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ON THE GROTHENDIECK AND NIKODYM PROPERTIES FOR ALGEBRAS OF BAIRE, BOREL AND UNIVERSALLY MEASURABLE SETS

W.H. GRAVES AND R.F. WHEELER

ABSTRACT. Let $\mathscr{A}$ be a Boolean algebra, represented as the algebra of clopen subsets of a zero-dimensional compact Hausdorff space T. Let $C(T)$ be the Banach space of continuous scalar-valued functions on T, $M(T)$ its dual space. Then $\mathscr{A}$ has the Grothendieck property (G) if every weak* convergent sequence in $M(T)$ is weakly convergent; $\mathscr{A}$ has the Nikodym property (N) if every subset of $M(T)$ which is setwise bounded on $\mathscr{A}$ is uniformly bounded.

Recently Schachermayer has shown that the algebra of Jordan measurable subsets of [0, 1] has (N), but fails (G). In this paper the universal measure space of Graves is used to place Schachermayer's result in a more general context. Various "Jordan algebras" of subsets of a given T, for example the algebra J of Baire sets with scattered boundary, are examined with respect to properties (G) and (N). The results include: (a) If T is an F-space, then J has both (G) and (N); (b) If T is first countable, then J has (N); (c) If T is metrizable and not scattered, then J fails (G); and (d) If $T = 2^A$, A uncountable, then J fails both (G) and (N). The Alexandroff duplicate of a given T, and the notion of a quasi-F-space introduced by Dashiell play a prominent role in the discussion. Some applications to vector measures with range in a Fréchet space are also given.

1. Introduction. This paper studies Boolean algebras for which analogues of the classical Nikodym and Vitali-Hahn-Saks Theorems hold for sequences of bounded additive measures. The Grothendieck property (that every weak*-convergent sequence in the dual of a Banach space should be weakly convergent) is also investigated in this context. The work builds on recent, fundamental progress in the area due to Schachermayer **[29]**. The approach is via the universal measure space of Graves **[14]**, using continuity and orthogonality properties of vector measures studied by Brook **[2, 3]**. The results for scalar measures then follow from topological properties of the Stone space of a given Boolean algebra.

A history of the Grothendieck, Nikodym, and Vitali-Hahn-Saks properties can be found in the book by Diestel and Uhl **[9]**. See **[5, 7, 8, 10, 11, 35]** for recent work in this area. A starting point for this paper is Schachermayer's discovery **[29]** that the algebra of Jordan measurable sub-

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sets of [0, 1] has the Nikodym property, but fails the Grothendieck and Vitali-Hahn-Saks properties. We build on this as follows. Let $\mathscr{A}$ be a Boolean algebra with Stone space T. Then one can define various "Jordan algebras" of subsets of T, for example the algebra J of members of the Baire σ-algebra which have scattered boundary. The boundaries of members of J are assigned measure zero by every purely non-atomic regular Borel measure on T; thus they imitate the behavior of the Jordan measurable subsets of [0, 1] with respect to Lebesgue measure. The results of this study include the following properties. If T is an F-space, then J has both the Grothendieck and Nikodym properties. If T is first countable, then J is still Nikodym, but if T is metrizable and not scattered, then J is not Grothendieck. A particularly interesting example, investigated in some detail, is the Cantor space 2^N. If $T = 2^A$ for uncountable A, then both the Grothendieck and Nikodym properties fail for J.

It is also shown that if Φ is a purely non-atomic, strongly bounded vector measure on $\mathscr{A}$, with range in a Fréchet space, then Φ gives rise in a natural way to an algebra J_Φ of subsets of T which has the Nikodym property.

In the final portion of the paper, we consider a large collection of "Jordan algebras" of subsets of T—the algebras of Baire, Borel, or universally measurable subsets of T which have scattered, countable, or finite boundary, or which differ by a finite set from a clopen set. We analyze (not exhaustively) the relationships among these algebras, and the situations in which they do or do not satisfy the Grothendieck and Nikodym properties. The Alexandroff duplicate **[16]** of a given T, and the notion of a quasi-F-space **[5, 6]** play a prominent role in this discussion. We conclude with a list of open questions.

We wish to thank the University of Missouri, Columbia, for its generous hospitality during the period when this research was conducted.

2. Some notation and identifications. The background for the measure theory and topology needed in this work can be found in **[9, 13, 32]**. Throughout, $\mathscr{A}$ is a Boolean algebra, and $\mathscr{C}$ is the algebra (Boolean isomorphic to $\mathscr{A}$) of all clopen subsets of the Stone space T of $\mathscr{A}$, a zero-dimensional compact Hausdorff space. Elements of $\mathscr{A}$ will be identified with their clopen counterparts in $\mathscr{C}$, but the Boolean operation $\vee$, $\wedge$, and $'$ in $\mathscr{A}$ will be distinguished from the set-theoretic operations $\bigcup$, $\bigcap$, and $\sim$ in $P(T)$, the power set of T. In general $\bar{A}$, A°, ∂A, and $\tilde{A}$ represent the closure, interior, boundary ($\bar{A} - A^\circ$), and complement of a subset A of T. If $A, B \subset T$, then $A \Delta B = (A - B) \bigcup (B - A)$.

$C(T)$ denotes the Banach space of continuous scalar-valued functions on T, with the supremum norm. A subset A of T is a zero-set [resp., cozero set] if it has the form $\{t : f(t) = 0\}$ [resp., $\{t : f(t) \neq 0\}$] for some $f \in C(T)$.

The σ-algebra of Baire sets is the least σ-algebra $\mathrm{Ba}(T)$ of subsets of T containing the zero-sets. Any Baire set in a compact space is Lindelöf. Any closed Baire set in T is a zero-set; if U is open in T, then the following are equivalent: (a) $U \in \mathrm{Ba}(T)$; (b) U is a cozero-set; and (c) U is a countable (disjoint) union of clopen sets. Unfortunately the closure or interior of a Baire set need not be a Baire set, even for first countable T (Example 8.4), but this does hold in case T is a compact group or a product of separable metric spaces [**26**].

