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# ASYMPTOTIC BEHAVIOUR OF A REACTION-DIFFUSION EQUATION IN HIGHER SPACE DIMENSIONS

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ABSTRACT. The reaction-diffusion equation considered has a travelling wave solution in one space dimension for which strong stability results have been proved by Fife and McLeod [3]. In this paper it is proved that a certain class of solutions of this equation, in higher space dimensions, approach this one-dimensional travelling wave when followed out along any ray.

**1. Introduction.** In this paper I extend a theorem of Jones [4]. The result concerns the reaction-diffusion equation:

$$u_t = \Delta u + f(u), \tag{1.1}$$

where $u \in \mathbf{R}$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$. Here $f$: $\mathbf{R} \to \mathbf{R}$ is assumed to be smooth and to have the cubic-like form depicted in Fig. 1.

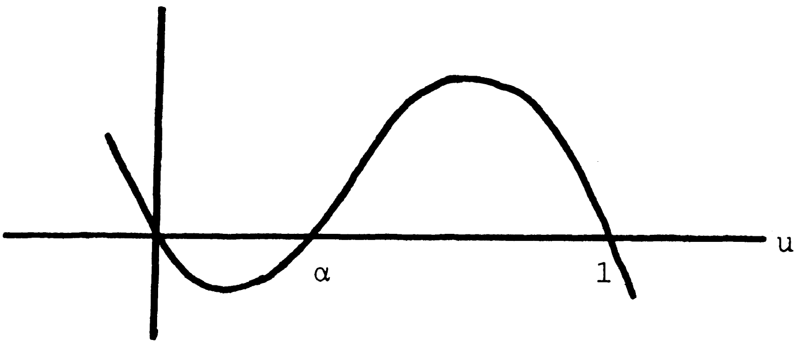


Fig. 1

Specifically it has three zeroes 0, $\alpha$ and 1, with $f'(0) < 0$, $f'(1) < 0$ and $\int_0^1 f(u)du > 0$. An initial value problem is naturally associated with (1.1):

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a solution $u(x, t)$, for $t \geqq 0$, should be determined by an initial condition

$$u(x, 0) = u(x)\,. \tag{1.2}$$

This equation lives under a variety of names; one of the most appropriate mathematically is the 'Bistable Equation'. It is so named because the constant states 0 and 1 are both stable in the supremum norm.

The case of interest here is when the spatial domain is all of $\mathbf{R}^n$ and $n > 1$. If $n = 1$, there is a travelling wave solution for which Fife and McLeod [3] have proved a strong stability result. A travelling wave is a solution which has the special form $u(x, t) = u(x - ct)$ for some fixed $c$. For (1.1) there is a travelling wave with $c > 0$ which satisfies the boundary conditions $u(-\infty) = 1$ and $u(+\infty) = 0$. I shall call this $\bar{u}$ and its speed $\bar{c}$.

In [4] the following result was proved for spherically symmetric solutions (any $n$). If $u(r, t)$ is a spherically symmetric solution with $\lim_{r\to+\infty} u(r, 0) < \alpha$ (the middle zero of $f$), $u(r, 0)$ nonincreasing in $r$, and $u(r, t) \to 1$ uniformly on compact sets as $t \to +\infty$, then there is a function $g(t) = o(t)$ such that

$$\lim_{t\to+\infty} \|u(\xi + \bar{c}t + g(t), t) - \bar{u}(\xi)\| = 0 \tag{1.3}$$

where the norm is the supremum over $\xi + \bar{c}t + g(t) \geqq 0$. This says that if followed out in a radial direction at the correct speed, the solution approaches the one-dimensional travelling wave, at least in shape.

The theorem proved in this paper extends this result to solutions that are not necessarily spherically symmetric but initially have compact support.

