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ON ROTA'S MODELS FOR LINEAR OPERATORS

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ABSTRACT. In this note Rota type models for linear operators in terms of weighted shifts are obtained.

The role of shift operators in the structure theory of Hilbert space operators is well known. It was first pointed out by Rota [6] that they can be regarded as 'universal' operators, see also de Branges and Rovnyak [1] and Foias [2]. The object of this note is to obtain Rota type theorems for weighted shift operators and to generalize some of the known results in this direction. For a beautiful and almost exhaustive account of the literature on weighted shifts through 1973, we refer to Shields [7]. However, the aspect of their study which forms the subject of this note has not been touched there. A model for quasinilpotent operators in terms of weighted shifts has been obtained by Foias, and Pearcy [3].

Let H be an infinite-dimensional, separable complex Hilbert space with an orthonormal basis $\{e_n\}_{n=0}^{\infty}$. We shall denote by $B(H)$ the algebra of all bounded operators on H. Let $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ be a bounded sequence of complex numbers. The operator S_α on H defined by

$$S_\alpha e_0 = 0 \text{ and } S_\alpha e_n = \alpha_n e_{n-1} \text{ for } n \geqq 1$$

is called the backward weighted shift with the weight sequence α. If $\alpha_n = 1$ for all n, then S_α is simply called the backward shift which we shall denote by S. We may and shall take α to be a bounded sequence of positive real numbers [4], [5]. If $\ell^2(H)$ denotes the Hilbert space of all square-summable sequences $x = \{x_0, x_1, \ldots, x_n, \ldots\}$ of vectors x_n's in H, then S_α on $\ell^2(H)$ appears as

$$S_\alpha(x) = \{\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_{n+1} x_{n+1}, \ldots\}.$$

We write $\beta_n = \alpha_1\alpha_2 \ldots \alpha_n$ for $n \geqq 1$ and $\beta_0 = 1$. We shall denote by $r(T)$ the spectral radius of an operator T in $B(H)$. A subspace M of H is

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invariant under an operator T if $TM \subseteqq M$. By a part of T we shall mean the restriction $T|_M$ of T to its invariant subspace M. T is said to be power bounded if there exists a real number $\delta > 0$ such that $\|T^n\| \leqq \delta$ for all $n \geqq 0$.

THEOREM 1. *If T is in $B(H)$ and $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ is a bounded sequence of positive real numbers such that*

$$(1) \qquad \sum_{n=0}^{\infty} \beta_n^{-2} \|T^n\|^2 < \infty,$$

then T is similar to a part of S_α on $\ell^2(H)$.

PROOF. Define $A: H \to \ell^2(H)$ by

$$Ax = \{\beta_0^{-1}x, \beta_1^{-1}Tx, \beta_2^{-1}T^2x, \ldots, \beta_n^{-1}T^n x, \ldots\}.$$

It is easy to see that A is one-to-one and linear. A is also bounded, for it follows from (1) that

$$\begin{aligned} \|Ax\|^2 &= \sum_{n=0}^{\infty} \|\beta_n^{-1}T^n x\|^2 \\ &\leqq \Big(\sum_{n=0}^{\infty} \beta_n^{-2}\|T^n\|^2\Big) \|x\|^2 \\ &< \infty. \end{aligned}$$

Since

$$\|Ax\|^2 = \sum_{n=0}^{\infty} \|\beta_n^{-1}T^n x\|^2 \geqq \|x\|^2$$

for each x in H, A is bounded below, and hence its range M is a closed subspace of $\ell^2(H)$.
Now

$$\begin{aligned} (S_\alpha A)(x) &= S_\alpha\{\beta_0^{-1}x, \beta_1^{-1}Tx, \beta_2^{-1}T^2x, \ldots, \beta_n^{-1}T^n x, \ldots\} \\ &= \{\alpha_1(\beta_1^{-1}Tx), \alpha_2(\beta_2^{-1}T^2x), \alpha_3(\beta_3^{-1}T^3x), \ldots, \alpha_{n+1}(\beta_{n+1}^{-1}T^{n+1}x), \ldots\} \\ &= \{\beta_0^{-1}(Tx), \beta_1^{-1}T(Tx), \beta_2^{-1}T^2(Tx), \ldots, \beta_n^{-1}T^n(Tx), \ldots\} \\ &= A(Tx) \\ &= (AT)x \end{aligned}$$

for each x in H. Thus $S_\alpha A = AT$. This implies that M is an invariant subspace of S_α and T is similar to $S_\alpha|_M$.

