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A BRIEF SURVEY AND HISTORY OF ASYMPTOTIC PRIME DIVISORS

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ABSTRACT. Recently there has been considerable interest in high powers of ideals, their integral closures, and the associated primes of these ideals, and many of these results are closely related to other areas of current research interest. The purpose of this paper is to summarize these new results and to give a brief historical overview of their development.

0. Introduction. Powers of ideals have played an important role in commutative algebra. (For example, in completions, quadratic transformations, associated graded rings, Rees rings, asymptotic closure and integral closure of ideals, Hilbert function and multiplicity, etc.) Recently, quite a few results have shown that large powers of ideals and their prime divisors are of considerable interest in their own right and are also closely connected to other areas of current research interest, such as going-down between prime ideals and the catenary chain conjectures. In this paper a brief survey of this area together with a sketch of the historical antecedents of several of the results is given.

§1 contains a review of the terminology and a summary of the notation used in this paper. §2 is mainly historical; it contains a summary of various generalizations of a 1916 theorem of Macaulay. §3 contains some additional (but less closely related) generalizations of Macaulay's result. The first few of these are quite new and yield characterizations of Cohen-Macaulay rings and of unmixed and quasi-unmixed Noetherian rings, and the others are very useful older generalizations. §4 contains numerous results on the prime divisors of I^n and of $(I^n)_a$ with n large. Included are characterizations of these sets of prime ideals, characterizations of various kinds of rings in terms of these sets, and results showing when a given prime ideal P is a prime divisor of all large powers of all ideals of a certain type contained in P. Several results concerning the analytic spread of an

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ideal and its relationship to asymptotic prime divisors are mentioned in §5. The uses of generalized analytically independent elements to characterize R-sequences, asymptotic sequences, and unmixed and quasi-unmixed Noetherian rings are mentioned in §6. §7 continas a few of the important results that developed from the 1952 paper of Samuel on asymptotic powers of ideals together with their new asymptotic versions, and some open problems are listed and briefly discussed in §8.

I am indebted to the referee for his many comments and suggestions that helped to order the subject matter and to clarify the narrative in many places.

Hopefully this paper can be used to quickly familiarize the reader with the important new results in an old (but newly active) area of research.

1. Notation and terminology. For ease of reference, this section contains most of the notation and terminology that will be used in this paper. Generally, the terminology is standard, but to keep the paper reasonably self-contained most of the important definitions needed in what follows will be briefly recalled in this section.

R', R^*, and $\mathscr{R}(R, I)$. A standard assumption is that all rings are commntative with an identity element. R' will consistently be used to denote the *integral closure* of a ring R in its total quotient ring, and if R is local (Noetherian) with maximal ideal M, then R^* denotes the *M-adic completion of R*. Also, $\mathscr{R}(R, I)$ denotes the *Rees ring of R with respect to an ideal I in R*, so $\mathscr{R}(R, I) = R[tI, u]$ (t an indeterminate and $u = 1/t$) is a graded subring of $R[t, u]$ and $u^n\mathscr{R}(R, I) \cap R = I^n$ for all $n \geqq 1$.

Spec R, chain of primes ideals, length, height, depth, dimension. Spec R is the set of all prime ideals of R, considered here only as a partially ordered set. *A chain of prime ideals of length s in R* is simply a collection of $s + 1$ elements of Spec R which is ordered by inclusion. If $P \in$ Spec R, then *height* P ($=$ rank P) is the supremum of the lengths of chains of prime ideals in R descending from P, and *depth* P ($=$ dimension P) is defined analogously by using chains ascending from P. *Dimension* R ($=$Krull dimension R) is the supremum of the heights of the maximal ideals in R or, equivalently, the supremum of the depths of the minimal prime ideals in R. For an ideal I in R, *height* I is the infimum of the heights of the prime ideals in R that contain I, and *depth* I is the supremum of the depths of these prime ideals, so depth $I =$ dimension R/I.

I_a, $\ell(I)$, $A^*(I)$, $B^*(I)$, $\hat{A}^*(I)$, principal class, height-unmixed, grade, and R-sequence. If I is an ideal in R, then I_a denotes the *integral closure of I in R*, so $I_a = \{x \in R : x$ is a root of a polynomial of the form $T^k + a_1T^{k-1} + \cdots + a_k$, where $a_i \in I^i\}$ is an ideal in R, by [**36**, §6], and $I \subseteqq I_a \subseteqq$ Rad I. Also, $\ell(I)$ denotes the *analytic spread of I*, so $\ell(I) = \sup\{$dimension $\mathscr{R}(R, I)/(u, M)\mathscr{R}(R, I)$: M is a maximal ideal in R that contains $I\}$ If R is

Noetherian, then the sets Ass R/I^n, Ass I^{n-1}/I^n, and Ass $R/(I^n)_a$ stabilize for large n, by [**3**] and [**48**] (see (4.1) and (4.2)), so we denote these stabilized sets by $A^*(I)$, $B^*(I)$, $\hat{A}^*(I)$, respectively. I is said to be of the *principal class* in case I can be generated by h elements, where h = height I. If R is Noetherian, then I is *height-unmixed* in case all prime divisors (= associated primes) of I have the same height (in the literature this is often simply called unmixed, but in this paper unmixed will have a different meaning (see below)), and *grade* (I) is the common length of all maximal R-sequences contained in I. (The ordered sequence of elements $b_1, \ldots, b_h$ in R is an *R-sequence* in case $(b_1, \ldots, b_h)R \neq R$ and for $i = 1, \ldots, h$ the residue class in $R/(b_1, \ldots, b_{i-1})R$ of b_i is a regular element. Therefore it is always the case that grade (I) $\leq$ height I.)

Cohen-Macaulary, unmixed, and quasi-unmixed. Finally, a local ring R is a *Cohen-Macaulay local ring* in case there exists an R-sequence of length d = altitude R in R; equivalently (by [**56**, Thm. 4.3]), grade (I) = height I for all ideals I in R. R is an *unmixed* (resp., *quasi-unmixed*) *local ring* in case all (resp., all minimal) prime divisors of zero in the completion R^* of R have the same depth. Finally, a Noetherian ring R is a *Cohen-Macaulay* (resp., *unmixed, quasi-unmixed*) *ring* in case R_P is a Cohen-Macaulay (resp., unmixed, quasi-unmixed) local ring for all $P \in$ Spec R. (Most known examples of local domains are quasi-unmixed; the only known exceptions are all closely related to M. Nagata's examples, [**30**, Example 2, pp. 203–205], or to T. Ogoma's recent example of an integrally closed noncatenary local domain [**34**]. Thus local domains which are not quasi-unmixed are rather rare and somewhat pathological.)

2. A theorem of F. S. Macaulay. (This section is mainly historical. The history starts with a theorem of Macaulay, (2.1), and various generalizations of this theorem through 1957 are mentioned. In 1957 the best possible direct generalization appeared, (2.4), but a new and closely analogous generalization has been developed since then, and this is described in (2.6)–(2.7).)

We begin by noting that the subject has somewhat venerable roots, since it can be traced back at least to the following 1916 result of Macaulay.

THEOREM 2.1. [**19**, (50)]. *If K is a field and $X_1, \ldots, X_m$ are indeterminates, then each power I^n of an ideal I of the principal class in $K[X_1, \ldots, X_m]$ is height-unmixed.*

This result seems to have been at least 30 years ahead of its time, since it was not generalized in any form until 1946. (For historical perspective, it was in 1928 in [**18**] that W. Krull proved the Principal Ideal Theorem and the Generalized Principal Ideal Theorem.) In 1946, I. S. Cohen partly generalized this by proving the following theorem.

