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ON THEOREMS OF B.H. NEUMANN CONCERNING FC- GROUPS, II

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ABSTRACT. B.H. Neumann characterized central-by-finite and finite-by-abelian groups. In this paper, we generalize these results by characterizing those members of a large class of groups that are central-by $(<m)$ or $(<m)$-by-abelian. Here we mean that a group is $(<m)$ if its cardinality is less than m for some infinite cardinal m.

1. Introduction. B.H. Neumann [8] characterized central-by-finite and finite-by-abelian groups. A group G is central-by-finite if and only if each subgroup has only finitely many conjugates or, equivalently, U/U_G is finite for each subgroup U of G. Here U_G denotes the core of U; that is, the largest normal subgroup of G contained in U. We use U^G to denote the normal closure of U in G; then G is finite-by-abelian if and only if $|U^G: U|$ is finite for each subgroup U of G.

Eremin [3] indicated that it is only necessary to consider the abelian subgroups of G in the first of these results. A corrected form of Eremin's proof can be found in the book by Gorčakov [7].

In [13], one of us considered the extension of these results to FC-groups in which $|G/Z(G)| < m$ or $|G'| < m$, where m denotes an infinite cardinal. Here we go further and consider the extent to which the FC-condition can be relaxed.

To describe our results, we define the following classes of groups. If m is an infinite cardinal, the class mC consists of those groups G in which $|G: C_G(x)| < m$ for each $x \in G$. Z_m is the subclass of mC consisting of those groups G in which $|G: C_G(S)| < m$ for each subset $S \subseteqq G$ such that $|S| < m$. See [4] and [5] for theorems concerning the abelian subgroup structure of mC-groups.

In the case $m = \aleph_0$, both these classes coincide with the class of FC-groups and so either class may be considered as a generalization of the class of FC-groups. As was shown in [13], the condition on Z_m-groups makcs these groups much easier to work with, and here we are able to prove the following results.

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THEOREM A. *Let $G \in Z_m$. Then the following are equivalent*:

(i) $|G/Z| < m$,

(ii) $|U/U_G| < m$ *for each* $U \leqq G$,

(iii) $|\mathrm{Cl}(U)| < m$ *for each* $U \leqq G$,

(iv) $|A/A_G| < m$ *for each abelian subgroup* A *of* G, *and*

(v) $|\mathrm{Cl}(A)| < m$ *for each abelian subgroup* A *of* G.

THEOREM B. *Let $G \in Z_m$. Then $|G'| < m$ if and only if $|A^G: A| < m$ for each abelian subgroup A of G.*

In Theorem A, it is reasonable to restrict our attention to Z_m-groups because the condition $|G/Z| < m$ clearly implies that $G \in Z_m$. In the case of groups in which $|G'| < m$, this restriction seems somewhat unsatisfactory. In **[13]**, the most interesting result showed that for *FC*-groups (without a Z_m-condition), $|G'| < m$ if and only if $|U_G: U| < m$ for each $U \leqq G$. We can improve this result when m is uncountable in the following form.

THEOREM C. (GCH). *Let n and m be infinite cardinals such that $n < m$ and let G be an nC-group. Then $|G'| < m$ if and only if $|U^G: U| < m$ for each $U \leqq G$.*

Since the condition $|G'| < m$ only implies that G is an mC-group, we might hope to prove the result for mC-groups and change the condition $n < m$ to $n \leqq m$. However, we construct an example that proves the following.

THEOREM D. ($2^m = m^+$). *There is a group $G \in m^+ C$ such that $|G'| = m^+$ but $|U^G: U| \leqq m$ for each $U \leqq G$.*

The proofs given here are usually rather different from and a considerable improvement upon those given in **[13]**. In certain results, the case in which $m = \aleph_0$ has to be treated separately from the uncountable case and we do not always include this special case. Sometimes the countable case is purely a matter of detail and would only divert the reader from the main arguments. In one proof (Lemma 5.3), the countable case is considerably more complicated; it can be found in **[13]**.

2. Notation. Let S and T be sets, $S < T$ always means strict inclusion. The cardinality of S is denoted by $|S|$. If m is an infinite cardinal, m^+ is the first cardinal greater than m and $\exp m = 2^m$. The *cofinality* of an ordinal γ (respectively, cardinal m) is the first cardinal n such that γ (respectively, m) is the sum of n smaller ordinals (respectively, cardinals); we denote this relationship by $n = \mathrm{cf}(\gamma)$ (respectively, $\mathrm{cf}(m)$); m is a *regular* cardinal if $\mathrm{cf}(m) = m$ and is *singular* otherwise. A cardinal m is a *strong* *limit* if $\exp n < m$ for every $n < m$. The first infinite ordinal is denoted by ω. Additional terminology can be found in [11].