The σ-algebra of Borel sets is the least σ-algebra $\mathrm{Bo}(T)$ of subsets of T containing the closed sets. A subset B of T is universally measurable if for each finite non-negative regular Borel measure μ on T, there exist B_1, B_2 (depending on μ) in $\mathrm{Bo}(T)$ with $B_1 \subset B \subset B_2$ and $\mu(B_1) = \mu(B_2)$. The σ-algebra of universally measurable subsets of T is denoted by $U(T)$. Clearly $\mathrm{Ba}(T) \subset \mathrm{Bo}(T) \subset U(T)$. $\mathrm{Ba}(T) = \mathrm{Bo}(T)$ if and only if every closed subset of T is a G_δ-set, in particular if T is metrizable.

A compact Hausdorff space S is scattered if every non-empty closed subset contains an isolated point. Equivalently, S admits no non-zero, non-atomic regular Borel measures [**27**, **19**] and so $U(S) = P(S)$. Every scattered space is sequentially compact [**21**, **23**], and has a dense set of isolated points. Any countable compact S is scattered; if S is first-countable, the converse is true [**32**]. See [**24**, **33**] for additional information.

Let $\mathscr{B}$ be an algebra of subsets of T containing the clopen sets. Then $J_f(\mathscr{B})$, $J_c(\mathscr{B})$, and $J_{na}(\mathscr{B})$ denote, respectively, the algebras of members of $\mathscr{B}$ which have finite, countable, or scattered boundaries, while $J_\Delta(\mathscr{B}) = \{B \in \mathscr{B}: B \,\Delta\, C \text{ is finite for some clopen set } C\}$. Thus $J_\Delta(\mathscr{B})$ is the algebra generated by the clopen sets and the finite members of $\mathscr{B}$. Clearly $\mathscr{C} \subset J_\Delta(\mathscr{B}) \subset J_f(\mathscr{B}) \subset J_c(\mathscr{B}) \subset J_{na}(\mathscr{B})$, and in general all inclusions are proper. We shall only consider $\mathscr{B} = \mathrm{Ba}(T)$, $\mathrm{Bo}(T)$, or $U(T)$. The symbol $\mathrm{St}(J_\Delta(\mathscr{B}))$ denotes the Stone space of $J_\Delta(\mathscr{B})$ considered as a Boolean algebra; in general this will not coincide with $T = \mathrm{St}(\mathscr{C})$.

For the duality theory of locally convex spaces (LCS) see Schaefer [**30**]. If E is an LCS, then E' denotes the space of continuous linear functionals on E. Let $\mathrm{ba}(\mathscr{A})$ denote the space of scalar-valued, bounded, finitely additive set functions on a Boolean algebra $\mathscr{A}$, $M(T)$ the space of scalar-valued regular Borel measures on T. The space $S(\mathscr{C})$ of $\mathscr{C}$-simple functions on T is dense in the Banach space $C(T)$, and we have the usual identifications

$$\mathrm{ba}(\mathscr{A}) = \mathrm{ba}(\mathscr{C}) = (S(\mathscr{C}), \| \|)' = (C(T), \| \|)' = M(T).$$

Each $\mu \in M(T)$ has a unique extension to a regular measure on $U(T)$.

We also require a few results about vector-valued measures. For any complete LCS X, $\mathrm{sb}(\mathscr{A}, X)$ is the space of finitely additive maps $\Phi\colon A \to X$ which are strongly bounded (i.e., if (A_i) is a disjoint sequence in $\mathscr{A}$, then

$\Phi(A_i) \to 0$). If X is the scalar field, we write simply sb($\mathscr{A}$) (= ba($\mathscr{A}$)).

Graves [**14**] has introduced and studied the universal measure topology τ on $S(\mathscr{C})$. The completion $L(\mathscr{C})$ of $(S(\mathscr{C}), \tau)$ is a semi-reflexive Mackey space, and may be identified with $(M(T)', \tau(M(T)', M(T))$. For subsets of $M(T)$, the following are equivalent: (a) τ-equicontinuity; (b) relative $\sigma(M(T), M(T)')$-compactness; and (c) uniform boundedness (in the total variation norm) and uniform strong boundedness on $\mathscr{C}$ [**14**, 11.6].

Brook [**2**, **3**] has shown that $L(\mathscr{C})$ with an Arens multiplication derived from the normed algebra $S(\mathscr{C})$ is a commutative C^*-algebra (the norm coincides with the natural dual norm of $M(T)'$). There is a natural embedding of $U(T)$ in the complete Boolean algebra $\mathscr{P}$ of all projections (idempotents) in $L(\mathscr{C}) = M(T)'$. The topologies $\sigma(M(T)', M(T))$ and $\tau(M(T)', M(T))$ agree on $\mathscr{P}$ [**4**, Th. 3.3]. Boolean limits and τ-limits are the same for monotone nets in $\mathscr{P}$ [**2**, Cor. 4.13]. These results allow us to identify sb($\mathscr{C}, X$) with $M(T, X)$, the space of regular, countably-additive, X-valued Borel measures on T, and to extend every $\Phi \in M(T, X)$ uniquely to a regular countably-additive, X-valued measure on $U(T)$. From Theorems 1.4 and 1.5 of [**14**].

$$\mathscr{L}((S(\mathscr{C}), \tau), X) = \mathscr{L}(L(\mathscr{C}), X) = \mathrm{sb}(\mathscr{C}, X) = \mathrm{sb}(\mathscr{A}, X) = M(T, X).$$

In the scalar case,

$$(S(\mathscr{C}), \tau)' = (S(\mathscr{C}), \|\ \|)' = \mathrm{ba}(\mathscr{C}) = (L(\mathscr{C}), \tau)'.$$

Note that τ is Mackey on $L(\mathscr{C})$, but not on $S(\mathscr{C})$ unless $\mathscr{C}$ is finite. Context rather than distinguishing notation will indicate the domain of any measure (= continuous linear map) under these identifications.