THEOREM 2.2. [**10**, Theorem 21]. *Ideals of the principal class in regular local rings are height-unmixed.*

Ten years later in [**32**, Lemma 2] D.G. Northcott extended Cohen's result to ideals of the principal class in Cohen-Macaulay local rings, and as noted below, D. Rees completely generalized (2.1) by proving the following theorem.

THEOREM 2.3. [**56**, Theorem 2.3]. *If R is a Noetherian ring and I is an ideal generated by an R-sequence, then each power I^n of I is grade-unmixed; that is, the prime divisors of all powers of I all have the same grade.*

Because of the concepts of grade and R-sequence, (2.3) captures the essence of Macaulay's theorem and yields the complete direct generalization of it. To explain this comment, recall that a Cohen-Macaulay ring can be defined as a Noetherian ring R in which grade (I) = height I for all ideals I (and then ideals of the principal class are generated by R-sequences, by [**17**, Theorem 125]). Thus (2.3) implies the following theorem.

THEOREM 2.4. [**56**, Theorem 3.2]. *Powers of ideals of the principal class in Cohen-Macaulay rings are height-unmixed.*

And (2.4) is the complete direct generalization of (2.1), since a standard result (and sometimes the definition) is that the condition on ideals of the principal class in (2.4) characterizes Cohen-Macaulay rings as in the following theorem.

THEOREM 2.5. [**28**, Theorem 5] and [**30**, (25.6)]. *A Noetherian ring is a Cohen-Macaulay ring if and only if each ideal of the principal class is height-unmixed.*

Nagata proved the local version of (2.5) in 1955 in [**28**, Theorem 5], and he gave the global version in 1961 in [**30**, (25.6)]. (In 1957 in [**56**, Theorem 3.1] Rees gave another (but equivalent) formulation of (2.5), and in 1957 he and Northcott gave another equivalent version of it in [**33**, Theorem 2.2].)

Thus by 1957 Macaulay's result had been extended as far as it is possible to directly extend it. Of course, various other generalizations of this result have appeared in the literature, and this section will be closed by considering one of these. (Other generalizations are mentioned in §3.)

Specifically, in 1974 this author proved a result, (2.6), whose statement suggests that there is a close connection between Cohen-Macaulay rings and R-sequences on the one hand and quasi-unmixed Noetherian rings and integral closures of ideals of the principal class on the other hand. The ramifications of this relationship have not been fully investigated, but a start on this was recently made, as is partly explained following the partial proof of (2.6). ((2.6) is stated to bring out the close analogy between [39, Theorem 2.29] and (2.4)–(2.5). However it is clear that this version is equivalent to the one in [39], since if $I = (b_1, \ldots, b_h)R$ is of the principal class, then $(I^n)_a = ((b_1^n, b_2^n, \ldots, b_h^n)R)_a$.)

THEOREM 2.6. [39, Theorem 2.29]. *A Noetherian ring R is quasi-unmixed if and only if the integral closure of each ideal of the principal class is height-unmixed. If R is quasi-unmixed and I is an ideal of the principal class, then all the ideals $(I^n)_a$ are height-unmixed.*

Since the proof of the "necessary" part of (2.6) is easy and the method used in the proof is quite useful, it will now be sketched. To avoid certain complicating details, only the integral domain case will be considered.

PARTIAL PROOF OF (2.6). Let R be a quasi-unmixed Noetherian domain, let $I = (b_1, \ldots, b_h)R$ be an ideal of the principal class in R of height h, fix $n \geqq 1$, and let P be a prime divisor of $(I^n)_a$. Let $\mathscr{R} = \mathscr{R}(R, I)$ be the Rees ring of R with respect to I. Then there exists a (height one) prime divisor p' of $u^n\mathscr{R}'$ such that $P = P' \cap R$, since $\mathscr{R}$ is a Noetherian domain and since it is readily seen that $u^n\mathscr{R}' \cap R = (I^n)_a$ for all $n \geqq 1$. Now R satisfies the altitude formula, by [37, Theorem 3.6] (since R is quasi-unmixed), so with $p = p' \cap \mathscr{R}$ and $t = \text{trd}\,(\mathscr{R}/p)/(R/P)$, it follows that height $p = 1$ and height $p + t =$ height $P +$ trd $\mathscr{R}/R$; that is, height $P = t$. Also height $P \geqq h$, since I is of the principal class and has height h, and $t \leqq h$, since $\mathscr{R}$ is generated over R by $tb_1, \ldots, tb_h$, u and $u \in p$. Therefore $t =$ height $P = h$, so $(I^n)_a$ is height-unmixed.

The close relationship of (2.6) to (2.4)–(2.5) suggests that there may be an analog of R-sequences which plays the same role in quasi-unmixed rings that R-sequences play in Cohen-Macaulay rings and which similarly illuminates the study of general Noetherian rings. (2.6) shows us the connecting link, ideals of the principal class. In Cohen-Macaulay rings, ideals of the principal class are generated by R-sequences. In quasi-unmixed rings, ideals of the principal class are generated by *asymptotic sequences*; that is, by elements $b_1, \ldots, b_h$ in R such that $(b_1, \ldots, b_h)R \neq R$ and $((b_1, \ldots, b_{i-1})^n R)_a : b_i R = ((b_1, \ldots, b_{i-1})^n R)_a$ for $i = 1, \ldots, h$ and for all large n. (It seems natural to call such elements an asymptotic sequence, because of P. Samuel's asymptotic closure of an ideal (which is defined in terms of large powers of the ideal and which in Noetherian rings is equal to the integral closure of the ideal; see §7).) It is readily seen that an R-sequence is an asymptotic sequence, but not conversely, and that an ideal of the principal class in a quasi-unmixed Noetherian ring is generated by an asymptotic sequence.

In the summer of 1980, this author briefly considered such elements, but was not able to do much with this concept at that time. However, in the summer of 1981 this author found that Rees had used them in

[**59**, Theorem 4.2] to prove a result concerning the analytic spread of an ideal (see (5.2)). This renewed my interest in such elements and this author was then able to show that they are an excellent analogue of R-sequences. In fact, it is shown in [**46**] that most of the known basic results concerning R-sequences and the grade of an ideal (for example, those in I. Kaplansky's book [**17**]) have a valid version concerning asymptotic sequences and asymptotic grade, where the *asymptotic grade of an ideal* I, denoted *grade*$^*(I)$, is defined as the common length of all maximal asymptotic sequences contained in I.

Two results that illustrate this and that show the close parallel between the standard and asymptotic theories should be mentioned in this section. The first of these is the asymptotic version of the fact a Noetherian ring R is Cohen-Macaulay if and only if grade (I) = height I holds for all ideals I in R. Namely, a Noetherian ring R is quasi-unmixed if and only if grade$^*(I)$ = height I holds for all ideals I in R, [**46**, (4.1)]. And the second of these is the following asymptotic version of (2.1).

THEOREM 2.7. [**46**, (3.7)]. *If R is a Noetherian ring and I is an ideal generated by an asymptotic sequence in R, then each of the ideals $(I^n)_a$ is grade*-unmixed; that is, the prime divisors of all the ideals $(I^n)_a$ all have the same asymptotic grade.*

Much of the beauty of the concepts developed to understand Macaulay's theorem is that they interlock so well with the theory of prime divisors, an outgrowth of the Primary Decomposition Theorem. As we shall see later, asymptotic sequences play a role in the study of asymptotic prime divisors much as R-sequences do in the study of prime divisors.