THEOREM. *If* $u(x, t)$ *satisfies* (1.1), (1.2) *such that*

(a) $u(x)$ *has compact support and*

(b) $u(x, t) \to 1$ *as* $t \to +\infty$ *uniformly on compact sets,*

*then for any* $v \in S^n$, *there is a* $g(t) = o(t)$ *such that*

$$\lim_{t\to+\infty} \|u(\{\xi + \bar{c}t + g(t)\}v, t) - \bar{u}(\xi)\| = 0 \tag{1.4}$$

*where the norm is the supremum over* $\xi + \bar{c}t + g(t) \geqq 0$.

A remark should be made on when (b) is satisfied. Aronson and Weinberger [1] give conditions on $u(x)$ that guarantee $u(x, t) \to 1$ uniformly on compact sets. They describe a family $U$ of comparison functions such that if $u(x) \geqq v(x)$ for some $v(x) \in U$, then $u(x, t) \to 1$ uniformly on compact sets. The construction of these comparsion functions is fairly complicated and I shall not give it here.

An alternative class of comparison functions is implicit in [4]. In that work a non-trivial solution to the steady-state spherically symmetric equation

$$u_{rr} + \frac{n-1}{r} u_r + f(u) = 0 \tag{1.5}$$

is constructed. This solution itself serves as a comparison function, but so does any function $v(x)$ for which the associated solution $v(x, t)$ tends to such a steady-state. In other words if $u(x) \geqq v(x) \quad (u(x) \not\equiv v(x))$ and $v(x, t) \to \bar{u}(r)$, where $\bar{u}$ satisfies (1.5), then $u(x, t) \to 1$ uniformly on compact sets.

Aronson and Weinberger [1] have proved that solutions satisfying (a) and (b) propagate with an asymptotic speed. If followed slower than $\bar{c}$, a solution tends to 1; if faster than $\bar{c}$, a solution tends to 0, This theorem strengthens their result.

The theorem says that under the conditions mentioned $u(x, t)$ is an expanding lump that approaches the one-dimensional travelling wave in each direction. There is, however, more information in the proof of the theorem about the shape of the front of this lump, namely that it has to be roughly spherical.

The extra information is contained in the proof of Lemma 1, see §3. It is proved there that the normal line to a level surface $u(x_0, t) =$ constant at $x_0$ ($t$ is fixed) must intersect the support of $u(x, 0)$. Consider then, for large fixed $t$, the level surface $u(x, t) = \alpha$ in $\mathbf{R}^n$ (it needs to be proved that this exists globally if $t$ is large but this is possible). This gives the shape of the front at that $t$. All the normals to this surface must intersect the support of $u(x, 0)$, see Fig. 2.

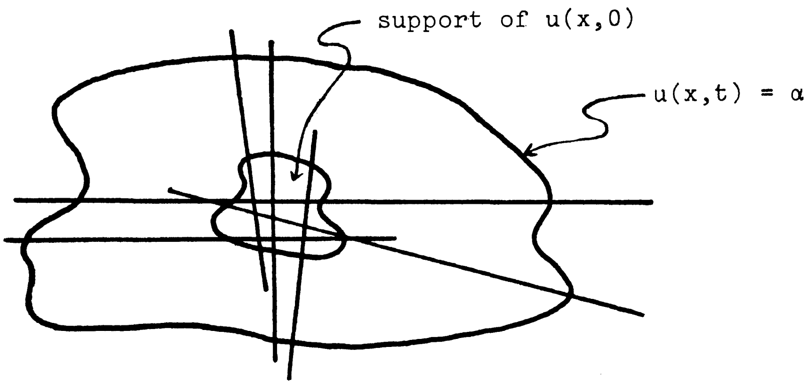


Fig. 2

As $t$ becomes large the support of $u(x, 0)$ looks progressively smaller and this condition on the normals forces the shape of $u(x, t) = \alpha$ to become rounder. Exactly what this says about the shape of the level surface is not addressed here; it would be an interesting problem to formulate consequences of this feature.

As in [4], the proof of the theorem is based on techniques of dynamical systems. In §2, the language and definitions I need from dynamical systems will be presented. §3 contains the proof of the theorem.