Let G be a group and S and T be subsets. $C_G(S)$ denotes the centralizer of S in G. We denote the center of G by $Z(G) = C_G(G)$. If $S = \{1\}$, the trivial subgroup, we often denote S by 1. If $x, y \in G$, the commutator of x and y is $[x, y] = x^{-1}y^{-1}xy$. We denote by $[S, T]$ the set $\{[s, t]: s \in S, t \in T\}$. The derived (commutator) group of G is denoted by G'. The conjugate class of S in G is $\mathrm{Cl}(S) = \mathrm{Cl}_G(S) = \{g^{-1}Sg: g \in G\}$. If G_i, $i \in I$, is a family of groups, their direct sum (restricted direct product) is denoted by $\mathrm{Dr}_{i \in I}G_i$. If $g = \pi_{i \in I}g_i$ is an element of $\mathrm{Dr}_{i \in I}G_i$, then $\mathrm{supp}(a_i)$ denotes the set of coordinates $i \in I$ such that a_i is nontrivial; that is, $\mathrm{supp}(a_i) = \{i \in I: a_i \neq 1\}$. Additional terminology can be found in [9] and [10].

3. Preliminary Results. We shall see that problems may arise with the class of mC-groups when m is singular. The following result shows that these difficulties do not arise with Z_m-groups.

LEMMA 3.1. *If $G \in Z_m$, then $G \in nC$ for some regular cardinal $n \leqq m$.*

PROOF. Let $G \in Z_m$; then certainly $G \in mC$. Let $n = \sup\{|\mathrm{Cl}(x)|: x \in G\} \leqq m$. If $n = m$ and n is singular, then $n = m = \sum_{\alpha < \mathrm{cf} m} m_\alpha$, with $m_\alpha < m$. By the definition of n, there are elements $x_\alpha \in G$ with $|\mathrm{Cl}(x_\alpha)| \geqq m_\alpha$. Let $S = \{x_\alpha: \alpha < \mathrm{cf}(m)\}$: then $|G: C_G(S)| \geqq \sum m_\alpha = m$ although $|S| = \mathrm{cf}(m) < m$. This is contrary to $G \in Z_m$ so, if $n = m$, n must be regular. If $n < m$, then $G \in n^+C$, n^+ is regular and $n^+ \leqq m$. This proves the lemma.

LEMMA 3.2. *Let m be uncountable and let either $n < m$ or $n = m$ be regular. If $G \in nC$ and $S \subseteqq G$ such that $|S| < m$, then $|\langle S^G \rangle| < m$.*

PROOF. $\langle S^G \rangle = \bigcup\{\langle x^G \rangle: x \in S\}$. Now $|\langle x^G \rangle| = n_x$, where $n_x < n$ if n is uncountable and $n_x \leqq n$ if $n = \aleph_0$, and so $|\langle S^G \rangle| \leqq \sum_{x \in S} n_x$. If $n < m$, then $\sum_{x \in S} n_x \leqq n|S| < m$. If $n = m$ is uncountable and regular, then $n_x < n$ for all x and so, since $|S| < n$, $\sum_{x \in S} n_x < n = m$.

COROLLARY 3.3. *Let m be a strong limit cardinal. Then $G \in Z_m$ if and only if $G \in nC$ for some regular $n \leqq m$.*

PROOF. One direction is just Lemma 3.1. So we assume that $G \in nC$ for some regular $n \leqq m$ and let $S \subseteqq G$ with $|S| < m$. For m uncountable, $|\langle S^G \rangle| < m$ by Lemma 3.2. Hence $|G: C_G(S)| \leqq |G: C_G(\langle S^G \rangle)| \leqq |\mathrm{Aut}\,\langle S^G \rangle| \leqq \exp |\langle S^G \rangle| < m$. If $m = \aleph_0$ then, as remarked earlier, $G \in Z_m$ if and only if $G \in mC$.

At this stage, we note the following elementary result (see Lemma 2.1 in [13]).

LEMMA 3.4. *Z_m and mC are both QS-closed classes of groups.*

We now need to consider the elementary relationships between the different cardinalities $|U^G\colon U|$, $|U/U_G|$, and $|\mathrm{Cl}(U)|$.

LEMMA 3.5. *Let m be an infinite cardinal and let either $n < m$ or $n = m$ be regular. Let U be a subgroup of the nC-group G.*

(i) If $|U/U_G| < m$, *then* $|U^G\colon U| < m$.

(ii) If $|\mathrm{Cl}(U)| < m$, *then* $|U^G\colon U| < m$.

PROOF. (m uncountable) (i) Let T be a transversal to U_G in U; then $U = \bigcup_{t \in T} U_G t$ and $|T| < m$. By Lemma 3.2, $|\langle T^G \rangle| < m$ and clearly $U^G = U\langle T^G \rangle$; hence $|U^G\colon U| < m$.

(ii) Let T be a transversal to $N_G(U)$ in G so that $|T| < m$. For each $t \in T$, $U^t \leqq U\langle t^G \rangle$ and so $U^G \leqq U\langle T^G \rangle$. By Lemma 3.2, $|\langle T^G \rangle| < m$ and hence $|U^G\colon U| < m$.

The following example shows that if n is singular then we must have $n < m$ in this result and also in Lemma 3.2.

EXAMPLE 3.6. Let A_n and B_n be elementary abelian 3-groups of cardinality $\aleph_n$ and let $\varphi_n\colon A_n \to B_n$ be an isomorphism.