In closing this section, it should be noted that a number of other useful generalizations of R-sequences have appeared in the literature. Among these, the concept of a d-sequence has been considerably developed and utilized by C. Huneke, and in particular in [**16**, Theorem 3.1] he shows that all powers of certain ideals generated by such sequences are height-unmixed.

3. Other generalizations of Macaulay's theorem. (This section begins by describing a few very recent generalizations of (2.1) which lead to some interesting characterizations of Cohen-Macaulay (respectively, unmixed, quasi-unmixed) rings, (3.1)–(3.4). Then two older and very useful generalizations of a different type are recalled in (3.5) and (3.6), and the section is closed with a much newer asymptotic analogue of (3.6).)

(2.1) is such a nice result that quite a few generalizations of it besides those in §2 have appeared in the literature. In this section we consider several of these which, although not as closely related to (2.1) as those in §2, still yield useful and interesting results. We begin with the following 1980 result.

THEOREM 3.1. [**44**, (2.5)–(2.7)]. *If I is an ideal of the principal class in a Noetherian ring R such that all powers of I are integrally closed, then*

(3.1.1) *I is generated by an R-sequence,*

(3.1.2) *I is height-unmixed and R_P is an analytically irreducible Cohen-Macaulay local domain for each prime divisor P of I, and*

(3.1.3) *for each $P \in \operatorname{Spec} R$ such that $I \subseteqq P$ and height $P/I \leqq 1$, it holds that R_P is a reduced Cohen-Macaulay local ring.*

Closely related to this is the next theorem.

THEOREM 3.2. [**44**, (3.4) and (3.5)]. *If I is an integrally closed ideal of the principal class in a quasi-unmixed Noetherian ring R, then for all ideals $Z \subseteqq \operatorname{Rad} R$ it holds that I/Z is generated by an R/Z-sequence and R_P/ZR_P is a Cohen-Macaulay local ring for all $P \in \operatorname{Spec} R$ such that $I \subseteqq P$ and* height $P/I \leqq 1$.

This last result can be used to give the following characterization of Cohen-Macaulay local domains.

THEOREM 3.3. [**43**, (16)]. *A local domain (R, M) of dimension $d + 1 > 1$ is Cohen-Macaulay if and cnly if R is quasi-unmixed and there exists a depth one integrally closed ideal of the principal class in*

$$R(X_1, \ldots, X_d) = R[X_1, \ldots, X_d]_{MR[X_1,\ldots,X_d]}.$$

Thus the existence of a certain kind of ideal in $R(X_1, \ldots, X_d)$ shows that R is Cohen-Macaulay. Extending this idea, it follows from (3.4) and the comment following it that there are times when a Noetherian domain can be shown to be Cohen-Macaulay (or unmixed or quasi-unmixed) by showing that the powers (or their integral closures) of just one ideal on the principal class are all height-unmixed.

THEOREM 3.4. [**43**, (6)]. *Let R be a local domain of dimension $d + 1 > 1$, let $b_0, b_1, \ldots, b_h$ be elements in R, and let $K = (b_0X_1 - b_1, \ldots, b_0X_h - b_h) R[X_1, \ldots, X_h]$. Then the following statements are equivalent.*

(3.4.1) *R is quasi-unmixed.*

(3.4.2) *Whenever $b_0, b_1, \ldots, b_h$ $(h \geqq 1)$ is a subset of a system of parameters, it holds that K_a is prime and $(K^n)_a$ is K_a-primary for all $n \geqq 1$.*

(3.4.3) *There exists a system of parameters $b_0, b_1, \ldots, b_d$ in R such that (with $h = d$) K_a is prime and $(K^n)_a$ is K_a-primary for all $n \geqq 1$.*

In (3.4), if $(K^n)_a$ is replaced by K^n (and K_a by K), then this characterizes a Cohen-Macaulay local domain [**43**, (4)]; and if, instead, it is added that there exists $i \geqq 1$ such that $(K^{i+n})_a \subseteqq K^n$ for all $n \geqq 1$, then this characterizes an unmixed local domain [**43**, (11)].

The last three results in this section are a different type of generalization of (2.1). Namely, they are concerned with when a given $P \in \operatorname{Spec} R$ is a prime divisor of all ideals of a certain type contained in P. The first of these was proved in 1955 by Nagata.

THEOREM 3.5. [**27**, p. 300]. *Let P be a prime ideal in a Noetherian domain R and let $0 \neq b \in P$.*

(3.5.1) *If P is a prime divisor of bR, then P is a prime divisor of cR for all nonzero c in P.*

(3.5.2) *If $P = P' \cap R$ for some height one prime ideal P' in R', then P is a prime divisor of bR.*

He had proved a related version of (3.5.1) in 1955 in [**26**, Corol. 4, p. 76], and in 1961 in [**30**, (12.6)] this result was generalized to regular principal ideals in an arbitrary Noetherian ring. (A result closely related to this is given in (4.7.1) below.) In 1957 Rees generalized (3.5.1) by proving the following theorem.

THEOREM 3.6. [**57**, Theorem 1.3]. *Let R be a Noetherian ring and let $P \in \operatorname{Spec} R$. If P is a prime divisor of an ideal I generated by an R-sequence of length h and J is another such ideal contained in P, then P is a prime divisor of J.*

(3.6) is a corollary of the following important and closely related theorem.

THEOREM. [**55**, Theorem 3.1(ii)]. *If I and J are ideals generated by R-sequences of the same length and K is an ideal containing $I + J$, then $(I{:}\ K)/I \cong (J{:}\ K)/J$.*

In 1976 in [**40**] this author conjectured another generalization of (3.5.1) which will be discussed in §4.

The final result in this section is the following asymptotic version of (3.6).

THEOREM 3.7. [**50**]. *Let R be a Noetherian ring and let $P \in \operatorname{Spec} R$. If $P \in \hat{A}^*(I)$ for some ideal I generated by an asymptotic sequence of length h and J is another such ideal contained in P, then $P \in \hat{A}^*(J)$.*

Concerning (3.7), it is not true that if P is a prime divisor of I_a, then P is a prime divisor of J_a. The reason this asymptotic version of (3.6) does not hold lies in the difference in the definitions of R-sequences and asymptotic sequences (the residual division for asymptotic sequences is for all large n).

(4.11) has two results which are closely related to (3.7).

4. Asymptotic prime divisors. (The results in this section are concerned with $\hat{A}^*(I)$, $A^*(I)$, and $B^*(I)$, and they show many things concerning these sets, such as: their existence and containment relationships, (4.1)–(4.3); a characterization of them, (4.4), (4.7.2), and (4.8); information on when a given prime ideal is in $\hat{A}^*(I)$ or $A^*(I)$ for all ideals $I \subseteqq P$ of a certain type, (4.6), (4.7.1), and (4.9)–(4.11); their use in characterizing certain types of rings, (4.12)–(4.15), and in determining when going-down holds, (4.16); and some of their deeper properties, (4.5), (4.17), and (4.18).)

In 1976 in **[40]** this author tried to generalize to all ideals in a Noetherian ring Nagata's 1956 result that says a prime ideal P in a Noetherian domain R is in $A^*(bR)$ if and only if P is a prime divisor of b^nR for some $n \geqq 1$ (see (3.5.1)). (Some such generalization was somewhat suggested by Rees' result in 1956 in **[52**, Thm. 6.7]; if I is an ideal in a Noetherian domain R and x is in the center of a valuation associated with I (see (7.1)), then $I^n\colon xR \neq I^n$ for infinitely many n.) This author was not successful (and M. Brodmann showed in [3] that in fact this is not true), but this author did show that it holds for large powers of ideals that contain regular elements [**40**, (2.11)]. This author also showed in [**40**, (2.5)] that the corresponding statement for the integral closure of an ideal does hold. This result was recently sharpened as follows.