**2. Dynamical systems framework.** Let $C(\mathbf{R}^n, \mathbf{R}) = C$ be the space of bounded uniformly continuous functions $u$: $\mathbf{R}^n \to \mathbf{R}$ endowed with the supremum norm. It is well known that the solutions of (1.1), (1.2) generate a local semiflow on $C$; denote this by $S$. So $S$ is a function on $C \times [0, \infty)$ into $C$ and $S(u(x), t) = u(x, t)$ where $u(x, t)$ satisfies (1.1) and (1.2). That $S$ is a semiflow means (a) $S(S(u(x), \tau), t) = S(u(x), \tau + t)$ and (b) $S$ is continuous. If $S$ is a local semiflow, the domain need not be all of $C \times [0, \infty)$, but will be open and have the property that for each $u \in C$ there is an $s$ so that $\{u\} \times [0, s)$ is in the domain. The conditions for a semiflow are merely a restatement of existence, uniqueness and continuity. A semiflow is local if only local existence is true.

Consider the following subset of $C$: $M = \{u \in C \mid 0 \leqq u(x) \leqq 1\}$. The constant states $u \equiv 0$ and $u \equiv 1$ are solutions of (1.1), so by the maximum principle if $u(x, 0) \in M$, then $u(x, t) \in M$ for all $t \geqq 0$. From this it follows that $S$ is defined on all of $M \times [0, \infty)$. In the following I shall use the notation $u \cdot t$ for $S(u, t)$.

The compact-open topology on $C$ (an open neighbourhood of 0 is a set of the form $\{u \in C \mid \ |(u(x)| < \varepsilon \text{ for } x \in K\}$ where $K$ is a compact set) is more appropriate for the present purpose than the sup-norm. It is proved in [4] that if the topology on $M$ is replaced by the compact-open topology, $S$ is still a semiflow. From here on, $M$ will be assumed to carry this topology.

From standard derivative estimates, the semiflow $S$ on $M$ has the property that $M \cdot [\tau, \infty)$ is precompact for all $\tau > 0$. The estimates are of the form

$$\sup_{\substack{x \in \mathbf{R}^n \\ t \in [\tau, \infty)}} |\nabla u(x, t)| \leqq k \sup_{\substack{x \in \mathbf{R}^n \\ t \geqq 0}} |u(x, t)| \tag{2.1}$$

where $k$ depends on $\tau$, see for instance [4] or, as an application of the Schauder estimates, Fife [2]. The precompactness of $M \cdot [\tau, \infty)$ then follows from the Arzela-Ascoli Theorem. A semiflow with this property I shall call compact.

For the purpose of introducing some definitions, suppose that $S$ is a compact semiflow on a metric space $Y$. A subset $B \subset Y$ is invariant if $B \cdot t = B \cap (Y \cdot t)$ for all $t \geqq 0$, and positively invariant if $B \cdot t \subset B$ for all $t \geqq 0$. The notion of an $\omega$-limit set is used to locate asymptotic behaviour. Again let $B \subset Y$ then

$$\omega(B) = \bigcap_{t \geq 0} \mathrm{cl}(B \cdot [t, \infty)) \tag{2.2}$$

is called the $\omega$-limit set of $B$.

An isolated invariant set is a compact invariant set that is the maximal invariant set in some neighbourhood of itself. An attractor is an isolated invariant set that is the $\omega$-limit set of some neighbourhood of itself. The neighbourhoods performing these functions will be called isolating and attracting neighbourhoods respectively. They will always be closed sets with the relevant invariant set in their interior.

For viewing the asymptotic behaviour of solutions of (1.1) while moving our along a ray, at a certain speed, a two-parameter family of semiflows will be needed. Let $c \in \mathbf{R}$, $v \in S^n$, and define the translation operator $T[ct, v]$ for any $t \in \mathbf{R}$, as operating on elements $u(x) \in C(\mathbf{R}^n, \mathbf{R})$, by

$$T[ct, v]\, u(x) = u(x + (ct)v)\,. \tag{2.3}$$

This translates $u$ by an amount "$-ct$" in the direction "$v$".