3. The properties G, G′, N, and VHS. If $\mathscr{A}$ is a Boolean algebra, and (μ_n) is a sequence of measures in ba($\mathscr{A}$) = $M(T)$, we always use the terms "weak*" and "weak" to refer to $\sigma(M(T), C(T))$ and $\sigma(M(T), M(T)')$ convergence, respectively. $M(T)$ is sequentially complete with respect to each of these topologies. If $\mathscr{B}$ is a family of subsets of T, then (μ_n) is said to be a $\mathscr{B}$-Cauchy sequence if $\lim_n \mu_n(B)$ exists (finitely) for all $B \in \mathscr{B}$. If there is a member μ of $M(T)$ such that $\lim_n \mu_n(B) = \mu(B)$ for all $B \in \mathscr{B}$, the sequence is said to be $\mathscr{B}$-convergent. If $\mathscr{B}$ is an algebra of subsets of T, this simply describes convergence in the topology $\sigma(m(T), S(\mathscr{B}))$. A $\mathscr{B}$-Cauchy sequence need not be $\mathscr{B}$-convergent. For example, let $\mathscr{A}$ be the Boolean algebra of finite-cofinite subsets of N. The Stone space of $\mathscr{A}$ is $\hat{N} = N \cup \{\infty\}$, the one-point compactification of N. If $\mathscr{B} = \mathscr{C}$ is the family of clopen subsets of $\hat{N}$, and $\mu_n = \sum_{i=1}^{n} \delta(i) - n \cdot \delta(\infty)$, then the sequence (μ_n) is $\mathscr{B}$-Cauchy, but not $\mathscr{B}$-convergent (cf. [**9**, p. 3]).

DEFINITION 3.1. A Boolean algebra $\mathscr{A}$ has the Grothendieck property (G) if every weak*-convergent sequence in $M(T)$ is weakly convergent.

$\mathscr{A}$ has (G) if and only if every continuous linear map from $C(T)$ to a separable Banach space is weakly compact **[15]**.

DEFINITION 3.2. A Boolean algebra in $\mathscr{A}$ has property (G′) if every $\mathscr{C}$-convergent sequence in $M(T)$ is weakly convergent (again, $\mathscr{C}$ = clopen subsets of T).

DEFINITION 3.3. A Boolean algebra $\mathscr{A}$ has the Nikodym property (N) if every $\mathscr{C}$-bounded subset of$M(T)$ is uniformly bounded.

$\mathscr{A}$ has (N) if and only if $(S(\mathscr{C}), \| \ \|)$ is barrelled **[29]**.

DEFINITION 3.4. A Boolean algebra $\mathscr{A}$ has the Vitali-Hahn-Saks property (VHS) if every $\mathscr{C}$-Cauchy sequence is uniformly strongly bounded on $\mathscr{C}$ (i.e., $\mu_n(C_i) \to 0$ uniformly in n for every pairwise disjoint sequence (C_i) of clopen sets).

A σ-complete Boolean algebra has all four of these properties **[1]**. Substantial extensions of this result can be found in **[5, 29, 31]**. The finite-cofinite subsets of N fail all four.

The equivalence of (2) and (3) in the following proposition has been established in **[8, 29]**.

THEOREM 3.5. *The following are equivalent for a Boolean algebra* $\mathscr{A}$: (1) G′, (2) G + N, *and* (3) VHS.

PROOF. (1) ⇒ (2): (G′) ⇒ (G) is obvious, and (G′) ⇒ (N) follows from the usual $\mu_n \to \mu_n / \sqrt{\|\mu_n\|}$ argument.

(2) ⇒ (3): Since $S(\mathscr{C})$ is norm dense in $C(T)$, the Nikodym property implies that the weak*-and $\mathscr{C}$-topologies on $M(T)$ have the same Cauchy (=convergent) sequences. Thus, from (G), and $\mathscr{C}$-Cauchy sequence is weakly convergent, hence uniformly strongly bounded on $\mathscr{C}$ **[14**, 11.6].

(3) ⇒ (1): Combining 4.1 and 11.6 of **[14]**, if a subset H of $M(T)$ is uniformly strongly bounded on $\mathscr{C}$, then it is relatively weakly compact if and only if it is $\mathscr{C}$-bounded. Thus if (VHS) holds, every $\mathscr{C}$-convergent sequence is $\mathscr{C}$-bounded and uniformly strongly bounded on $\mathscr{C}$, hence weakly convergent. Hence (G′) holds.

By modifying the final argument, one can show that the three parts of the theorem are also equivalent to (G″): every $\mathscr{C}$-Cauchy sequence in $M(T)$ is weakly convergent. Schachermayer's example (the Jordan measurable subsets of [0, 1]) shows that (N) need not imply (G). It is not known whether (G) implies (N) (and, consequently, (VHS)).

4. Orthogonally converging sequences and property G′. In this section we show that if there is an $\mathscr{A}$-convergent sequence in ba($\mathscr{A}$), with terms orthogonal in an appropriate sense, then $\mathscr{A}$ fails (G′). One application arises when a sequence of atomic measures is $\mathscr{A}$-convergent to a non-atomic measure.

Since $\mathscr{A}$ is not assumed to be σ-complete, the $\varepsilon - \delta$ rather than the 0-θ versions of continuity and orthogonality are appropriate. These notions will be needed later in the context of vector measures and so are stated here in full generality.

DEFINITION 4.1. Let $\mathscr{A}$ be a Boolean algebra, X and Y complete LCS with 0-neighborhood bases $\mathscr{M}$ and $\mathscr{N}$, respectively. Let $\Phi \in \text{sb}(\mathscr{A}, X)$, $\Psi \in \text{sb}(\mathscr{A}, Y)$. Then

(a) Ψ is Φ-continuous ($\Psi \ll_t \Phi$) on $\mathscr{A}$ if for each $N \in \mathscr{N}$ there exists $M \in \mathscr{M}$ such that $\Psi(A) \in N$ whenever $A \in \mathscr{A}$ and $\{\Phi(B)\colon B \leqq A\} \subset M$, and

(b) Φ and Ψ are topologically orthogonal ($\Phi \perp_t \Psi$) on $\mathscr{A}$ if for each pair of 0-neighborhoods $M \in \mathscr{M}$, $N \in \mathscr{N}$, there is an $A \in \mathscr{A}$ such that $\{\Phi(B)\colon B \leqq A\} \subset M$ and $\{\Psi(C)\colon A \wedge C = 0\} \subset N$.

THEOREM 4.2. *If there is a sequence* (μ_n) *in* $\text{ba}(\mathscr{A})$, $\mathscr{A}$*-convergent to* μ, *and if there is* $\nu \in \text{ba}(\mathscr{A})$ *such that* $\nu \neq 0$, $\nu \ll_t \mu$, *and* $\nu \perp \mu_n$ *for all* n, *then* $\mathscr{A}$ *fails* (G′).