Let $X_n = A_{n+1} \times B_n$ and $X = \mathrm{Dr}_{n=0}^{\infty} X_n$. Let $Y = \mathrm{Dr}_{n=1}^{\infty} \langle y_n \rangle$ be an elementary abelian 2-group and form the split extension G of X by Y such that $[x, y_n] = 1$ if $x \in X_m$ and $m \neq n$ and $[x, y_n] = x$ if $x \in X_n$. Then $|G/X| = \aleph_0$, $G' = X$, $Z(G) = 1$, and $G \in \aleph_\omega C$.

Now $|Y/Y_G| = \aleph_0 < \aleph_\omega$ but $Y^G = G$ and so $|Y^G\colon Y| = \aleph_\omega$ showing that Lemma 3.5(i) is false for mC-groups if m is singular. This also shows that Lemma 3.2 is false if the condition $n < m$ is weakened to $n = m$ when m is singular.

Let $U_n = \langle aa^{\phi_n}\colon a \in A_n \rangle$ and $U = \mathrm{Dr}_{n=1}^{\infty} U_n$. Then $N_G(U) = X$ and so $|\mathrm{Cl}(U)| = \aleph_0 < \aleph_\omega$, but $U^G = X$ and, since $X = U \times \mathrm{Dr}_{n=0}^{\infty} B_n$, we have $|X\colon U| = \aleph_\omega$. Thus Lemma 3.5(ii) is also false for mC-groups with m singular.

It was observed in [13] that the converses of Lemma 3.5(i) and (ii) are false even for $Z_m \cap \aleph_0 C$-groups.

LEMMA 3.7. *Let U be a subgroup of the Z_m-group G. Then $|\mathrm{Cl}(U)| < m$ if and only if $|U/U_G| < m$.*

PROOF. (m uncountable). If $|U/U_G| < m$ then, since $G/U_G \in Z_m$ (Lemma 3.4), $|G\colon C_G(U/U_G)| < m$. Hence $|G\colon N_G(U)| < m$.

If $|\mathrm{Cl}(U)| < m$, let T be a transversal to $N_G(U)$ in G so that $|T| < m$. By Lemmas 3.1 and 3.2, $|\langle T^G \rangle| < m$. Since $G \in Z_m$, $|G\colon C_G(\langle T^G \rangle)| < m$ and so $|U\colon C_U(\langle T^G \rangle)| < m$. But $C_U(\langle T^G \rangle)$ is normalized by $\langle T^G \rangle$ and $N_G(U)$ and so $C_U(\langle T^G \rangle) \lhd G$. Thus $U_G \geqq C_U(\langle T^G \rangle)$ and hence $|U/U_G| < m$.

LEMMA 3.8. *Let m be an infinite cardinal and let either $n < m$ or $n = m$*

be regular. If U is a subgroup of the nC-group G with $|G: U| < m$ and $|U'| < m$, then $|G'| < m$.

PROOF. (m uncountable). By Lemma 3.2, there is a normal subgroup N such that $NU = G$ and $|N| < m$, $G' \leqq U'N$ and so $|G'| \leqq |U'|\,|N| < m$.

LEMMA 3.9. *Let U be subgroup of the Z_m-group G such that $|G:U| < m$ and $|U/Z(U)| < m$. Then $|G/Z(G)| < m$.*

PROOF. (m uncountable). By Lemmas 3.1 and 3.2, there is a normal subgroup N of G such that $NZ(U) = G$ and $|N| < m$. Since $G \in Z_m$, $|G: C_G(N)| < m$ and hence $|G: Z(U) \cap C_G(N)| < m$. But $Z(U) \cap C_G(N) \leqq Z(G)$.

We saw in [13] (following Lemma 2.3) that Lemma 3.9 is false without the Z_m-condition even if G is an FC-group. In Lemma 3.8 we need n to be regular if $n = m$ as is shown by Example 3.6 if we take $U = X$.

4. Existence of N-subgroups. Neumann's original proof [8] and the generalizations in [13] both depend on obtaining subgroups of a rather special type and then discussing these groups in some detail. An N-group of cardinality m is a group generated by elements a_α, b_α, $\alpha < \rho$, where ρ is the least ordinal of cardinality m, satisfying the conditions

$$(1) \qquad [a_\alpha, a_\beta] = [b_\alpha, b_\beta] = [a_\alpha, b_\beta] = 1, \text{ if } \alpha \neq \beta,$$

and

$$[a_\alpha, b_\alpha] = c_\alpha \neq 1.$$

It is a consequence of these conditions that

$$(2) \qquad a_\alpha a_\beta^{-1} \text{ is noncentral if } \alpha \neq \beta.$$

The N-group $\langle a_\alpha, b_\alpha; \alpha < \rho\rangle$ of cardinality m is called an N_1-group if it satisfies the additional condition

$$(3) \qquad c_\alpha \neq c_\beta, \text{ whenever } \alpha \neq \beta.$$

THEOREM 4.1. *If G is a Z_m-group with $|G'| \geqq m$, then G contains an N_1-subgroup of cardinality m.*