Fixing $c \in \mathbf{R}$ and $v \in S^n$, a semiflow can now be constructed by following $S$ with $T[ct, v]$, call this $S_{c,v}$. Restriciting to $M$, $S_{c,v}\colon M \times [0, \infty) \to M$ is therefore given by the formula

$$\begin{aligned} S_{c,v}(u(x), t) &= T[ct, v]S(u(x), t) \\ &= u(x + (ct)v, t)\,. \end{aligned} \tag{2.4}$$

Using the continuity of translation in the compact topology and the translation invariance of the estimates (2.1) it is not hard to check that $S_{c,v}$ is also a compact semiflow. It is the evolution of the equation as seen while moving out in the direction $v$ with speed $c$.

Define the set $M_v \subset M$ as

$$M_v = \{u \in M \,|\, u(x) = u(x + \eta) \text{ for any } \eta \text{ such that } \eta \cdot v = 0\}\,. \tag{2.5}$$

Elements of $M_v$ can be expressed as functions of a scalar times $v$; denoting that scalar variable by $\xi$, we obtain that $u \in M_v$ implies $u = u(\xi v)$. $M_v$ is invariant relative to $M$ under the semiflow $S_{c,v}$ and in the variable $\xi$ this semiflow is associated with the equation

$$u_t = u_{\xi\xi} + cu_\xi + f(u)\,. \tag{2.6}$$

Let $c_1$, $c_2$ satisfy the inequalities

$$0 < c_1 < \bar{c} < c_2\,. \tag{2.7}$$

With $v$ fixed, the semiflows $S_{c,v}$ can be piled together to form yet another semiflow on $[c_1, c_2] \times M$; call this $S_v$ (*I* shall soon fix $c_1$ and $c_2$ so they need not appear in the name of this semiflow). The action of $S_v$ on $[c_1, c_2] \times M$ is

$$S_v((c, u(x)), t) = (c, S_{c,v}(u(x), t)) . \tag{2.8}$$

Recall that $\bar{u}(\xi)$ is the travelling wave of (1.1) with $n = 1$ and $\bar{u}(-\infty) = 1$, $\bar{u}(+\infty) = 0$. Let

$$W = \{0, 1\} \cup \left\{ \bigcup_{p \in \mathbf{R}} \bar{u}(\xi + p) \right\}.$$

Then $W$ is a compact curve in $M_v$ running from 0 to 1. Denote by $\bar{M}_v$ the subset of $M_v$ consisting of functions that are non-increasing in $\xi$. Theorem 5.2 of [4] asserts that $W$ is an attractor in $\bar{M}_v$ under the semiflow $S_{c,v}$ for $c \in [c_1, c_2]$, with some $c_1$, $c_2$ satisfying (2.7). Furthermore, if $S_v$ is restricted $[c_1, c_2] \times \bar{M}_v$, $[c_1, c_2] \times W$ is an attractor and it carries the flow depicted in Fig. 3.

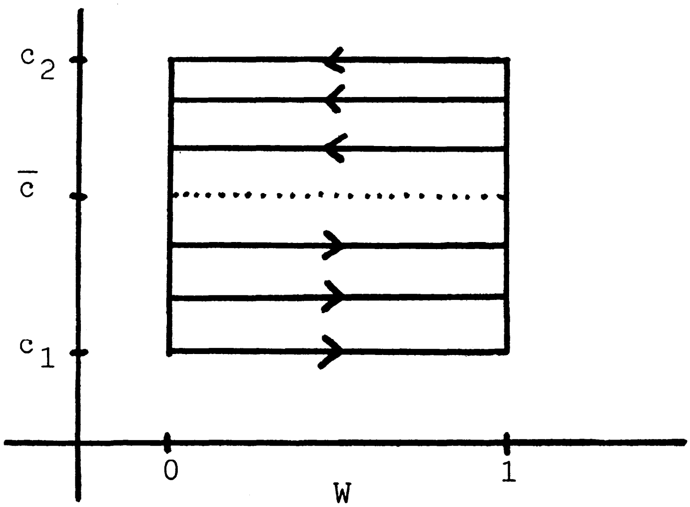


Fig. 3

Henceforth fix $c_1$ and $c_2$ so that the above statements are true. These are the tools needed to prove the theorem.

