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ON HAAR MEASURE OF CERTAIN HYPERCOMPLEX UNITARY GROUPS

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ABSTRACT. By using the theory of hypercomplex matrices some results of Toyama [5] on Haar measure of orthogonal and unitary groups are extended to similar results for Haar measure on the unitary groups of hypercomplex matrices. A formula for Haar measure of unitary and unitary symplectic hypercomplex matrix groups of the division algebras is derived, in terms of the Cayley parametrization of these matrices.

1. Introduction. In his celebrated paper on the theory of invariants, Hurwitz [2] introduced the notion of invariant measures on group manifolds. He gave an explicit expression for Haar measure of unitary unimodular groups, and also for orthogonal groups by using the theory of generalized polar coordinates. Later Weyl [6], p. 169, and p. 217] obtained other expressions for the same measures. Toyama [5] obtained different expressions for these measures by using Cayley's parametric representations of orthogonal and unitary matrices. Toyama's [5] results are as follows.

THEOREM 1. *The infinitesimal volume element $d\Omega$ of Haar measure on the unitary group of $n \times n$ unitary matrices Ω is*

$$(1) \qquad d\Omega = |I + H^2|^{-n} dH.$$

Here H is a $n \times n$ Hermitian matrix, $H = (h_{ij})$, $dH = \pi_{i<j}\, dh_{ij} = \pi_{i<j}\, da_{ij}\, \pi_{i<j} db_{ij}$, $H = A + iB$, $A = (a_{ij})$, $B = (b_{ij})$, A is symmetric and B is skewsymmetric, and $n \times n$ Ω is represented by Cayley's parametric representation as

$$(2) \qquad \Omega = (I + iH)(I - iH)^{-1}.$$

Note that Toyama [5] considers only the group of those unitary matrices which can be represented by (2).

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The second result of Toyama [5] obtains the infinitesimal Haar measure on the unitary symplectic group of $2n \times 2n$ unitary matrices. Let U be a $2n \times 2n$ unitary symplectic matrix, then U may be represented by Cayley's parametrization as

$$U = (I + iH)(I - iH)^{-1}, \tag{3}$$

where the Hermitian matrix H now has the form

$$H = \begin{bmatrix} A & B \\ \bar{B}' & -A \end{bmatrix}. \tag{4}$$

Here A is an $n \times n$ Hermitian matrix, B is a complex symmetric matrix, and $\bar{B}'$ is the complex conjugate of B. Now Toyama [5] proves the following theorem.

THEOREM 2. *For unitary symplectic group of $2n \times 2n$ matrices U, the infinitesimal Haar measure dU on this group is*

$$dU = |I + H^2|^{-(2n+1)/2} dH. \tag{5}$$

Here $dH = dAdB$, $dB = dB_1 dB_2$, $B = B_1 + iB_2$, and B_1 and B_2 are symmetric matrices. Note that B_1, B_2 can be diagonalized by a single orthogonal matrix, and also note that B may have complex roots while A always has real roots. However, the roots of H are always real. For any symmetric matrix A the differential $dA = \pi_{i<j} da_{ij}$.

The purpose of the present paper is to generalize the results (1) and (5) to the case of hypercomplex matrices.

The relevant theory of hypercomplex matrices is stated in the next section, and the main results of the paper are derived in §3.

Sometimes the same symbol denotes different quantities, however, its meaning is made explicit in the context.

2. Hypercomplex Matrices. The hypercomplex matrices are divided into four categories: 1) quaternion hypercomplex matrices, 2) octonion hypercomplex matrices, 3) biquaternion hypercomplex matrices, 4) bioctonion hypercomplex matrices. The hypercomplex numbers do not form a field as the real and complex numbers, and several results of real variable and complex variable theory cannot be generalized to the hypercomplex case. However, certain integration results derived for the groups of real and complex matrices do generalize to the hypercomplex case, see e.g., Kabe [3], and this paper presents some such results. Here we show that the integration results derived by Toyama [5] for the real orthogonal matrices and the complex unitary matrices do generalize to the hypercomplex case. Toyama [5] derives his results separately for orthogonal matrices and for the unitary matrices. We give a single unified result which holds for orthogonal matrices, unitary matrices, and hypercomplex unitary matrices. For the purpose of deriving this unified formula we use a particular notation, which although very common in multivariate statistical analysis (see, Kabe [3]) is of rare occurrence in group theory.