PROOF. Brook [3] has shown that an idempotent $P_\nu \in L(\mathscr{C}) = \text{ba}(\mathscr{A})'$ can be canonically associated with ν. Then P_ν induces a Lebesgue decomposition of vector measures with respect to ν; hence $P_\nu(\mu_n) = 0 \forall n$, while $P_\nu(\mu) \neq 0$. Thus (μ_n) cannot be weakly convergent to μ.

THEOREM 4.3. *If there is a sequence* (μ_n) *in* $\text{ba}(\mathscr{A})$, $\mathscr{A}$*-convergent to* 0, *such that* $\|\mu_n\| \geqq \varepsilon$ *for some* $\varepsilon > 0$ *and all* n, *and* $\mu_m \perp_t \mu_n$ *for* $m \neq n$, *then* $\mathscr{A}$ *fails* (G′).

PROOF. Let $P = \bigvee P_{\mu_n}$ in the complete Boolean algebra $\mathscr{P}$. Then

$$P(\mu_n) = \lim_k \Big(\bigvee_{i=1}^{k} P_{\mu_i}\Big)(\mu_n) = P_{\mu_n}(\mu_n) = \|\mu_n\| \geqq \varepsilon \ \forall n .$$

Thus (μ_n) is not weakly convergent to 0.

Theorems 4.2 and 4.3 can also be proved by passing to the corresponding regular Borel measures on T, and using the notions of absolute continuity and orthogonality appropriate to that setting. In 4.2, fix n, and choose a sequence (C_m) in $\mathscr{C}$ with $|\mu_n|(C_m) < 1/2^m$, $|\nu|(\tilde{C}_m) < 1/2^m$ for all m. Let

$$D_n = \bigcap_{m=1}^{\infty} \Big(\bigcup_{k=m}^{\infty} C_k\Big);$$

then $|\nu|(\tilde{D}_n) = |\mu_n|(D_n) = 0$. Now let $D = \bigcap_{n=1}^{\infty} D_n$; then $|\nu|(D) = |\nu|(T)$, while $|\mu_n|(D) = 0$ for all n. Since $0 \neq \nu \ll_t \mu$, there is a Borel set $B \subset D$

with $\mu(B) \neq 0$. Thus $\mu_n(B) \nrightarrow \mu(B)$, so weak convergence fails. In 4.3, similar reasoning produces a pairwise disjoint sequence (B_m) of Borel sets with $|\mu_n|(B_m) = \delta_{mn} \cdot |\mu_n|(T)\ \forall m, n$. For each m, choose a Borel set $D_m \subset B_m$ with $|\mu_m(D_m)| \geqq \varepsilon/4$, and let $D = \bigcup_{m=1}^{\infty} D_m$. Then $|\mu_n(D)| \geqq \varepsilon/4$ for all n, and so (μ_n) does not converge weakly to 0.

5. Generalized Jordan algebras. Here we begin to consider specific algebras of subsets of T which do or do not possess properties (G) and (N). If T is an F-space (disjoint cozero-sets have disjoint closures), then Seever [**31**] has shown that $\mathscr{C}$ has both properties. In the other direction, we have the following proposition.

PROPOSITION 5.1. *If T admits a single non-trivial convergent sequence (t_n), then $\mathscr{C}$ fails both* (G) *and* (N).

PROOF. Consider the sequences of measures $\delta(t_n) - \delta(t_{n+1})$ and $n(\delta(t_n) - \delta(t_{n+1}))$, respectively.

REMARK. The converse of 5.1 is not true. From Theorems 8.1 and 8.12, the Alexandroff duplicate of an infinite F-space has no non-trivial convergent sequences, and every member of ba($\mathscr{C}$) is (Boolean) countably-additive, yet $\mathscr{C}$ fails both (G) and (N).

If T is infinite, then Proposition 5.1 applies for T scattered, first countable, or supercompact [**36**] (in particular, if T is a compact group). It is therefore noteworthy that for first countable T, one can construct "reasonably small" algebras of universally measurable subsets of T which do possess the Nikodym property. The work of Schachermayer [**29**] provides the basic motivation for this development.

Let $\mathscr{B}$ be an algebra of subsets of T with $\mathscr{C} \subset \mathscr{B} \subset U(T)$. Let $\Phi \in$ sb$(\mathscr{C}, X) = M(T, X)$ for a complete LCS X, so that Φ defines a strongly bounded, countably-additive measure on $\mathscr{B}$.

DEFINITION 5.2. The Jordan sub-algebra of $\mathscr{B}$ with respect to Φ is $J_\Phi(\mathscr{B}) = \{B \in \mathscr{B}: \Phi(D) = 0$ for all $D \in U(T)$ such that $D \subset \partial B\}$.

It is easily seen that $J_\Phi(\mathscr{B})$ is an algebra of subsets of T, with $\mathscr{C} \subset J_\Phi(\mathscr{B}) \subset \mathscr{B}$. The next result allows us to lift sequential convergence in $M(T)^+$ from $\mathscr{C}$ to $J_\Phi(\mathscr{B})$.

THEOREM 5.3. *If (μ_n), $\mu \in M(T)^+$, $\mu \ll_t \Phi$, and (μ_n) is $\mathscr{C}$-convergent to μ, then (μ_n) is $J_\Phi(\mathscr{B})$-convergent to μ.*

PROOF. This is very similar to Schachermayer's proof [**29**, V.1.1] for the classical Jordan algebra of subsets of [0, 1], so we only sketch the details. We may assume that $\|\mu\| = 1$. Let $B \in J_\Phi(\mathscr{B})$ and $\varepsilon > 0$ be given. Since $\mu \ll_t \Phi$, $\mu(B^\circ) = \mu(B) = \mu(\bar{B})$. Since μ is regular, there are clopen sets $A \subset B^\circ$ and $C \subset T - \bar{B}$ with $\mu(B^\circ) - \mu(A) < \varepsilon/2$, and $\mu(T - \bar{B}) -$

$\mu(C) < \varepsilon/2$. Since $\mu_n(A) \to \mu(A)$ and $\mu_n(C) \to \mu(C)$, a short calculation shows that $|\mu(B) - \mu_n(B)| < \varepsilon$ for sufficiently large n.