THEOREM 4.1. [**48**, (2.4) and (2.8)]. *If I is an ideal in a Noetherian ring R and if $P \in \operatorname{Spec} R$ is a prime divisor of $(I^k)_a$ for some $k \geqq 1$, then P is a prime divisor of $(I^n)_a$ for all $n \geqq k$. Also P is a prime divisor of I^m for all large m.*

The proof of this is perhaps informative enough to warrant sketching it.

PROOF OF (4.1). By hypothesis there exists $b \in R$ such that $(I^k)_a\colon bR = P$. Let $\mathscr{R} = \mathscr{R}(R, I)$. Then $P\mathscr{R} = ((I^k)_a\colon bR)\mathscr{R} \subseteqq (I^k)_a\mathscr{R}\colon b\mathscr{R} \subseteqq (u^k\mathscr{R})_a\colon b\mathscr{R}$ and $((u^k\mathscr{R})_a\colon b\mathscr{R}) \cap R = (I^k)_a\colon bR = P$, by [**65**, p. 220], since $(u^k\mathscr{R})_a \cap R = (I^k)_a$. Therefore there exists a prime divisor p or $(u^k\mathscr{R})_a$ such that $(u^k\mathscr{R})_a\colon b\mathscr{R} \subseteqq p$ and $p \cap R = P$. Now either height $P = 0$ or height $P > 0$ and $tI \nsubseteq p$ by [**48**, (2.3)]. If height $P = 0$, then the conclusions are clear, so assume that height $P > 0$ and let $c \in I$ such that $ct \notin p$. Then $(u^{k+j}\mathscr{R})_a\colon bc^j\mathscr{R} = (u^k\mathscr{R})_a\colon bc^jt^j\mathscr{R} \subseteqq p$ for all $j \geqq 0$, and contracting to R it follows that $(I^{k+j})_a\colon bc^jR = P$, so P is a prime divisor of $(I^n)_a$ for all $n \geqq k$. Finally, since $p = p' \cap \mathscr{R}$ for some prime divisor of p' of $u\mathscr{R}'$ (since $(u^k\mathscr{R})_a = u^k\mathscr{R}' \cap \mathscr{R}$ and the prime divisors of $u^k\mathscr{R}'$ are the prime divisors of $u\mathscr{R}'$), p is also a prime divisor of $u\mathscr{R}$ by [**38**, Thm. 2.15], so there exists a homogeneous element dt^m in $\mathscr{R}$ such that $u\mathscr{R}\colon dt^m\mathscr{R} = p$. Then $u^{m+1+j}\mathscr{R}\colon dc^j\mathscr{R} = u\mathscr{R}\colon dt^mc^jt^j\mathscr{R} = p$ for all $j \geqq 0$, so it follows that $I^{m+1+j}\colon dc^jR = P$, so P is a prime divisor of I^m for all large m.

The extent to which the sets Ass R/I^n are monotonically increasing is an open problem. In this regard, see (8.4).

It follows from $(I^n)_a = u^n\mathscr{R}(R, I)' \cap R$ that $\{P \in \text{Spec } R: P$ is a prime divisor of $(I^n)_a$ for some $n \geqq 1\}$ is a finite set, so it follows from this and (4.1) that $\hat{A}^*(I)$ is a well defined set [**48**, (2.7)]. (This fact was first explicitly stated for ideals I such that height $I \geqq 1$ in [**20**, Prop. 7].) Related to this, it follows from $I^n = u^n\mathscr{R}(R, I) \cap R$ that $\{P \in \text{Spec } R: P$ is a prime divisor of I^n for some $n \geqq 1\}$ is a finite set; Rees essentially noted this in 1956 in [**53**, Lemma 3.1].

In 1979 Brodmann considerably sharpened my result concerning the prime divisors of large powers of I by proving the following theorem.

THEOREM 4.2. [**3**, (1), (3), and (7)]. *$A^*(I)$ and $B^*(I)$ are well defined; that is, if I is an ideal in a Noetherian ring, then* Ass R/I^n *and* Ass I^{n-1}/I^n *are constant for large n. Moreover, $B^*(I) \subseteqq A^*(I)$ and equality holds when I contains a regular element.*

Therefore, since $\hat{A}^*(I)$ and $A^*(I)$ are well defined sets, the next theorem follows from (4.1).

THEOREM 4.3. [**48**, (2.8)]. *If I is an ideal in a Noetherian ring R, then $\hat{A}^*(I) \subseteqq A^*(I)$.*

In 1979, S. McAdam and P. Eakin characterized $B^*(I)$ and $A^*(I)$ as follows.

THEOREM 4.4. [**20**, Corollary 17, Proposition 10, and Corollary 13]. *$B^*(I) = \{P \in \text{Spec } R$: $P = p \cap R$ for some prime divisor p of $u\mathscr{R}(R, I)$ such that $tI \nsubseteq p\}$ and $A^*(I) = B^*(I) \cup \{P: I \subseteqq P \in \text{Ass } R\}$.*

Unfortunately, there is at present no nice characterization of $B^*(I) \cap \{P: I \subseteqq P \in \text{Ass } R\}$. (4.7.2) contains a characterization of $\hat{A}^*(I)$.

In 1980 in [**35**] J. Okon considered what could be said along the same lines as (4.1)–(4.4) for filtrations $\{I_n\}$ in a Noetherian ring (and with I_n/I_{n+k} replacing I^n/I^{n+1}). He showed that for "nice" filtrations results quite analogous to (4.1)–(4.4) hold in this more general situation.

The determination of what prime ideals are in $\hat{A}^*(I)$ or in $A^*(I)$ is usually difficult. The next few results ((4.5)–(4.11)) give some very useful information on this. The first of these that will be mentioned is actually the newest. It was proved in 1981, and it is mentioned first since some of its preliminary versions led to some of the other results that are mentioned.

THEOREM 4.5. [**45**, (2.9)] and [**46**, (2.4)]. *Let R and S be Noetherian rings such that S is R-flat and let P be a minimal prime divisor of $(I, z)S$, where I is an ideal in R and $z \in$ Ass S. Then*

(4.5.1) *$P \in A^*(IS)$, $P \cap R \in A^*(I)$, and $P \in A^*((P \cap R)S)$; and*

(4.5.2) *if z is minimal, then $P \in \hat{A}^*(IS)$, $P \cap R \in \hat{A}^*(I)$, and P is a prime divisor of $(I^n)_a S$ for all large n.*

The version of (4.5.2) in **[45]** has the additional hypothesis $(S_P/(\text{Rad } S_P))^*$ has no nonzero nilpotent elements. However, it is explained in **[46]** that a recent result of L. Burch **[9]** shows that this assumption is superfluous.

Specializing (4.5) to the case when (R, M) is local and S is the completion R^* of R leads to the following interesting result.

THEOREM 4.6. [**42**, Theorems 0 and 1]. *If (R, M) is a local ring, then*

(4.6.1) *there exists an integer n such that M is a prime divisor of all regular ideals $I \subseteq M^n$ if and only if there exist a depth one $z \in$ Ass R^*; and,*

(4.6.2) *there exists an integer m such that M is a prime divisor of I_a for all ideals $I \subseteq M^m$ such that* height $I \geqq 1$ *if and only if there exist a depth one minimal $z \in$ Ass R^*.*

(4.7) is a rather immediate consequence of (4.6). It should be noted that (4.7.1) is closely related to (3.5) and that (4.7.2) characterizes $\hat{A}^*(I)$.