PROOF. Suppose that we have defined the elements a_α, $b_\beta \in G$ for all $\beta < \alpha$ (some $\alpha < \rho$) such that $[a_\beta, a_\gamma] = [b_\beta, b_\gamma] = [a_\beta, b_\gamma] = 1$ if $\beta \neq \gamma$, and $[a_\beta, b_\beta] = c_\beta \notin \langle a_\gamma, b_\gamma; \gamma < \beta\rangle$. Let $S_\alpha = \{a_\beta, b_\beta; \beta < \alpha\}$ and let $C_\alpha = C_G(S_\alpha)$. Since $G \in Z_m$ and $|S_\alpha| < m$, $|G; C_\alpha| < m$. By Lemmas 3.1 and 3.8. $|C_\alpha'| \geqq m$. If m is uncountable, $|\langle S_\alpha\rangle| < m$; if $m = \aleph_0$, then G is an FC-group, G' is periodic, and $|\langle S_\alpha\rangle \cap G'|$ is finite. In both cases, $|\langle S_\alpha\rangle \cap G'| < m$ and so C_α contains elements a_α, b_α such that $[a_\alpha, b_\alpha] = c_\alpha \notin \langle S_\alpha\rangle$. Thus we can construct the N_1-subgroup $\langle a_\alpha, b_\alpha: \alpha < \rho\rangle$.

To obtain the more interesting case required for Theorem C, we use the result proved in **[12]** in the following form. (The result also follows from the Marczewski-Erdös-Rado Theorem on Δ-systems **[2]**).

THEOREM 4.2. (GCH). *Let n and m be infinite cardinals with $n < m$. Let S be a family of sets such that $|S| = m$ and $|A| < n$ for each $A \in S$. Then there is a subfamily F of S such that $|F| = m$ and $|\bigcup_{A \neq B \in F}(A \cap B)| < m$.*

COROLLARY 4.3. (GCH). *Let n and m be infinite cardinals with $n < m$ and let $G \in nC$.* If *$S \subseteqq G$ such that $\|[S, G]\| = m$, then there is a subset T of S such that $\|[T, G]\| = m$, but $\|[T, T]\| < m$.*

PROOF. For each $s \in S$, $|\{[s, g]: g \in G\}| < n$ and so we have m subsets $X_s = \{[s, g]: g \in G\}$ of cardinality less than m. (This is clear if m is uncountable; if $m = \aleph_0$ then G is an FC-group, G' is periodic and $\{[s, g]: s \in S, g \in G\}$ is infinite.) There is a subset S_1 of S such that $|S_1| = m$ and there are distinct elements $x_s \in X_s$, $s \in S_1$. The commutator $[s, t]$ is in the intersection $X_s \cap X_t$. By Theorem 4.2, there is a subset T of S_1, such that $|T| = m$ and hence $\|[T, G]\| = m$ but $|\bigcup_{s \neq t \in T}(X_s \cap X_t)| < m$. Hence $|\{[t_1, t_2]: t_1, t_2 \in T\}| < m$ and, if $m = \aleph_0$ again is considered separately, it follows that $\|[T, T]\| < m$.

THEOREM 4.4. (GCH), *Let n and m be infinite cardinals with $n < m$. If $G \in nC$ and $|G'| \geqq m$, then there is a normal subgroup F of G with $|F| < m$ such that G/F contains an N_1-subgroup of cardinality m.*

PROOF. If $m = \aleph_0$, then $G \in Z_m$ and Theorem 4.1 gives the required result by taking $F = 1$.

Suppose m is uncountable. By Theorem 4.3, there is a subset $T \subseteqq G$ such that $\|[T, G]\| = m$ and $\|[T, T]\| < m$. By Lemma 3.2, $|[T, T]^G| < m$. Factoring out $F_1 = [T, T]^G$, we have an abelian subgroup $A = \langle T \rangle F_1/F_1$ such that $|[A, G]| = m$ (writing F in place of G/F_1).

We can choose elements a_i, $i \in I$, of A and b_i $i \in I$, of G such that $|I| = m$ and $[a_i, b_i] = c_i$ with $c_i \neq c_j$ whenever $i \neq j$. By Theorem 4.3, there is a subset $B = \{b_i; i \in I_1\}$ of $\{b_i: i \in I\}$ such that $|B| = m$ and $\|[B, B]\| < m$. Again $|[B, B]^G| < m$, and factoring out $F_2 = [B, B]^G$, we have (taking subsets if necessary) two abelian subgroups $A_1 = \langle a_i; i \in I_1 \rangle$ and $B_1 = \langle b_i; i \in I_1 \rangle$ with the commutators $c_i = [a_i, b_i]$, $i \in I_1$, being distinct.

Now let $X_i = [a_i, B_1] \cup [A_1, b_i]$; then $|X_i| < n$ and $c_i \in X_i$. There is a subset J of I_1 such that $|J| = m$ and $|\bigcup_{j \neq k \in J}(X_j \cap X_k)| < m$. Factoring out $F_3 = \langle \bigcup_{j \neq k \in J}(X_j \cap X_k) \rangle^G$, we have (taking subsets if necessary) elements a_j, b_j, $j \in J$, such that $[a_j, a_k] = [b_j, b_k] = [a_j, b_k] = 1$ if $j \neq k$ and $[a_j, b_j] = c_j$ and $c_j \neq c_k$, whenever $j \neq k$.