Let $X_1, X_2, \ldots, X_{4t}$, $t = 1/4, 1/2, 1, 2, 4$, be $4t$ $p \times n$ real matrices. Then the $p \times n$ matrix

$$Y = X_1 + iX_2 + jX_3 + kX_4 + lX_5 + mX_6 + nX_7 + oX_8, \tag{6}$$

where the special (generator) octonions i, j, k, l, m, n, o satisfy the multiplication rule

$$\begin{aligned} i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1 \\ = ijk = iln = ion = jln = jmo = klo = kmn, \end{aligned} \tag{7}$$

is termed the octonion hypercomplex matrix. The matrix

$$\bar{Y} = X_1 - iX_2 - jX_3 - kX_4 - lX_5 - mX_6 - nX_7 - oX_8, \tag{8}$$

is the octonion hypercomplex conjugate of Y.

The matrix $Y\bar{Y}' = H$ has a special structure and H is termed the octonion hypercomplex Hermitian matrix. Note that H is positive definite. Thus if Σ is a $p \times p$ octonion hypercomplex Hermitian matrix, then Σ can be written as

$$\Sigma = \Sigma_1 + i\Sigma_2 + j\Sigma_3 + k\Sigma_4 + l\Sigma_5 + m\Sigma_6 + n\Sigma_7 + o\Sigma_8, \tag{9}$$

where Σ_1 is a $p \times p$ real symmetric matrix, and $\Sigma_2, \Sigma_3, \ldots, \Sigma_8$ are $p \times p$ real skew symmetric matrices. If Σ_1 is positive definite, then Σ is positive definite, $\bar{\Sigma}' = \Sigma$, and hence Σ has real roots.

The octonion and bioctonion hypercomplex theory used here is due to Hamilton, see [1]. There exists a similar system of octonions and bioctonions due to Cayley. The equations (6), (7), (8), (9) assume $t = 2$. When $t = 1$, we assume l, m, n, o to be zero, and in this case the equations (6), (7), (8), (9) yield the results for the quaternion case. When $t = 1/2$ we have the complex case, and when $t = 1/4$ the above equations trivially reduce to the real case.

The bioctonion hypercomplex case is just a complex copy of the octonion case. Let $Z_1, Z_2, \ldots, Z_8$ be eight $p \times n$ complex matrices, then the bioctonion hypercomplex matrix Z is defined by

$$Z = Z_1 + iZ_2 + jZ_3 + kZ_4 + lZ_5 + mZ_6 + nZ_7 + oZ_8. \tag{10}$$

The bioctonion hypercomplex conjugate of Z is

$$\bar{Z} = Z_1 - iZ_2 - jZ_3 - kZ_4 - lZ_5 - mZ_6 - nZ_7 - oZ_8. \tag{11}$$

The matrix $Z\bar{Z}' = G$ has a special structure and G is said to be the bioctonion hypercomplex Hermitian matrix. Note that G is positive definite. Thus if Σ is a $p \times p$ bioctonion hypercomplex matrix, then

$$(12) \qquad \Sigma = \Sigma_1 + i\Sigma_2 + j\Sigma_3 + k\Sigma_4 + l\Sigma_5 + m\Sigma_6 + n\Sigma_7 + o\Sigma_8,$$

where Σ_1 is $p \times p$ Hermitian, and $\Sigma_2, \ldots, \Sigma_8$ are skew Hermitian matrices. If Σ_1 is positive definite then Σ is positive definite, $\bar{\Sigma}' = \Sigma$, and Σ has real roots. The bioctonion case corresponds to $t = 4$. If in (10), (11), (12) we set l, m, n, o to be zero, then the bioctonion case reduces to the biquaternion case. The biquaternion case corresponds to the case $t = 2$, and the octonion case also corresponds to case $t = 2$. However, if in (10), (11), (12), we set j, k, l, m, n, o to be zero then the bioctonion case reduces to the complex case.