THEOREM 4.7. [**48**, (2.6) and (2.7)]. *The following statements hold for a Noetherian ring R.*

(4.7.1) *If b is a regular nonunit in R and P′ is a prime divisor of bR', then $P' \cap R \in \hat{A}^*(I)$ for all ideals $I \subseteq P' \cap R$ such that* height $I \geqq 1$.

(4.7.2) *If I is an ideal in R, then $\hat{A}^*(I) = \{p' \cap R : p'$ is a (height one) prime divisor of $u\mathcal{R}(R, I)'\}$.*

The Noetherian domain version of (4.7) was given in 1981 in [**42**, Corols. 2 and 3]. Prior to this, McAdam and Eakin had shown (4.7.2) for the case when R is a locally quasi-unmixed Noetherian domain in [**20**, Prop. 18].

In 1980 K. Whittington gave another (but closely related to (4.7.2)) characterization of the elements in $\hat{A}^*(I)$.

THEOREM 4.8. [**64**, Theorem 2.4]. *If $I = (b_1, \ldots, b_h)R$ is an ideal in a Noetherian domain R, then $\hat{A}^*(I) = \{p' \cap R : p'$ is a (height one) prime divisor of $b_i(R[I/b_i])'$ for some $i = 1, \ldots, h\}$.*

Whittington used (4.8) to show two rather surprising results.

THEOREM 4.9. [**64**, Corollaries 2.8 and 2.9]. *Let I be an ideal in a Noetherian domain R and let $P \in \hat{A}^*(I)$. Then*

(4.9.1) *$P \in \hat{A}^*(IJ)$ for all nonzero ideals J in R, and*

(4.9.2) *there exists an integer n such that $P \in \hat{A}^*(J)$ for all ideals J in R such that $I \subseteq J \subseteq (I + P^n)_a$.*

The following result, which is closely related to (4.9.2), was proved by McAdam in 1980.

THEOREM 4.10. [**22**, Corollary 1.2]. *Let I be an ideal in a local domain (R, M), let S be an integral extension domain of R, and assume that S/IS contains a height zero maximal ideal. Then $M \in \hat{A}^*(J)$ for all ideas J in R such that $I \subseteqq J$.*

In 1980 in [**35**, (5.5) and (5.7)] Okon gave two fairly different versions of (4.10).

Another result along the same line as (4.6), (4.7.1), (4.9), and (4.10), but having to do with the asymptotic grade, grade*(I), is the following result proved in 1981.

THEOREM 4.11. [**50**]. *Let $I \subseteqq P$ be ideals in a Noetherian ring R such that P is prime.*

(4.11.1) *If* grade*(IR_P) = grade*(PR_P), *then $P \in \hat{A}^*(J)$ for all ideals J such that $I \subseteqq J \subseteqq P$.*

(4.11.2) *If I is generated by an asymptotic sequence and $P \in \hat{A}^*(I)$, then $P \in \hat{A}^*(J)$ for all ideals J such that $I \subseteqq J \subseteqq P$, and $P \in \hat{A}^*(K)$ for all ideals $K \subseteqq P$ such that* grade*(K) = grade*(I).

On a different but related subject, in 1979 McAdam and Eakin considered what could be said when $\hat{A}^*(I) = A^*(I)$ for all ideals I, and they proved the following theorem.

THEOREM 4.12. [**20**, Proposition 24]. *If R is a Noetherian domain of dimension two which is locally a UFD, then* Ass $R/(I^n)_a = \hat{A}^*(I) = A^*(I) =$ Ass R/I^n *for all ideals I in R and for all $n \geqq 1$.*

Actually, this property characterizes a local UFD of altitude two, as was shown in [**48**, (6.1)].

In 1980 McAdam proved the following related result.

THEOREM 4.13. [**21**, Theorem 6]. *If R is an integrally closed Noetherian domain of dimension two, then $\hat{A}^*(I) = A^*(I)$ for all ideals I in R.*

This author showed that a couple of other classes of local domains have this property in [**41**, Proposition 12] and about a year later essentially characterized the class of all such Noetherian domains by proving the following theorem.

THEOREM 4.14. [**42**, Theorem 4]. *Let $\mathscr{A}$ be the class of Noetherian domains R such that $\hat{A}^*(I) = A^*(I)$ for all ideals I in R.*

(4.14.1) *If* dimension $R > 3$, *then $R \notin \mathscr{A}$.*

(4.14.2) *If* dimension $R \leqq 3$ *and if one of the following three statements holds for each maximal ideal M in R, then $R \in \mathscr{A}$:*

(i) Height $M \leqq 1$;

(ii) Height $M = 2$ *and either* $R_M = (R_M)'$ *or there exists a height one maximal ideal in* $(R_M)'$; *or*

(iii) Height $M = 3$, *there exists a height one maximal ideal in* $(R_M)'$ *and for all height two prime ideals* $P \subset M$ *either* $R_P = (R_P)'$ *or there exists a height one maximal ideal in* $(R_P)'$.

(4.14.3) *The only other possible local domains in* $\mathscr{A}$ *are integrally closed local domains of dimension three which are not quasi-unmixed.*

It is still not known if there exist integrally closed local domains of dimension three in $\mathscr{A}$. However, T. Ogoma recently solved a long standing open problem by showing that there exist dimension three integrally closed local domains which are not quasi-unmixed [**34**]. Also, it is shown in [**48**, (7.1)] that if there exists a dimension three local UFD R which is not quasi-unmixed, then $R \in \mathscr{A}$. However, it is an important open problem if there exist such rings. It is important, since if they do exist, then the weakest of the catenary chain conjectures, the Normal Chain Conjecture (if the integral closure of a local domain R satisfies the first chain condition for prime ideals, then R satisfies the second chain condition for prime ideals) would be false.

Since the condition $\hat{A}^*(I) = A^*(I)$ is quite useful, it is somewhat disappointing that the set $\mathscr{A}$ in (4.14) is quite small. However, (4.15) shows that this idea can be used to give characterizations of larger classes of rings.

THEOREM 4.15. [**48**, (5.3)]. *Let* R *be a quasi-unmixed Noetherian ring. Then* R *is Cohen-Maculay if and only if* $\hat{A}^*(I) = A^*(I)$ *for all ideals* I *of the principal class in* R.

McAdam recently showed that asymptotic prime divisors can be used to give information on another subject that has been extensively studied, the Going-Down Theorem. For the statement of these results in (4.16), recall that prime ideals $P \subset Q$ in an integral domain R are said to satisfy *going-down* in case for all integral extension domains A of R and for all prime ideals Q^* in A that lie over Q there exists $P^* \in \operatorname{Spec} A$ such that $P^* \subset Q^*$ and $P^* \cap R = P$.

THEOREM 4.16. [**22**, Corollaries 2.2 and 2.3]. *The following statements hold for a prime ideal* P *in a Noetherian domain* R.

(4.16.1) *If* R *is local and quasi-unmixed and if* $\hat{A}^*(P) = \{P\}$, *then* $P \subset Q$ *satisfy going-down for all* $Q \in \operatorname{Spec} R$ *that contain* P.