Well-ordering the set J, we can relabel the elements a_j, b_j, $j \in J$, as a_α, b_α, $\alpha < \rho$.

5. Abelian subgroups of N_1-groups. We have shown in §4 that in Z_m-groups with $|G'| \geqq m$ there is a large N_1-group. Theorems B and C will follow if we can show that an N_1-group of cardinality m contains an abelian subgroup X such that $|X^G; X| = m$. Our proof of this fact for the uncountable case is considerably shorter than that given in [13], depending on the following result about abelian groups.

THEOREM 5.1. *Let A_i $i \in I$, be an uncountable family of finitely generated subgroups of an abelian group A with $|I| = m$. Then there is a subgroup $B \leqq A$ with $|B| < m$ and a subset J of I such that $|J| = m$ and $\langle A_i B/B$: $i \in J\rangle = \mathrm{Dr}_{i\in J}(A_i B/B)$.*

PROOF. Embed A in a divisible group $\mathrm{Dr}_{\lambda\in\Lambda} D_\lambda$ where each D_λ is locally cyclic. For each $i \in I$, let $X_i = \{\lambda \in \Lambda\colon \pi_\lambda(A_i) \neq 1\} = \bigcup_{\alpha\in A_i} \mathrm{supp}(a)$; then each X_i is finite. By Theorem 4.2, there is a subset $J \subseteqq I$ such that $|J| = m$ and $X = \bigcup_{i\neq j\in J}(X_i \cup X_j)$ has cardinality less than m. Let $B = A \cap \mathrm{Dr}_{\lambda\in X} D_\lambda$. If $a = \prod_{i\in J} a_i \in B$ with $a_i \in A_i$, then $a_i = a \prod_{j\neq i} a_j^{-1}$ and so $\mathrm{supp}(a_i) \subseteqq X_i \cap (X \cup \bigcup_{j\neq i} X_j) = X$. Thus $a_i \in B$ and so $\langle A_i B/B\colon i \in J\rangle = \mathrm{Dr}_{i\in J}(A_i B/B)$.

THEOREM 5.2. *Let G be an N_1-group of cardinality m.* [If $m = \aleph_0$ *assume further that G is an FC-group.*] *Then G contains an abelian subgroup X such that $|X^G\colon X| = m$.*

PROOF. (m uncountable). We may use the argument of the first two paragraphs of the proof of [13]. Theorem 3.2 to assume that G is nilpotent of class two.

Let $G_\alpha = \langle a_\alpha, b_\alpha\rangle$; then G_α is a finitely generated nilpotent group and so $C_\alpha = Z(G_\alpha)$ is finitely generated. The groups C_α are finitely generated subgroups of the abelian group $Z(G)$. By Theorem 5.1, there is a subgroup $H \leqq Z(G)$ with $|H| < m$ and a subset I of $\{\alpha\colon \alpha < \rho\}$ such that $|I| = m$ and $\langle C_\alpha H/H\colon \alpha \in I\rangle = \mathrm{Dr}_{\alpha\in I} C_\alpha H/H$. Now $c_\alpha \in C_\alpha$ and the c_α are distinct. Since $|H| < m$ we can therefore choose a subset J of I such that $|J| = m$ and $G_\alpha H/H$ is nontrivial for each $\alpha \in J$.

Let $x \in G_\alpha H \cap \langle G_\beta H\colon \beta \neq \alpha\rangle$: then $x = g_\alpha h = g h_1$ with $g \in \langle G_\beta\colon \beta \neq \alpha\rangle$. Thus $g_\alpha = g h_1 h^{-1} \in Z(G_\alpha) = C_\alpha$ and $g \in Z(\langle G_\beta\colon \beta \neq \alpha\rangle) = \langle C_\beta\colon \beta \neq \alpha\rangle$. Thus $x \in C_\alpha H \cap \langle C_\beta H\colon \beta \neq \alpha\rangle = H$. Hence $\langle G_\alpha H/H\colon \alpha \in J\rangle = \mathrm{Dr}_{\alpha\in J} G_\alpha H/H$.

Relabel the elements of J by ordinals β, $\beta < \rho$. The group $G_\beta G_{\beta+1} H/H$ is nonhamiltonian and so there is a cyclic subgroup X_β of $G_\beta G_{\beta+1} H$ such that $X_\beta^G H > X_\beta H$. Let $X = \langle H, X_\beta;\ \beta = \lambda + 2n\rangle$, where λ is a limit ordinal and n a natural number. Then $|X^G\colon X| = m$ as required.

Theorem 5.1 is not available for the case $m = \aleph_0$, and we have not been able to make any significant reduction to the proof included in Theorem

3.2 of **[13]**. The original proof of Neumann **[8]** also gives a proof for that case. Theorems B and C now follow by combining Theorem 5.2 with Theorems 4.1 and 4.4, respectively.