Brook [3] has shown that if $\Phi \in \mathrm{sb}(\mathscr{C}, X)$, $\Psi \in \mathrm{sb}(\mathscr{C}, Y)$, and $\Psi \ll_t \Phi$ on $\mathscr{C}$, then $\Psi \ll_t \Phi$ on every algebra $\mathscr{B}$ of subsets of T such that $\mathscr{C} \subset \mathscr{B} \subset U(T)$. Topological orthogonality shifts upward in the same fashion. With these observations, Theorems 4.2 and 5.3 yield the following corollary.

COROLLARY 5.4. *If there is a sequence* (μ_n) *in* $\mathrm{ba}(\mathscr{C})^+$, $\mathscr{C}$*-convergent to* μ, *where* $\mu \ll_t \Phi$ *on* $\mathscr{C}$, *and if there is* $\nu \in \mathrm{ba}(\mathscr{C})$ *such that* $\nu \neq 0$, $\nu \ll_t \mu$ *on* $\mathscr{C}$, *and* $\nu \perp_t \mu_n$ *on* $\mathscr{C}$ *for all* n, *then* $J_\Phi(\mathscr{B})$ *fails* (G′).

A strongly bounded map Φ on $\mathscr{C}$ is said to be purely non-atomic if every strongly bounded, Φ-continuous map on $\mathscr{C}$ is non-atomic. For $\Phi \in \mathrm{ba}(\mathscr{C})^+$, the notions of non-atomic and purely non-atomic coincide. Of particular importance in the sequel is the universal purely non-atomic strongly bounded map $\Phi_0 : \mathscr{C} \to (L(\mathscr{C}), \tau)$ studied in [4]. The corresponding idempotent P_0 in $\mathscr{P}$ is given by $P_0 = \bigvee P_\mu$, where μ runs over the non-negative non-atomic members of $\mathrm{ba}(\mathscr{C}) = M(T)$.

PROPOSITION 5.5. *Let* $\mathscr{C}$ *be an algebra of subsets of* T *such that* $\mathscr{C} \subset \mathscr{B} \subset U(T)$. *Then the following collections of sets are identical*: (a) $J_{\Phi_0}(\mathscr{B})$; (b) $\{B \in \mathscr{B} : \mu(\partial B) = 0$ *for all non-atomic* $\mu \in M(T)^+\}$; *and* (c) $J_{na}(\mathscr{B}) = \{B \in \mathscr{B} : \partial B$ *is scattered*$\}$.

PROOF. The equality of the sets mentioned in (b) and (c) is a consequence of a well-known result of Walter Rudin [27]. Since each non-atomic $\mu \in M(T)^+$ is Φ_0-continuous, $J_{\Phi_0}(\mathscr{B}) \subset J_{na}(\mathscr{B})$. If $B \in J_{na}(\mathscr{B})$, and D is a universally measurable subset of ∂B, then $\Phi_0(D) = \Phi_0(P_0 \cdot \chi(D)) = \tau\text{-}\lim\{P_\mu \cdot \chi(D) : \mu$ a non-atomic measure in $M(T)^+\} = 0$.

Members of $J_{na}(\mathscr{B})$ will be referred to simply as Jordan sets for the algebra $\mathscr{B}$. Henceforth we assume that $\mathscr{B} = \mathrm{Ba}(T)$, $\mathrm{Bo}(T)$, or $U(T)$, so that $\mathscr{B}$ is, in particular, a σ-algebra of sets. At this point we generalize Schachermayer's result [29, V. 1.2] that the classical Jordan algebra of [0, 1] has the Nikodym property.

THEOREM 5.6. *Assume that* $\Phi \in \mathrm{sb}(\mathscr{C}, X)$ *is purely non-atomic and satisfies* (*): *for each* $t \in T$ *there is a decreasing sequence* (A_i) *of clopen neighborhoods of* t *such that* $\Phi(F) = 0$ *for every* $F \in U(T)$ *with* $F \subset \bigcap A_i$. *Then* $J_\Phi(\mathscr{B})$ *has the Nikodym property for* $\mathscr{B} = \mathrm{Ba}(T)$, $\mathrm{Bo}(T)$, *or* $U(T)$.

PROOF. The argument is based on Schachermayer's adaptation of the proof of the Nikodym Theorem for σ-algebras given by Diestel and Uhl [9, pp. 14–16]. If the theorem is false, then there is a sequence (μ_n) in $\mathrm{ba}(J_\Phi(\mathscr{B}))$ such that $(\mu_n(\mathscr{B}))$ is a bounded sequence $\forall B \in J_\Phi(\mathscr{B})$, yet $\|\mu_n\| \to \infty$. Note that $\lambda_n = \mu_n|\mathscr{C}$ extends to a regular Borel measure ν_n on T, but $\nu_n|J_\Phi(\mathscr{B})$ need not agree with μ_n, since T is not, in general, the Stone space of $J_\Phi(\mathscr{B})$.

Combining compactness of T with the hypothesis of the theorem, there is a point $t_0 \in T$ such that $\sup_n|\mu_n|(A_i) = \infty\ \forall i$. As in [**29**, V.1.2] there is a sequence (T_k) of pairwise disjoint members of $J_\Phi(\mathscr{B})$, with $T_k \subset A_k \forall k$, and a strictly increasing sequence (n_k) of integers such that

$$|\mu_{n_k}(T_k)| > \sum_{j=1}^{k-1} |\mu_{n_k}(T_j)| + k + 1 .$$

If (T_{k_i}) is any subsequence of (T_k), then $W = \bigcup T_{k_i} \in \mathscr{B}$, since $\mathscr{B}$ is a σ-algebra. Moreover,

$$\bar{W} - W^\circ \subset \bigcup(\bar{T}_{k_i} - T_{k_i}^\circ) \cup (\bigcap A_{k_i}) .$$

Now Φ: $\mathscr{C} \to X$ is (trivially) countably-additive, by a compactness argument, and the unique extension to $U(T)$ retains this property. Since Φ annihilates universally measurable subsets of each $\bar{T}_{k_i} - T_{k_i}^\circ$ and $\bigcap A_{k_i}$, $W \in J_\Phi(\mathscr{B})$. Now, as observed by Schachermayer, the remainder of the proof in [**9**] proceeds without change to establish the desired contradiction.

Again $\mathscr{B} = \mathrm{Ba}(T)$, $\mathrm{Bo}(T)$, or $U(T)$ in the following corollaries.