**3. Proof of theorem.** The action of the semiflow $S_{c,v}$ on $M$ shall be denoted by "$\cdot_{c,v}$" and $\omega_{c,v}(B)$ refers to the $\omega$-limit set of $B$ under $S_{c,v}$. Let $u(x)$ satisfy the hypotheses of the theorem. I will show that $\omega_{c,v}(u(x)) \subset \bar{M}_v$. I use a technique due to Serrin [6] that this author learned from the paper by Nirenberg, Gidas and Ni [5].

The $\omega$-limit set $\omega_{c,v}(u(x))$ consists of limit points, in the compact-open topology, of sequences of the form $u(x + (ct_n)v, t_n)$. Let $K$ be a compact neighbourhood of 0 in $\mathbf{R}^n$. As far as the resting flow $S$ is concerned this is a neighbourhood of the point $(ct_n)v$. To see that $u(x + (ct_n)v, t_n)$ tends uniformly on such a set to an element of $M_v$, it is natural to consider $\nabla u(x, t)\cdot\eta$ on $K + (ct_n)v$ where $\eta\cdot v = 0$. I first consider $\nabla u$ on the complement of a large disc (note that all gradients are only in the spatial variables).

Let $D_R$ be the disc of radius $R$ and $D_R^c$ its complement. I will use the following lemma about nonsingular points in $D_R^c$.

LEMMA 1. *Let $x \in D_R^c$ such that $\nabla u(x, t) \neq 0$. Then*

$$\left| \frac{\nabla u}{|\nabla u|} + \frac{x}{|x|} \right| < \varepsilon(R) \tag{3.1}$$

*where $\varepsilon$ depends only on $R$ (not on $t$) and $\varepsilon(R) \to 0$ as $R \to +\infty$.*

PROOF. To prove this $I$ use the technique of Serrin [6] mentioned earlier. A hyperplane $P$ in $\mathbf{R}^n$ is determined by a vector $\lambda \in \mathbf{R}^n$ where $\lambda$ is its normal and $P$ sits a distance $|\lambda|$ from the origin in the direction of $\lambda$. The set $\mathbf{R}^n/P$ has two components; let $\Omega$ be one of the components. Also if $u$: $\mathbf{R}^n \to \mathbf{R}$, let $u^\lambda(x)$ be the reflection of $u(x)$ in $P$.

The function $w(x, t) = u^\lambda(x, t) - u(x, t)$ satisfies a parabolic equation of the form $w_t = \Delta w + a(x, t)w$ by the Mean Value Theorem; also $w(x, t) = 0$ on $\partial\Omega(=P)$. Therefore by the maximum principle if $w(x, 0) \geqq 0$ in $\Omega$, $w(x, t) \geqq 0$ in $\Omega$ for all $t \geqq 0$. If $P$ is any plane not intersecting the support of $u(x)$, each condition is satisfied on one side of $P$ and so in $\Omega$

$$u^\lambda(x, t) \geqq u(x, t) \tag{3.2}$$

for all $t \geqq 0$. Furthermore by the strong maximum principle

$$\nabla\{u^\lambda(x, t) - u(x, t)\}\cdot\lambda \neq 0 \tag{3.3}$$

for every $x \in P$ and $t > 0$.

Suppose now that $\nabla u(x_0, t) \neq 0$. Then the level surface $u(x_0, t) =$ constant has a well defined normal at $x_0$, given by $\nabla u/|\nabla u|$. I claim that the following two statements are true if $R$ is large enough, where $I$ is a ball containing the support of $u(x)$:

(a) the normal line determined by $\nabla u/|\nabla u|$ must intersect $I$, and

(b) the normal $\nabla u/|\nabla u|$ points towards $I$.