We also are able to give a proof of Theorem A which is considerably shorter than that given in **[13]**. The proof depends on the following rather simple result about abelian groups.

LEMMA 5.3. *Let B be a subgroup of the abelian group A such that $|B| < m$. Then there is a subgroup C of A such that $B \cap C = 1$,*

(i) $|A/C| < m$ if m is uncountable, and

(ii) A/C satisfies the minimal condition if $m = \aleph_0$.

PROOF. (i) Embed A in its divisible hull $\bar{A} = \mathrm{Dr}_{i\in I}D_i$, where each D_i is locally cyclic and hence is countable. Since $|B| < m$, the set $J = \bigcup_{b\in B} \mathrm{supp}(b) = \{i \in I;\ \pi_i(B) \neq 1\}$ has cardinality less than m. Let $C_1 = \mathrm{Dr}_{i\in I-J}D_i$ and $C = A \cap C_1$. Then $B \cap C \leqq \mathrm{Dr}_{i\in J}D_i \cap C_1 = 1$ and $A/C \leqq (A + \mathrm{Dr}_{i\in J}D_i)/C \cong \mathrm{Dr}_{i\in J}D_i/(C \cap \mathrm{Dr}_{i\in J}D_i) = \mathrm{Dr}_{i\in J}D_i$. Thus $|A/C| \leqq |\mathrm{Dr}_{i\in J}D_i| < m$.

(ii) For each $b \in B$, choose a subgroup M_b maximal subject to $b \notin M_b$. Then A/M_b is cyclic or of type C_{p^∞}. B is finite, and if $C = \bigcap_{b\in B}M_b$ then $B \cap C = 1$ and A/C satisfies minimal condition.

THEOREM 5.4. *Let $G \in Z_m$ and $|G/Z| \geqq m$. Then G has an abelian subgroup X such that $|X/X_G| = m$.*

PROOF. If $|G'| \geqq m$, then by Theorem B there is an abelian subgroup X such that $|X^G; X| = m$. It follows from Lemma 3.5(i) that $|X/X_G| \geqq m$. We may therefore assume that $|G'| < m$.

We now show that G contains an N-group of cardinality m. Suppose that we have constructed elements a_β. b_β for $\beta < \alpha$ (some $\alpha < \rho$) such that $[a_\beta, a_\gamma] = [b_\beta, b_\gamma] = [a_\beta, b_\gamma] = 1$, if $\beta \neq \gamma$, $[a_\beta, b_\beta] = c_\beta \neq 1$. Let $S_\alpha = \{a_\beta, b_\beta;\ \beta < \alpha\}$, then $|S_\alpha| < m$ and if $C_\alpha = C_G(S_\alpha)$, $|G\colon C(S_\alpha)| < m$. By Lemma 3.9, $|C_\alpha\colon Z(C_\alpha)| \geqq m$ and so there are elements $a_\alpha, b_\alpha \in C_\alpha$ such that $[a_\alpha, b_\alpha] = c_\alpha \neq 1$. Then $A = \langle a_\alpha, \alpha < \rho\rangle$ is an abelian subgroup of G such that $|A/A < Z| = m$ and $|A \cap G'| < m$.

If m is uncountable, then by Lemma 5.3 there is a subgroup X of A such that $|A/X| \cap m$ and $X \cap G' = X \cap (A \cap G') = 1$. It follows that $|X/X \cap Z| = m$ and also $[X_G, G] \leqq X \cap G' = 1$ so that $X_G \leqq X \cap Z$. Hence $|X/X_G| = m$.

If $m = \aleph_0$, then G is an FC-group and so G/Z is residually finite. Hence $A/A \cap Z$ is a countably infinite direct product of finite cyclic groups; in particular $\sum \mathrm{r}_p(A)$ is infinite where $r_p(A)$ denotes the p-rank of A. By Lemma 5.3, there is a subgroup X of A of A such A/X satisfies the the minimal condition and $X \cap G' = 1$. Since $\sum r_p(A/X)$ is finite, it follows that $X/X \cap Z$ is infinite. As above, $X_G \leqq X \cap Z$ and so $|X/X_G| = \aleph_0$.

The proof of Theorem A now follows easily. It is clear that if $|G/Z| < m$, then $|U/U_G| < m$ for each $U \leqq G$. Since the equivalence of statement (ii) with (iii) and statement (iv) with (v) was established in Lemma 3.7, and since statement (ii) obviously implies (iv) it is only necessary to show that (iv) implies (i). This is the content of Theorem 5.4.

6. The counterexample. The example that we give to prove Theorem D is based on the construction first used in **[1]** and again in **[6]**.

Let V and W be vector spaces over a field K and let ρ: $V \times V \to W$ be an alternating bilinear function. If γ: $V \times V \to W$ is any bilinear function such that $\rho(x, y) = \gamma(x, y) - \gamma(y, x)$, then $V \times W$ can be given the structure of a nilpotent group of class two, denoted by $G = V\gamma W$, by defining $(x, a)(y, b) = (x + y, a + b + \gamma(x, y))$. Note that $[(x, a), (y, b)] = (0, \rho(x, y))$.