COROLLARY 5.7. *If X is a Fréchet space, and $\Phi \in \mathrm{sb}(\mathscr{C}, X)$ is purely non-atomic, then $J_\Phi(\mathscr{B})$ has the Nikodym property.*

COROLLARY 5.8. *If T is metrizable, and μ is a non-atomic probability measure on T, then $J_\mu(\mathscr{B})$ has* (N), *but fails* (G).

PROOF. Choose a sequence (μ_n) of finite non-negative linear combinations of point mass measures so that $(\mu_n) \to \mu$ in $\sigma(M(T), C(T))$ [**37**, p. 188]. Then Corollary 5.4 (with $\mu = \nu = \Phi$) shows that $J_\mu(\mathscr{B})$ fails (G′). Now $J_\mu(\mathscr{B})$ has (N), by Theorem 5.6, so it fails (G), by Theorem 3.5.

The fact that $J_\mu(\mathscr{B})$ has (N) is observed by Schachermayer [**29**, V.1.3].

COROLLARY 5.9. *If T is metrizable, then $J_{na}(\mathscr{B})$ always has* (N), *but has* (G) *if and only if T is scattered* (*equivalently, countable*).

PROOF. Theorem 5.6, with $\Phi = \Phi_0$, shows that (N) holds. If T is scattered, then $J_{na}(\mathscr{B}) = \mathscr{B}$ is a σ-algebra, hence has (G). If T is not scattered, then T admits a non-atomic probability measure μ, and Corollary 5.4 (with $\mu = \nu$, $\Phi = \Phi_0$, (μ_n) as in Corollary 5.8) shows that $J_{na}(\mathscr{B})$ fails (G′). Hence it must also fail (G).

REMARK. If T is first countable, then Theorem 5.6 still implies that $J_{na}(\mathscr{B})$ has (N). If the space $P(T)$ of probability measures were first countable for the weak*-topology (equivalently, for $\sigma(M(T), S(\mathscr{C}))$, then the argument in Corollary 5.9 could be used to show that $J_{na}(\mathscr{B})$ fails (G) for non-scattered T. While first countability apparently does not pass from T to $P(T)$ in general, R. Pol [**25**] has discovered a class of compact spaces in which this is indeed true. This class includes the double arrow space (see [**18**]), which is separable, first countable, and perfectly normal, but not metrizable.

6. Jordan sets for metrizable spaces. The results of the previous section are equally valid for $\mathscr{B} = \mathrm{Ba}(T)$, $\mathrm{Bo}(T)$, or $U(T)$. In this section we begin to record differences in the measure theoretic behavior of their respective Jordan algebras. Now J_Δ, J_f, J_c, and J_{na} give rise to twelve algebras of sets; however, $J_\Delta(\mathrm{Bo}) = J_\Delta(U)$, $J_f(\mathrm{Bo}) = J_f(U)$, and $J_c(\mathrm{Bo}) = J_c(U)$ for every T. Note that any subset of T with a scattered (resp., countable) boundary is necessarily in $U(T)$ (resp., $\mathrm{Bo}(T)$). The resulting nine algebras can be arranged as follows (arrows denoting inclusion):

$$\begin{array}{ccccccccccc} \mathscr{C} & \to & J_\Delta(\mathrm{Ba}) & \to & J_f(\mathrm{Ba}) & \to & J_c(\mathrm{Ba}) & \to & J_{na}(\mathrm{Ba}) & & \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & J_\Delta(\mathrm{Bo}) & \to & J_f(\mathrm{Bo}) & \to & J_c(\mathrm{Bo}) & \to & J_{na}(\mathrm{Bo}) & \to & J_{na}(U) \end{array}$$

No two of these coincide for every T. In fact, if ω_0 and ω_1 denote the first infinite and first uncountable ordinals, respectively, then all except $J_c(\mathrm{Ba})$ and $J_{na}(\mathrm{Ba})$ are distinct for $T = [0, \omega_1]$, and those are distinct for $T = [0, \omega_1] \times [0, \omega_0]$.

For metrizable T, the situation is much simpler. Since $\mathrm{Ba}(T) = \mathrm{Bo}(T)$, and every closed scattered subset of T is countable, even homeomorphic to $[0, \alpha]$ for some countable ordinal α [**32**, 8.6.10], the diagram of inclusions reduces to

$$\mathscr{C} \to J_\Delta(\mathrm{Ba}) \to J_f(\mathrm{Ba}) \to J_{na}(\mathrm{Ba})\,.$$

PROPOSITION 6.1. *If T is metrizable and infinite, then*

(a) $\mathscr{C}$ *and* $J_\Delta(\mathrm{Ba})$ *always fail both* (G) *and* (N),

(b) $J_f(\mathrm{Ba})$ *has* (G) *or* (N) *if and only if the set of non-isolated points of* T *is finite, and*

(c) $J_{na}(\mathrm{Ba})$ *always has* (N), *but has* (G) *if and only if T is countable.*

PROOF. (a) T contains a non-trivial convergent sequence (t_n). If $A \in J_\Delta(\mathrm{Ba})$, then either A or $\tilde{A}$ contains all but finitely many t_n. Now argue as in Proposition 5.1.

(b) If the derived set is finite, then $J_f(\mathrm{Ba}) = \mathrm{Ba}(T)$ is a σ-algebra. If the derived set is infinite, it contains a sequence (t_n) of distinct points. Let $\mathscr{B}_n = \{A \in J_f(\mathrm{Ba}) \colon \partial A \cap \{t_k\}_{k>n} = \varnothing\}$. Then $(\mathscr{B}_n)$ is a strictly increasing sequence of algebras with $\bigcup \mathscr{B}_n = J_f(\mathrm{Ba})$. By [**29**, VI.2.1], $J_f(\mathrm{Ba})$ must fail (G) and (N).

(c) This follows from Corollary 5.9.

This result leads to the conjecture that $J_{na}(\mathrm{Ba})$ is ihe smallest "reasonable" sub-algebra of $U(T)$ which contains the clopen sets and has the Nikodym property. Theorem 6.3 lends some support to this conjecture for first countable T, but Example 6.4 shows it to be false in general. Theorem 6.5 shows why "reasonable" is needed in the above.

DEFINITION 6.2. Let T be first countable, $\mathscr{D}$ an algebra of subsets of T. Then $\mathscr{D}$ is weakly σ-complete if whenever $t_0 \in T$, with a nested clopen neighborhood base (V_n), and (B_n) is a sequence in $\mathscr{D}$ with $B_n \subset V_n - V_{n+1} \forall n$, then $\bigcup B_n \in \mathscr{D}$.