Inequality (3.1) can easily be deduced from (a) and (b) by elementary geometric considerations; so $I$ need only prove (a) and (b). To prove (a), suppose the normal line did not intersect $I$. Then a plane $P$ could be found which was also disjoint from $I$. But then (3.3) must hold on $P$. Since $\nabla u^\lambda(x, t)\cdot\lambda = -\nabla u(x, t)\cdot\lambda$ in $P$, $\nabla u(x, t)\cdot\lambda \neq 0$ for $x \in P$, but this contradicts the fact that the normal line to the surface $u(x_0, t) \doteq$ constant lies in $P$.

To prove (b), the tangent plane to the level surface $u(x_0, t) =$ constant does not intersect $I$ if $R$ is large (by (a)), and so can act as such a $P$. But if (b) were false, (3.2) would clearly be violated in the associated $\Omega$. This completes the proof of the lemma.

It can now be seen that as $t_n \to +\infty$, $\nabla u(x, t_n)\cdot\eta \to 0$ uniformly on $K$ (which is a neighbourhood of $(ct_n)v$). From (3.1) $\nabla u/|\nabla u| = -x/|x| + z$ where $|z| \to 0$ as $t_n \to +\infty$, and so

$$\begin{aligned}\nabla u\cdot\eta &= |\nabla u|\left[\frac{\nabla u}{|\nabla u|}\cdot\eta\right]\\ &= |\nabla u|\left[-\frac{x}{|x|}\cdot\eta\right] + |\nabla u|(z\cdot\eta)\,.\end{aligned}$$

Since $|\nabla u|$ is bounded by (2.1), $\nabla u\cdot\eta \to 0$ so long as $x/|x|\cdot\eta \to 0$. This last statement is easily seen to hold uniformly for $x \in K$ as $K$ is a neighbourhood of $(ct_n)v$ and $t_n \to \infty$.

Let $u(x + (ct_n)v, t_n) \to w(x)$ uniformly on compact sets as $t_n \to +\infty$. By (2.1) convergence in the compact-open topology actuallys entail convergence in the $C^1$-compact-open topology (an open neighbourhood of 0 is all $C^1$-functions satisfying $\sup\{|u(x)|, |\nabla u(x)|\} < \varepsilon$ for $x \in K$ where $K$ is a compact set and $\varepsilon > 0$). Therefore $\nabla u(x + (ct_n)v, t_n)\cdot\eta \to \nabla w(x)\cdot\eta$ as $t_n \to \infty$ uniformly on compact sets, and this is true for any $\eta$ perpendicular to $v$. But $\nabla u(x + (ct_n)v, t_n)\cdot\eta \to 0$ and so $\nabla w(x)\cdot\eta = 0$ for any such $\eta$, it follows that $w(x) \in M_v$. That $w(x)$ is in fact in $\bar{M}_v$ follows easily from (b) in Lemma 1.

By the same argument $\omega_v(u(x)) \subset [c_1, c_2] \times \bar{M}_v$. Also these statements all remain true if $u(x)$ is replaced by $u(x, t)$ or even $u(x + kv, t)$ where $k$ is any scalar.

Something more must be tapped from Lemma 1. Consider a neighbourhood $X$ of $\bar{M}_v$, given by a compact set $K$ and a number $\varepsilon > 0$, defined by $X = \{u \in M|$ there is a $v(x)$ in $\bar{M}_v$ such that $|u(x) - v(x)| < \varepsilon$ for every $x \in K\}$. It is trivial that $u(x + kv) \in X$ if $k$ is sufficiently large. But it is also true that $\varepsilon$ and $K$ can be chosen, depending on $k$, so that it remains true under application of $S$. Lemma 1 gives a uniform estimate on $\nabla u(x, t)\cdot\eta$ for $x \in K + kv$ and independent of $t \geqq 0$. As long as $\eta\cdot v = 0$, this can be used to get an estimate on how far $u(x + kv, t)$ is from $\bar{M}_v$ and it is independent of $t$. Moreover $\varepsilon \to 0$ as $k \to +\infty$. This means that given $X$, $k$ can be chosen so that $S(u(x + kv), t) \in X$ for all $t \geqq 0$. Applying the translation operator $T[ct, v]$ only improves the estimate, so this is also true with $S$ replaced by $S_{c,v}$.

