrum of T in $B(L^2(X_k))$ is given by $\sigma_k(T) = \{\lambda \text{ in } \mathbb{C} : 0 \in \text{essential range} [\det(E_T - \lambda I)]\}$.*

PROOF. E_T is a member of the *-algebra E_k which is a subalgebra of $M_{k+1}(L^\infty(X_k))$, the ring of all $(k + 1) \times (k + 1)$ matrices with entries

from $L^\infty(X_k)$. Now $M_{k+1}(L^\infty(X_k))$ is a von Neumann algebra in $B([L^2(X_k)]^{(k+1)})$. We show first that E_k is in fact weakly closed, and thus a von Neumann algebra in $B([L^2(X_k)]^{(k+1)})$. Once again we invoke the Kaplansky density Theorem and the weak compactness of the unit ball in the von Neumann algebra $M(L^\infty(X))$ to see that a weakly convergent bounded net in E_k converges to a member of E_k . Now, since E_k is a von Neumann subalgebra of $M_{k+1}(L^\infty(X_k))$, E_T is invertible in E_k if and only if it is invertible in $M_{k+1}(L^\infty(X_k))$. But the latter holds if and only if $\det E_T$ is invertible in $L^\infty(X_k)$. Thus $E_T - \lambda I$ is invertible (in E_k) if and only if 0 is not in the essential range of $\det(E_T - \lambda I)$. By Lemma 3.2, the above characterization of $\sigma_k(T)$ is established.

It follows of course that for T in A acting on $L^2(X)$,

$$\sigma(T) = \bigcup_{k \in J} \sigma_k(T|_{L^2(X_k)}).$$

Answers to the following questions would be of interest.

1. For k in J what is $\|T|_{L^2(X_k)}\|$?
2. What is a reasonable characterization of the unitary and projection operators in A ?

Theorem 3.1 assures uniqueness of the representation of operators from A on each $L^2(X_k)$, although there is in general not uniqueness on all of $L^2(X)$ (Indeed, one easily verifies that uniqueness holds if and only if $X = X_N$). However Theorem 3.1 does allow us to establish a canonical representation for operators in A .

THEOREM 3.4. *Let T be in A . Then T has a unique representation*

$$Tf(x) = \sum_{i=0}^N b_i(x)f(\tau^i(x))$$

where each b_i is bounded and $b_i = 0$ a.e. on X_j if $i > j$.

PROOF. Let T have representation $T = \sum_{i=0}^N a_i f \circ \tau^i$. We know that on $L^2(X_k)$

$$Tf = \sum_{i=0}^k a_{ki} f \circ \tau^i$$

where

$$a_{ik} = \sum_{j=i(k+1)} a_j 1_{X_k}.$$

Further, this representation (on $L^2(X_k)$) is unique and each a_{ki} is bounded. Let f be in $L^2(X)$. Then

$$\begin{aligned} Tf &= \sum_{k \in J} T(f 1_{X_k}) \\ &= \sum_{k \in J} \left(\sum_{i=0}^k a_{ki} f \circ \tau^i 1_{X_k \circ \tau^i} \right) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{k \in J} \left(\sum_{i=0}^k a_{ki} f \circ \tau^i \right) 1_{X_k} \quad (X_k \text{ is } \tau\text{-invariant}) \\
 &= \sum_{i=0}^N \left(\sum_{\substack{k \in J \\ k \geq i}} a_{ki} 1_{X_k} \right) f \circ \tau^i,
 \end{aligned}$$

Let $b_i = \sum_{\substack{k \in J \\ k \geq i}} a_{ki} 1_{X_k}$.

(These are well defined since at least $N \in J$) Then each b_i is bounded, $Tf = \sum_{i=0}^N b_i f \circ \tau^i$ and if $i > j$, then $b_i = 0$ on X_j .

REMARK. Although the above representation is canonical, it is not faithful under multiplication. One needs to use the representations on each individual $L^2(X_k)$ in order to manipulate products usefully in terms of the functional coefficients of operators in A .

We conclude this article with the presentation of a method for explicitly obtaining the coefficients of a linear transformation T acting on measurable functions on X , and known to have the action $Tf = \sum_{i=0}^N a_i f \circ \tau^i$ where each a_i is measurable. Here there are no assumptions about boundedness. However, we may still restrict T to act only on measurable functions on an individual X_k and decompose T accordingly.

Fix k in J . For each real valued measurable function f on X_k let B_f be the $(k + 1) \times (k + 1)$ function entried matrix

$$B_f = \begin{bmatrix} f & f \circ \tau & \dots & f \circ \tau^k \\ f \circ \tau & f \circ \tau^2 & \dots & f \\ f \circ \tau^k & f & \dots & f \circ \tau^{k-1} \end{bmatrix}$$