Note that the $i = (-1)^{1/2}$ of the complex case commutes with all the generator octonions. Further, from the unified formula which is given in this paper, special care must be taken to deduce the results for the particular case $t = 2$ as this case corresponds to both the biquaternion and octonion case.

We consider only the octonion case, i.e., $t = 2$. The results for the bioctonion case follow simply by setting t to be $2t$ in our formula and replacing all the octonion matrices by the corresponding bioctonion matrices. To derive the results for the biquaternion case we first derive the results for the quaternion case and then change t to $2t$ and replace the quaternion matrices by the corresponding biquaternion matrices.

The Jacobian of the transformation from $A = A_1 + iA_2 + \cdots + oA_8$ to $W = W_1 + iW_2 + \cdots + oW_8$, $A = f(W)$, is defined by

$$(13) \qquad J(A\colon W) = J(A_1, \ldots, A_8\colon W_1, \ldots, W_8),$$

where the differential dY of Y of (6) is defined by $dY = dX_1 dX_2 \cdots dX_{4t}$, i.e., in the octonion case $dY = dX_1 dX_2 \cdots dX_8$.

Thus, e.g., the Jacobian of the transformation $Y = GWH$, where Y is $p \times n$ hypercomplex and W is $p \times n$ hypercomplex, and G $p \times p$, H $n \times n$, are arbitrary is written as

$$(14) \qquad J(Y\colon W) = |G\bar{G}'|^{2nt}|H\bar{H}'|^{2pt}.$$

The Jacobian of the transformation $B = HR\bar{H}'$, where B is $p \times p$ hypercomplex Hermitian, R is $p \times p$ hypercomplex Hermitian, and H $p \times p$ is arbitrary is

$$(15) \qquad J(B\colon R) = |H\bar{H}'|^{2t(p-1)+1}.$$

Note that the results (14) and (15) are general results. Thus if $t = 1/4$, then (15) represents the Jacobian of the transformation from a symmetric matrix B to a symmetric matrix R, by the transformation $B = HRH'$. Again if $t = 4$, then (15) represents the Jacobian of the transformation from a $p \times p$ bioctonion hypercomplex Hermitian matrix B to a $p \times p$ bioctonion hyper complex Hermitian matrix R, by the transformation $B = HR\bar{H}'$. Some further results on such Jacobians are given by Kabe [3].

The definitions of octonion hypercomplex unitary matrices or the bioctonion hypercomplex unitary matrices are similar to those in the complex case. Thus if hypercomplex Ω is unitary, then we have that $\Omega\bar{\Omega}' = I$, $\bar{\Omega}'\Omega = I$ and $\Omega\bar{\Omega}' \leqq I$ implies that $\bar{\Omega}'\Omega \leqq I$. We say that $A \leqq B$ for two symmetric at least positive semi-definite A and B if $B - A$ is at least positive semi-definite.

We now proceed to generalize (1) and (5).

3. Main Results. We state the following generalization of (1).

THEOREM 3. *The infinitesimal volume element $d\Omega$ of Haar measure on the unitary group of hypercomplex unitary matrices Ω is*

$$d\Omega = |I + H^2|^{-(2nt+2t-1)}dH. \tag{16}$$

Here $H = (h_{ij})$ is an $n \times n$ hypercomplex Hermitian matrix having the structure (9) or (12). The meaning of dH is obvious, and $n \times n$ Ω is represented by

$$\Omega = (I + iH)(I - iH)^{-1}, \tag{17}$$

and the structure of Ω is either (6) or (10).