Consider the sets $U(a, \varepsilon) = \{u \in M|u(av) < \varepsilon\}$ and $V(b, \delta) = \{u \in M|u(bv)$

$> \delta\}$. In [4] it is shown that if $a < b$ is large enough and $\varepsilon < \alpha < \delta$, then $U(a, \varepsilon) \cup V(v, \delta)$ is an attracting neighbourhood for $W$ in $\bar{M}_v$ under any of the semiflows $S_{c,v}$, with $c \in [c_1, c_2]$ (see Theorem 5.2). Also if $c < \bar{c}$, $V(b, \delta)$ is an attracting neighbourhood for 1 and if $c > \bar{c}$, $U(a, \varepsilon)$ is an attracting neighbourhood for 0.

Unfortunately $U$ and $V$ do not necessarily have these properties when viewed in the whole space $M$. The following dynamical systems lemma will substitute for this. Suppose $S$ is a compact semiflow on a metric space $Y$. Let $A \subset Y$ be invariant relative to $Y$ and $B \subset A$ an attractor for the semiflow restricted to $A$. If $N$ is an attracting neighbourhood for $B$, let $\bar{N}$ be a neighbourhood in $Y$ so that $\bar{N} \cap A = N$. With this notation I can state the lemma.

LEMMA 2. *There is a neighbourhood $X$ of $A$ so that if $y \in X \cap \bar{N}$ satisfies the two properties* (1)$y \cdot t \in X$ *for all* $t \geqq 0$, *and* (2) $\omega(y) \subset A$, *then* $\omega(y) \subset B$.

PROOF. Suppose this were not true, then there would exist a decreasing sequence of neighbourhoods $X_i$ of $A$ so that $\bigcap X_i = A$ and for each $i$ an element $y_i \in X_i \cap \bar{N}$ which satisfies (1) and (2) but for which $\omega(y_i) \cap \{A \backslash \mathrm{Int}(N)\} \neq \varnothing$.

Let $T > 0$ be chosen so that $N \cdot [T, \infty) \subset \mathrm{Int}(N)$; such a $T$ exists because $N$ is an attracting neighbourhood. For each $y_i$ there is a point in $\bar{N}$, call it $\bar{y}_i$, on the orbit of $y_i$, for which the set

$$\{\bar{y}_i \cdot [T, 2T]\} \cap \{X_i \backslash \mathrm{Int}(\bar{N})\}$$

is non-empty, otherwise $\omega(y_i)$ would be entirely in $N$. Pick, for each $i$, $t_i$ so that $y_i \cdot t_i \in X_i \backslash \mathrm{Int}(\bar{N})$. By compactness and continuity of the semiflow $y_i \cdot t_i \to \bar{y} \cdot t$, where $t \in [T, 2T]$ and $\bar{y} \in N$. Since the $X_i$ are decreasing, $y_i \cdot t_i \in X_0 \backslash \mathrm{Int}(\bar{N})$ for all $t_i$, and since $X_0 \backslash \mathrm{Int}(\bar{N})$ is closed, $\bar{y} \cdot t \in X_0 \backslash \mathrm{Int}(\bar{N})$. But clearly $\bar{y} \cdot t \in A$ and so is in $N^c$. This contradicts the fact that $N \cdot [T, \infty) \subset \mathrm{Int}(N)$ and the lemma is proved.