COROLLARY 1.1. (ROTA'S THEOREM). *If an operator $T \in B(H)$ has spectral radius $r(T)$ less than 1, then T is similar to a part of S.*

PROOF. In this case it suffices to observe that

$$\lim_{n\to\infty} (\|T^n\|^2)^{1/n} = r(T)^2 < 1,$$

and hence (1) follows for $\alpha_n = 1$ for all n.

COROLLARY 1.2. *If T is a power bounded operator in $B(H)$, then T is similar to a part of S_α with $\sum_{n=0}^{\infty} \beta_n^{-2} < \infty$.*

PROOF. Let $\delta > 0$ be such that $\|T^n\| \leqq \delta$ for all $n \geqq 0$.
Then

$$\sum_{n=0}^{\infty} \beta_n^{-2} \|T^n\|^2 \leqq \delta^2 (\sum_{n=0}^{\infty} \beta_n^{-2}) < \infty.$$

THEOREM 2. *For every T in $B(H)$, there exists an S_α with $r(S_\alpha) \leqq r(T)$ such that T is similar to a part of S_α.*

PROOF. If T is not nilpotent, then choose

$$\alpha_n = (n+1)\|T^n\|/(n\|T^{n-1}\|)$$

for all $n \geqq 1$. Clearly $\beta_n^{-2}\|T^n\|^2 = (n+1)^{-2}$ for all n, and therefore Theorem 1 shows that T is similar to a part of S_α. Moreover,

$$\|S_\alpha^n\| = \sup(\alpha_k \alpha_{k+1} \cdots \alpha_{k+n-1}) \leqq (n+1)\|T^n\|,$$

which implies that $r(S_\alpha) \leqq r(T)$. If T is nilpotent, let $N = \inf\{n\colon T^n = 0\}$. Choose $\alpha_k = 1$ if $k < N$ and $\alpha_k = 0$ if $k \geqq N$. Then $r(S_\alpha) = 0 = r(T)$. Define $A_1\colon H \to \ell^2(H)$ by $A_1 x = \{x, Tx, \ldots, T^{N-1}x, 0, 0, \ldots\}$. Then A_1 is one-one and bounded and has closed range M_1. Also

$$\begin{aligned}(S_\alpha A_1)(x) &= S_\alpha\{x, Tx, \ldots, T^{N-1}x, 0, 0, \ldots\}\\ &= \{Tx, T(Tx), \ldots, T^{N-1}(Tx), 0, 0, \ldots\}\\ &= (A_1 T)x\end{aligned}$$

for each x in H. So M_1 is invariant under S_α and T is similar to $S_\alpha|_{M_1}$.

COROLLARY 2.1. (FORIAŞ-PEARCY) [3]. *Let T be a quasinilpotent operator in $B(H)$. Then there exists a quasinilpotent S_α on $\ell^2(H)$ such that T is similar to a part of S_α.*

REFERENCES

1. L. De Branges and J. Rovnyak, *Perturbation Theory and its Applications in Quantum Mechanics* Calvin H. Wilcox, ed., Wiley, New York, 1966.

2. C. Foiaş, *A remark on the universal model for contractions of G.C. Rota*, Comm. Acad. R.P. Române **13** (1963), 349–352.

3. ———, and C. Pearcy, *A model for quasinilpotent operators*, Michigan Math. J. **21** (1974), 399–404.

4. P.R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, NJ, 1967.

5. ———, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
6. G.C. Rota, *On models for linear operators*, Comm. Pure Appl. Math. **13** (1960), 469–472.
7. A.L. Shields, *Weighted shift operators and analytic function theory*, Survey No. 13, *Topics in Operator Theory*, A.M.S., Providence, R.I, 1974.

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