(4.16.2) *If* $Q \in \operatorname{Spec} R$, *If* $P \subset Q$ *fail to satisfy going-down, and if height* $Q/P = 2$, *then for all but finitely many prime ideals* p *in* R *such that* $P \subset P \subset Q$ *it holds that* $Q \in \hat{A}^*(P)$. *For those* P *with* $Q \notin \hat{A}^*(p)$ *it holds that* $p \in \hat{A}^*(P)$.

(4.16.3) *There exist at most finitely many* $Q \in \operatorname{Spec} R$ *such that* $P \subset Q$, height $Q/P = 1$, *and* $P \subset Q$ *do not satisfy going-down.*

(4.16.3) is not given in [**22**]; it was communicated to me by McAdam in some personal correspondence.

This section will be closed with two additional results on $\hat{A}^*(I)$. These results are somewhat deeper and more technical than the preceding ones, but they are very useful, and in particular they play an essential role in showing that asymptotic sequences are an excellent analogue of R-sequences [**46**]. The first of these shows that $\hat{A}^*(I)$ is known if and only if $\hat{A}^*((I + z)/z)$ is known for all minimal prime ideals z in R.

THEOREM 4.17. [**48**, (8.3)]. *Let* I *be an ideal in a Noetherian ring* R *and let* $P \in \operatorname{Spec} R$.

(4.17.1) *If* $P \in \hat{A}^*(I)$, *then there exists a minimal prime ideal* z *in* R *such that* $z \subseteq P$ *and* $P/z \in \hat{A}^*((I + z)/z)$.

(4.17.2) *If* z *is a minimal prime ideal in* R *and* $Q \in \hat{A}^*((I + z)/z)$, *then there exists* $P \in \hat{A}^*(I)$ *such that* $z \subseteq P$ *and* $P/z = Q$.

The final result shows that $\hat{A}^*(I)$ is known if $\hat{A}^*(IS)$ is, where S is a flat R-algebra.

THEOREM 4.18. [**48**, (8.5) and (8.8)]. *Let* $R \subseteq S$ *be Noetherian rings such that* S *is a flat* R*-module and let* I *be an ideal in* R.

(4.18.1) *If* $P \in \hat{A}^*(I)$ *and* P^* *is a minimal prime divisor of* PS, *then* $P^* \in \hat{A}^*(IS)$.

(4.18.2) *If* $P^* \in \hat{A}^*(IS)$, *then* $P^* \cap R \in \hat{A}^*(I)$.

5. Analytic spread. (This section contains information on $\ell(I)$, the analytic spread of an ideal I, and its relation to asymptotic prime divisors. It begins with an inequality connecting $\ell(I)$ to grade and an asymptotic sequence version of this result (5.1) and (5.2), and then the use of $\ell(I)$ to determine $\hat{A}^*(I)$ in quasi-unmixed local domains is mentioned (5.3). Finally, a converse and partial generalization of (5.3) are mentioned in (5.4) and (5.5)).

The results in this section had their inception in the important and very useful 1954 paper [**31**] of Northcott and Rees. It was in this paper that they introduced the concepts of a *reduction* of an ideal I (an ideal $H \subseteq I$ such that $I^{n+1} = HI^n$ for some $n \geqq 1$) and of the analytic spread $\ell(I)$ of I. They proved many interesting properties of these concepts, and in particular they showed that if R is a local ring whose residue field is infinite and if H is a reduction of an ideal I in R, then $H_a = I_a$ and $\ell(I)$ is the number of elements in a minimal basis of a minimal reduction of I [**31**, Corl. p. 155 and Thm. 1, p. 150].

A number of papers concerning reductions have appeared since 1954, but here mention will be made only of the powerful existence theorem obtained in 1975 by Eakin and A. Sathaye. Namely, in [**13**, Thm.] it was shown that if (R, M) is a local ring with infinite residue field and I is an ideal in R such that I^n can be generated by less than $C_{n+s,s}$ elements, then any s "generic" linear combinations $y_1, \ldots, y_s$ of the elements in a basis of I satisfy $(y_1, \ldots, y_s)I^{n-1} = I^n$.

The analytic spread of an ideal has also been considered in a number of papers since 1954. In particular, during the period 1968–1972 work of E. Böger, G. Valla, and especially L. Burch in [**2**, **63**, **8**] showed that if I is an ideal in a local ring (R, M), then $\ell(I) \leqq$ dimension $R -$ min{grade (M/I^n): $n \geqq 1$}. Brodmann noticed a close connection between this result and his theorems concerning $A^*(I)$ and $B^*(I)$ (mentioned in (4.2)), and in 1980–81 he considerably sharpened this inequality of $\ell(I)$ by proving the following theorem.

THEOREM 5.1. [**4**], [**5**, (2.8) and (2.9)], and [**6**, (3.2) and (3.4)]. *If I is an ideal in a local ring (R, M), then* grade(I^{n-1}/I^n) *stabilizes for large n. Calling the stable value t^*, it holds that if I is not nilpotent, then $\ell(I) \leqq$* dimension $R - t^*$.

The connection between $\ell(I)$ and asymptotic prime divisors is not apparent from (5.1). In 1981 in [**59**] Rees made the connection somewhat more apparent by proving the following asymptotic sequence version of (5.1).

THEOREM 5.2. [**59**, Theorem 4.2]. *If I is an ideal in a local ring (R, M), then $\ell(I) \leqq$* dimension $R - h$, *where h is the maximum length of asymptotic sequences over I, and equality holds when R is quasi-unmixed.*

(In (5.2), $b_1, \ldots, b_h$ in M are an asymptotic sequence over I in case

$$((I, b_1, \ldots, b_{i-1})^n R)_a : b_i R = ((I, b_1, \ldots, b_{i-1})^n R)_a$$

for $i = 1, \ldots, h$ and for all large n.)

Even with Rees' result the connection between $\ell(I)$ and asymptotic prime divisors is still not very clear. This connection does exist, however, and it was clearly shown in 1980 by McAdam when he proved the following theorem.

THEOREM 5.3. [**21**, Theorem 3]. *Let $I \subseteqq P$ be ideals in a Noetherian domain R such that P is prime. If R is quasi-unmixed, then $P \in \hat{A}^*(I)$ if and only if $\ell(IR_P) =$* height P.

It follows from (5.3) (and the fact that $\ell(I) \geqq \ell(IR_P)$ always holds) that if Q is a primary ideal in a quasi-unmixed local domain such that $\ell(Q) =$ height Q, then $\hat{A}^*(Q) = \{\text{Rad } Q\}$; that is, $(Q^n)_a$ is height-unmixed for all $n \geqq 1$ by [**41**, Corol. 11].

The quasi-unmixed hypothesis is necessary in (5.3) and the extent to which the theorem fails otherwise was captured precisely by Whittington in the following theorem.

THEOREM 5.4. [**64**, Theorem 3.12]. *If $I \subseteqq P$ are ideals in a Noetherian domain R such that P is prime, then the following numbers are equal*:

(5.4.1) min{depth z: *z is a minimal prime ideal in* $(R_P)^*$};

(5.4.2) min{$\ell(IR_P)$ $P \in \hat{A}^*(I)$};

(5.4.3) min{height I: *I is of the principal class and* $P \in \hat{A}^*(I)$}; *and*

(5.4.4) min{$\dim_{k(P)} IR_P/PIR_P$: $P \in \hat{A}^*(I)$}, *where* $k(P) = R_P/PR_P$.

One direction holds in general, however, and so (5.3) can be sharpened to the following theorem.