Using this construction, the result will follow by taking the V, W, and ρ provided by the following theorem.

THEOREM 6.1. ($2^m = m^+$). *Let K be a field with $|K| \leqq m$. Writing ω_γ, $\omega_{\gamma+1}$ for the least ordinals of cardinality m, m^+, let V, W be m^+-dimensional spaces with bases $\{v_\alpha: \alpha < \omega_{\gamma+1}\}$, $\{w_\alpha: \alpha < \omega_{\gamma+1}\}$. Let $V_\varepsilon = \langle V_\alpha: \alpha < \varepsilon\rangle$ and $W_\varepsilon = \langle W_\alpha;\ \alpha < \varepsilon\rangle$. Then there is an alternating bilinear function ρ: $V \times V \to W$ such that*

(i) *$\rho(V_\varepsilon \times V) \subseteqq W_\varepsilon$ for $\varepsilon < \omega_{\gamma+1}$ and*

(ii) *$\rho(L \times L) = W$ for each m^+-dimensional subspace L of V.*

If $G = V\gamma W$, then condition (i) shows that $G\varepsilon m^+C$. If $U \leqq G$, denote UG'/G' by $\bar{U}$, then $\bar{U}$ may be considered to be a subspace of $V = G/G'$. If $|\bar{U}| \leqq m$, then since m^+ is regular, $\bar{U} \leqq V_\varepsilon$ for some $\varepsilon < \omega_{\gamma+1}$ and so $U^G = U[U, G] = U\rho(\bar{U} \times V) \leqq UW_\varepsilon$. Hence $|U^G: U| \leqq m$. If $|\bar{U}| = m^+$, then condition (ii) shows that $U' \geqq G'$ and so $U \lhd G$.

PROOF. For our proof of Theorem 6.1, we use the following notation. If P is a subspace of V_ε and f: $V_\varepsilon \times V_\varepsilon \to W$ is a given alternating bilinear function, let $\delta(P) = \min\{\alpha: w_\alpha \notin f(P \times P)\}$. In fact, δ depends on f as well as P but no confusion will arise from our abbreviated notation.

LEMMA 6.2. *f: $V_\varepsilon \times V_\varepsilon \to W_\varepsilon (\varepsilon \geqq w_\gamma)$ be an alternating bilinear function such that*

(a) $$f(V_\varepsilon \times V_\varepsilon) \leqq W_\alpha, \textit{ for each } \alpha \leqq \varepsilon.$$

Let R be a family of subspaces of V_ε such that

(b) $$\textit{each } P \in R \textit{ satisfies } \operatorname{codim}_P(P \cap V_{\delta(P)}) = m$$

and

(c) $$|R| \leqq m.$$

Then there is an extension $f^*: V_{\varepsilon+1} \times V_{\varepsilon+1} \to W_{\varepsilon+1}$ *of* f *such that*

(1) f^* *is an alternating bilinear function,*

(2) $f(V_\alpha \times V_{\varepsilon+1}) \subseteqq W_\alpha$ *for each* $\alpha \leqq \varepsilon + 1$, *and*

(3) *for each* $v \varepsilon V_\varepsilon$ *and* $P \in R$, *there is an element* $u \in P$ *such that* $f(u, v + v_\varepsilon) = w_{\delta(P)}$.

PROOF. If $R = \varnothing$, then condition (3) is vacuous and we may define $f(v_\alpha, v_\varepsilon) = 0$, for all $\alpha \leqq \varepsilon$. So we may assume that $R \neq \varnothing$ and so $|V_\varepsilon \times R| = m$. We well-order $V_\varepsilon \times R$ with order type ω_γ and label the αth element $(v^{(\alpha)}, P^{(\alpha)})$.

We define inductively a set of vectors $\{u_\alpha: \alpha < \omega_\gamma\}$ such that

(i) $\operatorname{supp}(u_\alpha) \cap \operatorname{supp}(u_\beta) = \varnothing$, if $\alpha \neq \beta$

and

(ii) $u_\alpha \varepsilon P^{(\alpha)} - V_{\delta(P^{(\alpha)})}$.

We may define u_0 to be any element of $P^{(0)} - V_{\delta(P^{(0)})}$. Then for any $\theta < \omega_\gamma$ we may assume that u_α has been defined for each $\alpha < \theta$. Let $\Sigma = \bigcup_{\alpha<\theta} \operatorname{supp}(u_\alpha)$; then $|\Sigma| < m$. If $V_{\varepsilon-\Sigma} = \langle v_\alpha;\ \alpha < \varepsilon,\ \alpha \notin \Sigma\rangle$, then $\operatorname{codim}_{V_\varepsilon}(V_{\varepsilon-\Sigma}) < m$ and hence

$$\operatorname{codim}_{P^{(\theta)} \cap V_{\varepsilon-\Sigma}} (P^{(\theta)} \cap V_{\varepsilon-\Sigma} \cap V_{\delta(P^{(\theta)})}) = m.$$

Therefore we can choose $u_\theta \in (P_{(\theta)} \cap V_{\varepsilon-\Sigma}) - V_{\delta(P^{(\theta)})}$. This completes the inductive definition of the u_α's.