The proof follows exactly on the same lines as the one given by Toyama [5]. We form the differential of Ω of (17), namely

$$d\Omega = \{I + i(H + dH)\}\{I - i(H + dH)\}^{-1} - \{(I + iH)(I - iH)\}^{-1}. \tag{18}$$

We multiply (18) on left by $\{I - i(H + dH)\}$ and on the right by $(I - iH)$, and we find that

$$\{I - i(H + dH)\}d\Omega(I - iH) = 2idH. \tag{19}$$

In (19) we neglect the terms of second order, i.e., we omit $dHd\Omega$, and we have that

$$(I - iH)d\Omega(I - iH) = 2idH, \tag{20}$$

or that

$$d\Omega = (I - iH)^{-1}2idH(I - iH)^{-1}. \tag{21}$$

By left translation an infinitesimal element $d\Omega$ is transformed to $\Omega d\Omega$, i.e.,

$$d\Omega \rightarrow \Omega d\Omega \tag{22}$$

and by right translation

$$(23)\qquad dQ \to (dQ)(Q).$$

We now define the metric dQ by the quadratic differential

$$(24)\qquad ds^2 = tr(d\bar{Q}'dQ) = \sum_i\sum_j |dQ_{ij}|^2.$$

The metric ds^2 thus defined is clearly invariant under the left and right translations, because

$$(25)\qquad \begin{aligned} \operatorname{tr}(\overline{QdQ'}QdQ) \operatorname{tr}(\overline{dQ'}dQ) &= \operatorname{tr}((\overline{dQ})(Q)'(dQ)(Q)) \\ &= \operatorname{tr}(\bar{Q}'(dQ)'dQ(Q)) = \operatorname{tr}(\overline{dQ'}dQ). \end{aligned}$$

Thus ds^2 and hence dQ are translation invariant volume elements. Now our purpose is to calculate the Euclidean volume element dQ contained in the metric ds^2. We note that

$$(26)\qquad \begin{aligned} ds^2 &= \operatorname{tr}(\overline{dQ'}dQ) \\ &= \operatorname{tr}\{(I+iH)^{-1}(2idH)(I+iH)^{-1}(I-iH)(2idH)(I-iH)^{-1}\} \\ &= 4 \operatorname{tr}\{(I+H^2)^{-1}dH(I+H^2)^{-1}dH\}. \end{aligned}$$

We omit the constant 4 and following Toyama [5] write (26) as

$$(27)\qquad \begin{aligned} \mathrm{d}s^2 &= \operatorname{tr}[\{(I+H^2)^{-1}dH\}^2] = \operatorname{tr}[\{(I+\Lambda^2)^{-1}dH\}^2] \\ &= \sum_i\sum_j (1+\lambda_i^2)^{-1}(1+\lambda_j^2)^{-1}|dh_{ij}|^2, \end{aligned}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of the latent roots of H. Now we resort to either (9) or (12) and find that

$$(28)\qquad |dh_{ij}|^2 = d\sigma_{1ij}^2 + d\sigma_{2ij}^2 + \cdots + d\sigma_{4tij}^2,$$

where

$$(29)\qquad \Sigma_1 = (\sigma_{1ij}), \ldots, \Sigma_{4t} = (\sigma_{4tij}).$$

Further from (27) noting that $\Sigma_2, \ldots, \Sigma_{4t}$ are skew symmetric matrices, we have that

$$(30)\qquad \begin{aligned} ds^2 = &\sum_{i=1}^n (1+\lambda_i^2)^{-2}d\sigma_{1ii}^2 + 2\sum_{i<j}(1+\lambda_i^2)^{-1}(1+\lambda_j^2)^{-1}d\sigma_{1ij}^2 \\ &+ 2\sum_{i<j}(1+\lambda_i^2)^{-1}(1+\lambda_j^2)^{-1}d\sigma_{2ij}^2 + \cdots \\ &+ 2\sum_{i<j}(1+\lambda_i^2)^{-1}(1+\lambda_j^2)^{-1}d\sigma_{4tij}^2. \end{aligned}$$