(that is, the i, j entry is $f \circ \tau^{i+j}$). Then B_f is symmetric a.e. Let D_f be the set

$$D_f = \{x : (\det B_f)(x) \neq 0\}.$$

Noting that $(\det B_f) \circ \tau = \det B_{f \circ \tau} = \det(B_f \circ \tau)$ and that $B_{f \circ \tau}$ may be obtained from B_f by $k + 1$ row transpositions, we arrive at the fact that $(\det B_f) \circ \tau = (-1)^{k+1} \det B_f$. In particular, D_f is a τ -invariant set. Thus we may restrict attention to T acting on functions supported on D_f . On D_f , let the functions b_0, \dots, b_k be defined by $\text{col}[b_0, b_1, \dots, b_k] = B_f^{-1} \cdot \text{col}[Tf, T(f \circ \tau), \dots, T(f \circ \tau^k)]$ a.e. (where ‘‘col’’ indicates column vector), so that, in particular $Tf = \sum_{i=0}^k b_i f \circ \tau^i$ on D_f . Of course the b_i 's a priori depend on f . Let g be any measurable function on D_f . Then

$$\begin{aligned}
 \sum_{i=0}^k b_i g \circ \tau^i &= \text{col}[b_0, b_1, \dots, b_k] \cdot \text{col}[g, g \circ \tau, \dots, g \circ \tau^k] \text{ a.e.} \\
 &= (B_f^{-1} \text{col}[Tf, \dots, T(f \circ \tau^k)]) \cdot \text{col}[g, \dots, g \circ \tau^k] \\
 &\quad \text{col}[Tf, \dots, T(f \circ \tau^k)] \cdot (B_f^{-1} \text{col}[g, \dots, g \circ \tau^k]) (B_f^* = B_f \text{ a.e.}).
 \end{aligned}$$

Write $H = \text{col}[h_0, \dots, h_k] = B_f^{-1} \text{col}[g, \dots, g \circ \tau^k]$. Let R_i be the row vector in position i of B_f , i.e. $B_f = \text{col}[R_0, \dots, R_k]$. Then, $B_f \circ \tau = \text{col}[R_1, \dots, R_k, R_0]$ and consequently, $(B_f \circ \tau)(H \circ \tau) = \text{col}[R_1, \dots, R_k, R_0] \cdot \text{col}[h_0 \circ \tau, \dots, h_k \circ \tau] = \text{col}[g \circ \tau, \dots, g]$. Thus for $0 \leq i \leq k$, $R_i \cdot H \circ \tau = g \circ \tau^i$. But then $B_f(H \circ \tau) = \text{col}[g, \dots, g \circ \tau^k]$ and so $H \circ \tau = B_f^{-1} \text{col}[g, \dots, g \circ \tau^k] = H$; that is $h_i \circ \tau = h_i$ for $0 \leq i \leq k$. In particular multiplication by each h_i commutes with T , whence $\sum_{i=0}^k b_i g \circ \tau^i = \text{col}[Tf, \dots, T(f \circ \tau^k)] \cdot \text{col}[h_0, \dots, h_k] = \sum_{i=0}^k h_i T(f \circ \tau^i) = \sum_{i=0}^k T(h_i f \circ \tau^i) = T(\sum_{i=0}^k h_i f \circ \tau^i)$. However,

$$\begin{aligned} \sum_{i=0}^k h_i f \circ \tau^i &= \text{col}[h_0, \dots, h_k] \cdot \text{col}[f, \dots, f \circ \tau^k] \\ &= B_f^{-1} \text{col}[g, \dots, g \circ \tau^k] \cdot \text{col}[f, \dots, f \circ \tau^k] \\ &= \text{col}[g, \dots, g \circ \tau^k] \cdot B_f^{-1} \text{col}[f, \dots, f \circ \tau^k] \\ &= \text{col}[g, \dots, g \circ \tau^k] \cdot \text{col}[1, 0, 0, \dots, 0] \\ &= g. \end{aligned}$$

Thus $Tg = \sum_{i=0}^k b_i g \circ \tau^i$ a.e. on D_f for any measurable f and g .

Now take A to be a subset of X_k such that $A, \tau^{-1}(A), \dots, \tau^{-k}(A)$ are disjoint. Expansion by minors shows that $\det B_{1_A} = \sum_{i=0}^k \alpha_i 1_{\tau^{-i}(A)}$ where each $\alpha_i = \pm 1$. In particular $D_{1_A} = A \cup \tau^{-1}(A) \cdots \cup \tau^{-k}(A)$. Call this set X_A . For each A in D_k the above procedure yields functions a_{0A}, \dots, a_{kA} on X_A such that $Tg = \sum_{i=0}^k a_{iA} g \circ \tau^i$ on X_A for every measurable function g . Now, if A and B are in D_k , then either $m[A \cap B] = 0$ or $A \cap B$ is in D_k . It follows that on $A \cap B$, $a_{iA} = a_{iB}$ ($0 \leq i \leq k$). But it then follows from [1, Lemma 3.1] that for each i there is a measurable function a_i such that $a_{iA} = a_i$ a.e. on A for every A in D_k . Let g be any measurable function, and let $S = \{x \in X_k: (Tg)(x) \neq \sum_{i=0}^k a_i(x)g(\tau^i(x))\}$. Then if $m(S) \neq 0$ S contains a subset in D_k . As this is impossible, $Tg = \sum a_i g \circ \tau^i$ a.e. on X_k .

It follows from [3, p. 70] that there is a set A in D_k such that $X_k = \bigcup_{i=0}^k \tau^{-i}(A)$. Thus the coefficient functions are determined explicitly on X_k :

$$\text{col}[a_0, \dots, a_k] = B_{1_A}^{-1} \cdot \text{col}[T(1_A), \dots, T(1_{\tau^{-k}(A)})].$$

Further, the calculations are facilitated by the easily derived fact that $B_{1_A}^{-1} = B_{1_A}$.

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