THEOREM 5.5. [**48**, (9.4)]. *Let $I \subseteqq P$ be ideals in a Noetherian ring R such that P is prime. If* $\ell(IR_P) =$ height P, *then $P \in \hat{A}^*(I)$, and the converse holds when R is quasi-unmixed.*

6. I-independent elements. (This section contains a summary of the main results on generalized analytical independence. Among these are a characterization of: R-sequences, (6.1); ideals of the principal class, (6.2); asymptotic sequences, (6.3); unmixed Noetherian rings, (6.4); and, quasi-unmixed Noetherian rings, (6.5).)

Elements $b_1, \ldots, b_h$ in a proper ideal I in a ring R are *I-independent* in case every form f in $R[X_1, \ldots, X_h]$ such that $f(b_1, \ldots, b_h) = 0$ has all its coefficients in I. (Thus if (R, M) is a local ring, then $b_1, \ldots, b_h$ in M are M-independent if and only if they are analytically independent.)

Although this terminology was introduced by Valla in 1970 in [**63**], such elements had been studied prior to this in several papers. Perhaps their first use was by Rees in 1957 when he showed the following theorem.

THEOREM 6.1. [**56**, Corollary, p. 35], *If (R, M) is a local ring, then $b_1, \ldots, b_h$ in M are an R-sequence if and only if they are $(b_1, \ldots, b_h)$ R-independent.*

And in 1968 E.D. Davis used I-independence to characterize ideals of the principal class in the following theorem

THEOREM 6.2. [**12**, Theorem]. *An ideal I in a Noetherian ring R is of the principal class if and only if I can be generated by* (Rad I)*-independent elements.*

In 1981 this author showed the intimate connection between I-independent elements and asymptotic sequences by proving the following asymptotic version of (6.1).

THEOREM 6.3. [**49**]. *Elements $b_1, \ldots, b_h$ in the Jacobson radical of a Noetherian ring R are an asymptotic sequence if and only if $b_1^n, \ldots, b_h^n$ are $((b_1, \ldots, b_h)^n R)_a$-independent for all $n \geqq 1$.*

Related to (6.1) and (6.3), note that it follows immediately from (6.1) that $b_1, \ldots, b_h$ are $I = (b_1, \ldots, b_h)R$-independent if and only if their I-forms are algebraically independent over R/I in $\mathcal{F}(R, I)$, the associated graded ring of I. And it follows from [**45**, (2.1.8)] that $b_1 \ldots, b_h$ are I_a-independent if and only if their I-form images in $\mathcal{F}(R, I)/(\overline{I_a})$ are algebraically independent over R/I_a, where $(\overline{I_a})$ is the I-form ideal of I_a.

At least two other known results concerning I-independent elements are closely related to asymptotic prime divisors. The first of these (6.4) was proved by W. Bruns in 1980.

THEOREM 6.4. [**7**, Theorem 2 and Corollary 1].

(6.4.1) *If I is an ideal in a Noetherian ring R and if $i(I)$ denotes the maximum number of I-independent elements, then $i(I^n)$ stabilizes for large n and, with $i^*(I)$ denoting this stable value,* $i^*(I) = \min\{\text{height}(I(R_P)^* + z)/z$: *$P \in A^*(I)$ and $z \in$* $\text{Ass}(R_P)^*\}$.

(6.4.2) *A Noetherian ring R is unmixed if and only if* $i^*(I) =$ height *I holds for all ideals I in R.*

Because of several known properties of unmixedness and $A^*(I)$ and the analogous properties of quasi-unmixedness and $\hat{A}^*(I)$ this author suspected that the quasi-unmixed analogue of (6.4) should be true. After struggling with this for several weeks in 1980 this author finally proved the following theorem.

THEOREM 6.5. [**45**, (2.12.2) and (3.7)].

(6.5.1) *If I is an ideal in a Noetherian ring R and if $i(I)$ is as in* (6.4.1), *then $i((I^n)_a)$ stabilizes for large n and, with $\hat{\imath}^*(I)$ denoting this stable value,* $\hat{\imath}^*(I) = \min\{\text{height}(I(R_P)^* + z)/z$: *$P \in \hat{A}^*(I)$ and z is a minimal prime ideal in* $(R_P)^*\}$.

(6.5.2) *A Noetherian ring R is quasi-unmixed if and only if* $\hat{\imath}^*(I) =$ height *I holds for all ideals I in R.*

In closing this section, for some form of completeness it should be noted that in [**1**, Prop. 8] Cohen-Macaulay rings were characterized as Noetherian rings R such that $i(I) =$ grade (I) holds for all ideals I in R, where $i(I)$ is as in (6.4.1).

7. P. Samuel's paper on asymptotic properties of powers of ideals. (This section is mainly historical, but two recent results are mentioned; one concerning Rees' Valuation Theorem [**54**] and the other his characterization of analytically unramified local rings [**58**].)

In 1951 in [**62**] Samuel proved a number of results concerning large powers of ideals in a Noetherian ring R, and in particular he therein introduced a new equivalence relation, *asymptotic equivalence*, between two ideals in R by saying I s J in case there exist sequences $\{v(n)\}$ and $\{w(n)\}$ of positive integers such that $I^{v(n)} \subseteq J^n \subseteq I^{w(n)}$ for all n and $\lim\{v(n)/n: n \geqq 1\} = \lim\{w(n)/n: n \geqq 1\} = 1$. (This is Nagata's formulation in **[29]** of this equivalence relation.) It is shown in **[62]** and **[29]** that if I_1 s I_2 and J_1 s J_2, then I_1J_1 s I_2J_2 and $I_1 + J_1$ s $I_2 + J_2$, and if IK s JK and either K contains a regular element or $I + J \subseteq \text{Rad } K$, then I s J. By the Noetherian hypothesis it follows that each equivalence class contains a largest element, denoted I_s and called the *asymptotic closure* of I, and it follows from the properties listed above that $I \to I_s$ is a semi-prime operation on the ideals of R. (That is, for all ideals I and J: $I \subseteq I_s$; $I \subseteq J \Rightarrow I_s \subseteq J_s$; $(I_s)_s = I_s$; and $I_sJ_s \subseteq (IJ)_s$. From this it follows that $(I_sJ_s)_s = (IJ)_s$, $(\sum I_j)_s = (\sum(I_j)_s)_s$, and $\bigcap(I_j)_s = (\bigcap(I_j)_s)_s$.)

Samuel's paper created considerable interest and some of the important results obtained in subsequent papers bear directly on the subject of the present paper, while some others bear supplementally, so a few of these results will now be briefly mentioned.

Due to the work of H. T. Mulhy, Rees, M. Sakuma, Nagata, and J. W. Petro in **[23, 51, 61, 29, 36]** the following results emerged in the period 1954–64. For I an ideal in a ring R. (1) If R is an integral domain, then $I_s \supseteqq I_a = I_b\ (=(\bigcap IR_v) \cap R$, where the intersection is over all valuation rings R_v in the quotient field of R that contains R), and it is possible that $I_s \supset I_a$ even for finitely generated I when R is not Noetherian. (2) If R is a Noetherian ring, then $I_s = I_a = I_b$ (where the definition of I_b is suitably modified).