Following Toyama [5] we find that the invariant volume element in (30) is $\sqrt{g}\,dH$, where

$$(31)\qquad g = \prod_{i=1}^n (1+\lambda_i^2)^{-1} \prod_{i<j} 2(1+\lambda_i^2)^{-1}(1+\lambda_j^2)^{-1} \cdots \prod_{i<j} 2(1+\lambda_i^2)^{-1}(1+\lambda_j^2)^{-1}.$$

Note that the term $\prod_{i<j} 2(1 + \lambda_1^2)^{-1}(1 + \lambda_j^2)^{-1}$ occurs $4t$ times while the term $\prod_{i=1}^{n}(1 + \lambda_i^2)^{-1}$ occurs only once. The relationship between ds^2 and $d\Omega$ is made clear by Mass [4], which explains the passage from (30) to (31).

Now we must calculate the number of occurrences of the terms of the type $(1 + \lambda_i^2)$ in g. Concentrate on any particular term, say $(1 + \lambda_1^2)$. The number of occurrences of the term depends on the order (i.e., size) of the matrix, and hence we suppose that it occurs $f(n)$ times. Now consider the matrices of order $(n - 1)$, then in this case H has exactly $(n - 1)$ latent roots, and hence a term of the type must occur $f(n - 1)$ times. However, in passing from n-th order matricse to $(n - 1)$-*th* order matrices, we lose one root, i.e., we lose one term of the type $(1 + \lambda_1^2)$. The question is how many times do we lose this term. The answer is obtained by carefully examining (31). The contributions to the number of times $(1 + \lambda_1^2)$ occurs in (31) come from the $4t$ matrices on the right hand side of (9) or (12). Thus we must assume that there are exactly $4t$ factors on the right hand side of (31), and each factor has lost the term $(1 + \lambda_1^2)$, i.e.,

$$f(n) - 4t = f(n - 1), \tag{32}$$

which yields

$$f(n) = 4tn + 4t - 2, \tag{33}$$

and hence $(1 + \lambda_1^2)$ occurs exactly $(4tn + 4t - 2)$ times in (31). It follows that

$$\begin{aligned} d\Omega &= \sqrt{g}\, dH = \prod_{i=1}^{n} (1 + \lambda_1^2)^{-(2tn+2t-1)} dH \\ &= |I + H^2|^{-(2tn+2t-1)} dH. \end{aligned} \tag{34}$$

The solution to the difference equation (32) contains exactly one arbitrary constant. This constant is determined by setting $t = 1/2$, and then using Toyama's [5] result, namely Theorem 1, as the initial condition. Note that when $t = 1/4$, (34) yields Toyama's [5] result for the real case (i.e., for the orthogonal matrices case).

We now proceed with the generalization of (5) to the hypercomplex case, i.e., we consider the hypercomplex unitary symplectic group. A hypercomplex unitary symplectic matrix is represented by Cayley's parameters as

$$U = (I + iH)(I - iH)^{-1}, \tag{35}$$

where H is hypercomplex Hermitian and has the structure

$$H = \begin{bmatrix} A & B \\ \bar{B}' & -A \end{bmatrix}, \tag{36}$$

where A is hypercomplex Hermitian of the *n-th* order, and B is hypercomplex symmetric of order n. Note that B may have hypercomplex roots. Here H is $2n \times 2n$ and has $2n$ real roots. Although the calculations in this case also follow on similar lines, the equation (29) is no longer true in this case. However, following Toyama [**5**] we may derive an expression similar to (31), and hence (32) holds. We solve (32) by first setting $t = 1/2$, and then using (5) as the initial condition, and find that in this case (32) yields

$$f(n) = 4tn + 2t, \tag{37}$$

and hence

$$dU = |I + H^2|^{-t(2n+1)} dH. \tag{38}$$

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