This will be applied with $M_v = A$ and with the attractor either $W$, 0 or 1. "$\omega$" and "$\cdot_v$" refer to the flow $S_v$. First, consider the flow $S_v$ on $[c_1, c_2] \times M$. Corresponding to the neighbourhood $U(a, \varepsilon) \cup V(b, \delta)$ in $\bar{M}_v$, there is a neighbourhood $X$ of $[c_1, c_2] \times \bar{M}_v$ such that if $w(x) \in X \cap \{U \cup V\}$ for which $w(x) \cdot_v t \in X$ for all $t \geqq 0$ and $\omega_v(w(x)) \subset [c_1, c_2] \times \bar{M}_v$ then $\omega_v(w(x)) \subset [c_1, c_2] \times W$.

From the remarks preceding the lemma, the set

$$[c_1, c_2] \times \{u(x + kv, t)\} \tag{3.4}$$

lies in such an $X$ if $k$ is sufficiently large and stays in $X$ under the application of $S_v$. Since it has already been proved that $\omega_v(u(x + kv, t))$ is in

$[c_1, c_2] \times \bar{M}_v$, it follows that $\omega_v(u(x + kv, t))$ lies in $[c_1, c_2] \times W$ if $u(x + kv, t) \in U \cup V$. By hypothesis $\lim_{t\to+\infty} u(x, t) = 1$ uniformly on compact sets and so, if $t$ is large enough, $u(x + kv, t) \in V$.

I must next determine $\omega_{c,v}(u(x + kv, t))$ for $c < \bar{c}$ and $c > \bar{c}$. In $\bar{M}_v$, $V$ is an attracting neighbourhood of 1 under any of the semiflows $S_{c,v}$ for $c < \bar{c}$. Arguing just as above, we can show that $\omega_{c,v}(u(x + kv, t)) = 1$ if $k$ and $t$ are chosen sufficiently large. But then it must be true for any $k$ and $t$. The set $U$ is an attracting neighbourhood of 0 for $c > \bar{c}$, and $u(x + kv, t)$ is in $U(a, \varepsilon)$ if $k$ is large enough, $t$ being fixed. Thus repeating the argument we obtain $\omega_{c,v}(u(x + kv, t)) = 0$ if $c > \bar{c}$.

The $\omega$-limit set of the set (3.4) in the semiflow $S_v$ lies in $[c_1, c_2] \times W$ and contains $[c_1, \bar{c}] \times \{1\}$ and $[\bar{c}, c_2] \times \{0\}$. Since it is connected, it must include all of $\{\bar{c}\} \times W$ (see Fig. 2). Therefore there is a function $c(t)$ so that $u(x + kv + c(t)v, t)$ tends to the one-dimensional travelling wave $\bar{u}(\xi)$ uniformly on compact sets. Also $c(t) = \bar{c}t + o(t)$ because of the known behaviour for $c \neq \bar{c}$. But then restricting to the line determined by the vector $v$, we obtain

$$u(\{\xi + \bar{c}t + g(t)\}v, t) - \bar{u}(\xi) \to 0 \text{ as } t \to +\infty \tag{3.5}$$

uniformly on compact sets (in $\xi$), and $g(t) = o(t)$. However, $u(x, t) \to 1$ on compact sets and away from the origin $u$ is nonincreasing in $\xi$. So (3.5) is true in the supremum norm. This proves the theorem.

## BIBLIOGRAPHY

**1.** D.G. Aronson and H.F. Weinberger, *Multidimensional nonlinear diffusion arising in population genetics*, Advances in Math. **30** (1978).

**2.** P.C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics, 28, Springer-Verlag, Berlin, 1979.

**3.** ——— and J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rat. Mech. and Anal. **65** (1977).

**4.** C.K.R.T. Jones, *Spherically symmetric waves of a reaction-diffusion equation*, Technical Summary Report #2046, Mathematics Research Center, University of Wisconsin, Madison.

**5.** B. Gidas, W-M Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, preprint.

**6.** J. Serrin, *A Symmetry problem in potential theory*, Arch. Rat. Mech. and Anal. **43** (1971).

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