In 1955 in **[51]** Rees approached asymptotic equivalence and asymptotic closure in a different manner. Therein he defined for each ideal I in a ring R a function from R into the extended non-negative real numbers by $v_I(x) = n$ in case $x \in I^n$, but $x \notin I^{n+1}$, and $v_I(x) = \infty$ in case $x \in I^n$ for all n. He then showed that $(\lim v_I(x^n))/n$ exists for all $x \in R$ and that the function $\bar{v}_I$ defined by $\bar{v}_I(x) = (\lim v_I(x^n))/n$ is a homogeneous pseudo-valuation. When R is Noetherian, he showed that I s J if and only if $\bar{v}_I(x) = \bar{v}_J(x)$ for all $x \in R$ and that $I_s = \{x \in R\colon \bar{v}_I(x) \geqq 1\}$. This approach led him in this and three subsequent papers **[52, 53, 54]** to two beautiful and important theorems, namely the Valuation Theorem and the Strong Valuation Theorem.

THEOREM 7.1. (THE VALUATION THEOREM, **[54]**). *For each ideal I in a Noetherian ring R there exists a finite set of discrete integrally valued valuations $v_1(x), \ldots, v_k(x)$ on R and integers $e_1, \ldots, e_k$ such that $\bar{v}_I(x) = \min\{e_i^{-1}v_i(x)\colon i = 1, \ldots, k\}$ and $\{x \in R\colon v_i(x) = \infty\}$ is a minimal prime ideal in R.*

THEOREM 7.2. (THE STRONG VALUATION THEOREM, **[53]**). *If R is a local ring such that R^* has no nonzero nilpotent elements and if I is any ideal in*

R, then there exists an integer $t(I)$ such that $0 \leqq \bar{v}_I(x) \dot{-} v(x) \leqq t(I)$ for all nonzero x in R.

The definition of $\bar{v}_I(x)$, mentioned prior to (7.1), clearly indicates that there should be a close connection between the valuations $v_1(x), \ldots, v_k(x)$ in (7.1) and the asymptotic prime divisors of I, and this connection seems even more apparent when it is noted that the prime divisors of $u\mathscr{R}(R, I)'$ play an important role in proving (7.1) and (7.2) and in determining the asymptotic prime divisors of I. Of course, an explicit statement of this connection would be highly desirable, since results in each of these areas would then illuminate the other area. This connection was made explicit in the summer of 1981 when the following result was proved.

THEOREM 7.3. [**48**, (3.1)]. *If I is an ideal in a Noetherian ring R, then $\hat{A}^*(I) = \{Q$: Q is the center of some valuation associated with I in the Valuation Theorem$\}$.*

(The *center* in R of a valuation v on R is the set $\{r \in R: v(r) > 0\}$.)

One example of the usefulness of (7.3) will now be mentioned. (7.2) led to Rees' important and useful characterization [**58**, Thms. 1.2 and 1.5] in 1961 of analytically unramified local rings.

THEOREM. *A local ring R is analytically unramified (that is, R^* has no non-zero nilpotent elements) if and only if S′ is a finite S-module for all finitely generated rings $S = R[u_1, \ldots, u_h]$ over R contained in the total quotient ring of R.*

In the process of proving this characterization he showed that if there exists an open ideal I in R and an integer n such that $(I^{n+i})_a \subseteqq I^i$ for all $i \geqq 1$, then R is analytically unramified, and if R is analytically unramified, then every ideal I in R has this property for some n depending on I.

Using this as a guide, this author recently showed the following theorem.

THEOREM 7.4. [**47**, (13)]. *If R is a reduced local ring and there exists an ideal I in R generated by a system of parameters and an integer n such that $I^{[n+i]} \subseteqq I^i$ for all $i \geqq 1$, then R is unmixed. Conversely, if R is unmixed, then every ideal I in R has this property for some n depending on I.*

(In (7.4), $I^{[n]}$ is the intersection of the height one primary components of u^n in $\mathscr{R}(R, I)$ contracted to R.)

Using (7.4), an overring characterization of locally unmixed Noetherian rings analogous to Rees' characterization of analytical unramification was given in [**47**, (20)].

Samuel's paper, together with Rees' results mentioned above, led to several other results, and among these are the 1963 papers [**24**, **25**] of Muhly and Sakuma in which they generalized O. Zariski's factorization theory for integrally closed ideals in regular local rings of dimension two [**66**, Appendix 5]. This was a beautiful piece of mathematics, and methods developed there have certainly proved useful in the present subject matter.

The results mentioned in this section, together with the explanation in §2 of the definition of asymptotic sequences, show that Samuel's paper and the results it inspired have played a very important role in the development of the results on asymptotic prime divisors.

8. Final comments and some questions. In §2–§7 a particular theorem or paper was singled out as being in some way preeminent, and then several other results which chronologically followed were briefly discussed. Probably another author would have grouped the results in this area in a different way, but this author believes there would be a very large overlap in the results common to such a paper and the present one.

It is almost certain that this paper has not mentioned some important results, but those included give a fairly complete historical summary of the sub-topics discussed and also give the reader a good overview of this area.

A discussion of symbolic powers of prime ideals has been completely omitted, and, admittedly, this is a big omission. This subject is clearly closely connected to asymptotic prime division, and the reader would certainly find many useful results and ideas in such papers as [**11**, **14**, **15**, **16**, **60**] (to mention only a few of the many possible), but for the most part these papers seem to have very little overlap with the ideas and results discussed above. So, although this author considers symbolic powers a very interesting and important topic, his reason for not including them is that at present this area seems to have few results that blend in well with the other results in this paper.

There are quite a few open problems which could be mentioned here, but several of these go beyond the results mentioned above, so only the following nine specific questions will be listed.

(8.1) It is mentioned before (2.7) that asymptotic sequences are an excellent analogue of R-sequences. The literature contains many results on R-sequences and on grade (I). To what extend are the asymptotic versions of these results valid?

(8.2) (3.3) contains a characterization of Cohen-Macaulay local domains. Is the quasi-unmixedness assumption necessary? That is, if there exists a depth one integrally closed ideal of the principal class in $R(X_1, \ldots, X_d)$, then is R Cohen-Macaulay? (This was asked in [**43**, (18)].)

(8.3) If R is an unmixed local ring and $b_1, \ldots, b_h$ are an asymptotic sequence in R, then does there exist an integer n such that $b_1^{n+i}, \ldots, b_h^{n+i}$ are $(b_1, \ldots, b_h)^i R$-independent for all $i \geqq 1$? (This is closely related to (6.3).)

(8.4) It was shown in (4.1) that the sets Ass $R/(I^n)_a$ are monotonically increasing, and it was shown in [3] that this is not true for the sets Ass R/I^n. If $P \in A^*(I)$ and $P \in$ Ass R/I^k for some $k \geqq 1$, then is $P \in$ Ass R/I^n for all $n \geqq k$? (This was asked in [48, (4.1)].)

(8.5) Characterize the ideals in $B^*(I) \cap \{P: I \subseteqq P \in \text{Ass } R\}$. (See (4.4) and its succeeding comment.)

(8.6) (McAdam) If (R, M) is a local domain and $M \in A^*(I)$ for all nonzero ideals I in R, then does there exist a depth one prime divisor of zero in R^*? (The converse is true; see (4.6.1).)

(8.7) Are there any integrally closed local domains of dimension three in the set $\mathscr{A}$ of (4.14)? Does there exist a dimension three local UFD in $\mathscr{A}$? (See the comment following (4.14).)

(8.8) Does (7.4) hold without assuming R is reduced? If R is an unmixed local ring, then is $\bigcap \{R_{(p)} : (p)$ is the set of regular elements in $R - p$ and p is a height one prime ideal in $R\}$ a finite R-algebra? (If the answer to the second question is yes, then the answer to the first question is also yes.)

(8.9) Given a finite set S of positive integers, do there exist a Noetherian ring R and ideals $I \subseteqq P$ in R such that P is a prime divisor of I^k if and only if $k \in S$?

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