Inspection of the proof of Theorem 5.6 reveals that it is the weak σ-completeness of $J_{na}(\mathrm{Ba})$ which enables the Nikodym property to be established for that algebra.

THEOREM 6.3. *Assume that T is first countable, and that $\mathscr{D}$ is a weakly σ-complete algebra of subsets of T which contains the clopen sets. Then $\mathscr{D}$ contains all cozero-sets which have scattered boundary.*

PROOF. Let U be a cozero-set with scattered boundary which is not clopen. By a theorem of Mazurkiewicz and Sierpinski [**32**, 8.6.10], ∂U is homeomorphic to a space $[0, \alpha]$, where α is a countable ordinal, and $[0, \alpha]$ has the order topology. Call α the rank of U. The theorem is now proved by induction on α.

If $\alpha = 0$, then $\bar{U} - U$ is a singleton $\{t_0\}$. Choose a nested clopen neighborhood base (V_n) for t_0 with $V_1 = T$, and let $B_n = U \cap (V_n - V_{n+1})$. Then each B_n is clopen, and so $U = \bigcup_n B_n \in \mathscr{D}$.

Now assume that the assertion holds for each U with rank less than some countable ordinal β. Identify $\bar{U} - U$ with $[0, \beta]$. If $\beta = \gamma + 1$, then β is an isolated point of $\bar{U} - U$, so there is a clopen subset C of T with $\beta \in C$, $[0, \gamma] \subset \tilde{C}$. Then $U \cap C$ and $U \cap \tilde{C}$ are cozero-sets with boundaries $[0, \gamma]$ and $\{\beta\}$, respectively, hence in $\mathscr{D}$. Since $\mathscr{D}$ is an algebra, $U \in \mathscr{D}$.

Finally, assume that β is a limit ordinal, and let (V_n) be a nested clopen neighborhood base for β in T, with $V_1 = T$. Then $(V_n - V_{n+1}) \cap [0, \beta]$ is a compact subset of $[0, \beta)$, hence of order type less than β, for each n. Thus each set $B_n = U \cap (V_n - V_{n+1})$ is a cozero-set of rank less than β, hence in $\mathscr{D}$, and so $U = \bigcup B_n$ is in $\mathscr{D}$, since $\mathscr{D}$ is weakly σ-complete.

EXAMPLE 6.4. Let $T = 2^A$, where $2 = \{0, 1\}$, and A is an uncountable index set. It is well-known that any Baire set B of T depends on countably many co-ordinates i.e., there is a countable subset D of A and $B_1 \in \mathrm{Ba}(2^D)$ such that $B = B_1 \times 2^{A-D}$. Since $\partial B = \partial B_1 \times 2^{A-D}$, and the continuous image of a scattered space is scattered [32, 8.5.3], ∂B must be empty. Thus $J_{na}(\mathrm{Ba}) = \mathscr{C}$, but $\mathscr{C}$ fails (G) and (N), by Proposition 5.1.

REMARK. Theorem 8.12 yields another class of spaces for which $J_{na}(\mathrm{Ba})$ fails both (G) and (N).

We close this section by showing that no algebra properly containing $\mathscr{C}$ can be strictly minimal with respect to possessing the Nikodym property.

THEOREM 6.5. *Let $\mathscr{D}$ be an algebra of subsets of T (first-countable or not), properly containing the clopen sets $\mathscr{C}$. If $\mathscr{D}$ has* (N), *then there is a proper sub-algebra $\mathscr{E}$ of $\mathscr{D}$ which contains $\mathscr{C}$ and has* (N).

PROOF. Choose two distinct Boolean ultrafilters p, q on $\mathscr{D}$ such that $\{C \in \mathscr{C}\colon C \in p\} = \{C \in \mathscr{C}\colon C \in q\}$. Existence of such p, q is easily seen, for example, by considering the map from the Stone space of $\mathscr{D}$ onto T induced by the embedding of $\mathscr{C}$ into $\mathscr{D}$. Let $\mathscr{E} = \{D \in \mathscr{D}\colon D \in p \cap q$, or $\tilde{D} \in p \cap q\}$. Then $\mathscr{E}$ is a proper sub-algebra of $\mathscr{D}$ and contains $\mathscr{C}$. Fix $D_0 \in \mathscr{D}$ such that $D_0 \in q$, $D_0 \notin p$, and define $T\colon S(\mathscr{D}) \to S(\mathscr{E})$ by

$$T(f) = f + (f(p) - f(q))\cdot\chi(D_0),$$

where $\chi(D_0)$ is the characteristic function of D_0. It is easily seen that T is a (norm-continuous) projection of $S(\mathscr{D})$ onto $S(\mathscr{E})$ with one-dimensional kernel (the span of $\chi(D_0)$. Since $\mathscr{D}$ is Nikodym, $(S(\mathscr{D}), \|\ \|)$ is barrelled. Thus $(S(\mathscr{E}), \|\ \|)$, a subspace of co-dimension one, is barrelled [28, 34] and so $\mathscr{E}$ is Nikodym.

The projection T was introduced by Isbell and Semadeni [17] and later used by Badé and Seever [31]. Schachermayer [29, VI.1.1] gives a result in a similar spirit. In the other direction, if $\mathscr{C} \subset \mathscr{E} \subset \mathscr{D}$, $\mathscr{E}$ has (N), and $S(\mathscr{E})$ has finite co-dimension in $S(\mathscr{D})$, then $\mathscr{D}$ has (N). This follows because $(S(\mathscr{D}), \|\ \|)$ is topologically isomorphic to the product of its closed subspace $S(\mathscr{E})$ and a finite-dimensional LCS, and therefore is barrelled [30].

Particular choices for p and q in the preceding proof may lead to interesting algebras $\mathscr{E}$. For example, if $T = \hat{N}$, the one-point compactification of N, and $\mathscr{D} = J_{na}(\mathrm{Ba}) = P(T)$, let $p = \{D \in \mathscr{D}\colon \infty \in D\}$, and let q be a free ultrafilter of subsets of N. Then $D_0 = N$. Now $\mathrm{St}(\mathscr{D})$ is the Stone-Čech compactification of $\hat{N}$ with the discrete topology, and $\mathrm{St}(\mathscr{E})$ is obtained by identifying ∞ with a point of $\beta N - N$. Thus $\mathrm{St}(\mathscr{E})$ is homeomorphic to βN, and so $\mathscr{E}$ is complete as a Boolean algebra, although not a σ-algebra under set-theoretic operations.