Now let σ_α be the largest member of $\operatorname{supp}(u_\alpha) = \{\alpha_1, \ldots, \alpha_n, \sigma_\alpha\}$ then we may assume that $u_\alpha = \sum_{i=1}^n \xi_i v_{\alpha_i} + v_{\sigma_\alpha}$. We define

$$f^*(v_{\sigma_\alpha}, v_\varepsilon) = w_{\delta(P^{(\alpha)})} - f(u_\alpha, v^{(\alpha)})$$

and $f^*(v_\beta, v_\varepsilon) = 0$, if $\beta \neq \sigma_\alpha$ for any α.

Clearly f^* can be made bilinear and alternating by defining $f(v_\varepsilon, v) = -f^*(v, v_\varepsilon)$; thus condition (1) is satisfied.

$$f(u_\alpha, v_\varepsilon) = f(v_{\sigma_\alpha}, v_\varepsilon) = w_{\delta(P^{(\alpha)})} - f(u_\alpha, v^{(\alpha)}),$$

and so

$$f^*(u_\alpha, v_\varepsilon + v^{(\alpha)}) = w_{\delta(P^{(\alpha)})}.$$

If $v \in V_\varepsilon$, $P \in R$, then there is an α such that $v = v^{(\alpha)}$, $P = P^{(\alpha)}$ and so u_α is the element required for condition (3).

If $\beta \neq \sigma_\alpha$ for any α, then $f^*(v_\beta, v_\varepsilon) = 0 \in W_{\beta+1}$. Also

$$f^*(v_{\sigma_\alpha}, v_\varepsilon) = w_{\delta(P^{(\alpha)})} - f(u_\alpha, v^{(\alpha)}) \in w_{\delta(P^{(\alpha)})} + W_{\sigma_{\alpha+1}}$$

by hypothesis (a). But $u_\alpha \notin V_{\delta(P^{(\alpha)})}$, so $\delta(P^{(\alpha)}) < \sigma_\alpha$ and $W_{\delta(P^{(\alpha)})} \subseteqq W_{\sigma_{\alpha+1}}$. Thus $f^*(v_{\sigma_\alpha}, v_\varepsilon) \in W_{\sigma_{\alpha+1}}$, as required for condition (2).

This completes the proof of Lemma 6.2.

Proof of Theorem 6.1. For each $\varepsilon(\omega_\gamma \leqq \varepsilon < \omega_{\gamma+1})$ we can define $F(\varepsilon)$ to be an m-dimensional subspace of V_ε in such a way that every m-dimensional subspace of V occurs as some $F(\varepsilon)$.

Let f: $V_{\omega_\gamma} \times V_{\omega_\gamma} \to W_{\omega_\gamma}$ be any alternating bilinear function such that $f(V_\alpha \times V_{\omega_\gamma}) \subseteqq W_\alpha$, for each $\alpha \leqq \omega_\gamma$. [For example, let $f(v_\alpha, v_{\alpha+1}) = w_\alpha$ for all $\alpha < \omega_\gamma$ and $f(v_\alpha, v_\beta) = 0$ if $\beta > \alpha+1$.] We extend f as follows.

Suppose that we have defined the extensions f: $V_\varepsilon \times V_\varepsilon \to W_\varepsilon$ such that $f(V_\alpha \times V_\varepsilon) \subseteqq W_\alpha$, for each $\alpha \leqq \varepsilon$. We extend this to f: $V_{\varepsilon+1} \times V_{\varepsilon+1} \to W_{\varepsilon+1}$ using Lemma 6.2 with $R_\varepsilon = \{F(\tau)\colon \tau \leqq \varepsilon \text{ and } \operatorname{codim}_{F(\tau)}(F(\tau) \cap V_{\delta(F(\tau))}) = m\}$. If ν is a limit ordinal, then f: $V_\nu \times V_\nu \to W_\nu$ is defined coherently.

Let ρ: $V \times V \to W$ be the resulting alternating bilinear function.

Suppose that V has an m^+-dimensional subspace L such that $\rho(L \times L) \neq W$. Then let δ be minimal such that $w_\delta \notin \rho(L \times L)$. W_δ has dimension m and $W_\delta \subseteqq \rho(L \times L)$. So there is an m-dimensional subspace P of L such that $\rho(P \times P) \supseteqq W_\delta$ and $\operatorname{codim}_P(P \cap V_\delta) = m$. Clearly $\delta(P) = \delta$. Also, $P = F(\eta)$, for some η. Now L contains an element $v = \sum_{\lambda<\kappa}\xi_\lambda v_\lambda + v_\kappa$, for some $\kappa > \eta$. $P = F(\eta) \in R_\kappa$ and $\sum_{\lambda<\kappa}\xi_\lambda v_\lambda \in V_\kappa$. So there is an element $u \in P$ such that $\rho(u, v) = f(u, \sum_{\lambda<\kappa}\xi_\lambda v_\lambda + v_\kappa) = w_\delta$. This is a contradiction to $w_\delta \notin \rho(L \times L)$ and hence $\rho(L \times L) = W$ for all m^+-dimensional subspaces L of V.

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