7. Some remarks on the Cantor space. Given a subset of a space T, how must it be positioned in order to have a scattered boundary? If F

is a closed subset of T, and $F \Delta C$ is countable and nowhere dense for some clopen set C (necessarily unique), then ∂F is scattered. Indeed $C - F$ is open and nowhere dense, so $C \subset F^\circ$, but then $\partial F \subset F - C$ is countable, and therefore scattered.

Unfortunately the converse fails for any uncountable, first countable T. To see this, note that some $t_0 \in T$ has no countable neighborhood. Let (V_n) be a nested clopen neighborhood base at t_0, with $V_n - V_{n+1}$ uncountable for all n. Let U_n be an uncountable proper clopen subset of $V_n - V_{n+1}$, and let $F = \bigcup_n U_n \cup \{t_0\}$. Then F is a zero-set of T with one-point boundary. There does not exist a clopen set C with $F \Delta C$ countable and nowhere dense; for if so, then $C \subset F^\circ$, so that (by compactness) $C \subset \bigcup_1^m U_n$ for some m. But then $U_{m+1} \subset F \Delta C$.

C. Brook has observed to the authors that a subset A of metrizable T is in $J_{na}(\mathrm{Ba})$ if and only if $A = U \cup M$, where U is an open set with countable boundary, and M has countable closure.

If $T = 2^N$, the classical Cantor space, then a countable subset E is a member of $J_{na}(\mathrm{Ba})$ if and only if $\bar{E}$ is scattered. We describe precisely when this occurs. The approach is suggested by the work of Lacey and Hardy [20]. Let $X_n = \{t \in 2^N: t_n = 0\}$, $Y_n = \{t \in 2^N: t_n = 1\}$. The phrase "basic clopen set of rank p" denotes any set of the form $\bigcap_{n=1}^p Z_n$, where $Z_n = X_n$ or Y_n for each n. Let e_k denote the member of 2^N whose j-th component is δ_{jk}.

If $M = (n_i)$ is an infinite (increasing) subsequence of N, then M can be arranged as a tree with levels $\{n_1\}$, $\{n_2, n_3\}$, $\{n_4, n_5, n_6, n_7\}$, ... If $x = (x_k) \in 2^N$, then x picks out a path B_x in the tree, as follows: $n_1 \in B_x$; having chosen an integer in the k-th level, go left if $x_k = 0$, go right if $x_k = 1$ to pick a member of the $(k+1)$-st level. Explicitly, $B_x = \{n_{j_1}, n_{j_2}, \ldots\}$ where $j_1 = 1$ and

$$j_k = 2^{k-1} + \sum_{i=1}^{k-1} 2^{i-1} x_{k-i}, \; 2 \leqq k .$$

If $A \subset 2^N$, say that A is compatible with M and x if for each integer p, there is a member t of A whose n_{j_k}-th coordinate is x_k, $1 \leqq k \leqq p$.

THEOREM 7.1. *Let E be a countable subset of 2^N. Then the following are equivalent*: (1) *$\bar{E}$ is not scattered*; *and* (2) *there is an infinite subsequence M of N such that, for every $x \in 2^N$, E is compatible with M and x.*

PROOF. (1) $\Rightarrow$ (2): Let P be a non-empty perfect subset of $\bar{E}$. Let x and y be distinct points of P, and choose a basic clopen set C_1 of rank n_1 with $x \in C_1$, $y \in \tilde{C}_1$. Since $P \cap C_1$ is perfect, there is a basic clopen set $C_2 \subset C_1$ with rank $n_2 > n_1$ such that $C_2 \cap P$, $(C_1 - C_2) \cap P$ are non-empty; hence E meets both C_2 and $C_1 - C_2$. Similarly, $\tilde{C}_1$ is split by a basic clopen set of rank $n_3 > n_2$. Continuing inductively, we obtain an infinite subsequence $M = (n_i)$ of N, and it can be seen from the construction that (2) is satisfied.

(2) $\Rightarrow$ (1): We construct an interlocking sequence $\{(A_n, B_n)\}$ of relatively clopen, complementary subsets of $\bar{E}$, and apply [**20**, Prop. 9]. Let $M = \{n_1, n_2, \ldots\}$. Let $A_1 = X_{n_1} \cap \bar{E}$, $B_1 = Y_{n_1} \cap \bar{E}$. Then A_1 and B_1 are non-empty, since E is compatible with M and either 0 or e_1. Let

$$A_2 = (X_{n_1} \cap X_{n_2} \cap \bar{E}) \cup (Y_{n_1} \cap X_{n_3} \cap \bar{E}),$$
$$B_2 = (X_{n_1} \cap Y_{n_2} \cap \bar{E}) \cup (Y_{n_1} \cap Y_{n_3} \cap \bar{E}).$$

Then $A_2 \cup B_2 = \bar{E}$. Since E is compatible with M and any of 0, e_1, e_2, or $e_1 + e_2$, each of the two components of A_2 or B_2 is non-empty, and (A_1, B_1), (A_2, B_2) satisfy the interlocking property. Continuing in this way, A_n will be a union of 2^{n-1} non-empty disjoint clopen sets, as will B_n. A_{n+1} is a union of 2^n clopen sets, each obtained by splitting one of the components of $A_n \cup B_n$, using the tree for M as a guide, and letting the last entry of the intersection be an X in each case. Similarly, the 2^n components of B_{n+1} are intersections whose last entry is a Y. By induction we obtain an interlocking sequence of sets, and (1) follows.

8. F-spaces, quasi-F-spaces, and Alexandroff duplicates. The work of Seever [**31**] demonstrates the importance of F-spaces in the study of measures on Boolean algebras. Recently Dashiell, et al. [**5, 6**] have studied the larger class of quasi-F-spaces and the associated up-down semi-complete Boolean algebras (see also [**29**]). In this section we use an important topological construction (formation of the Alexandroff duplicate of a given space) to analyze the Jordan algebras of F-spaces and quasi-F-spaces.

The Alexandroff duplicate of a given space T is $A(T) = T \cup T'$, where T' is a disjoint copy of T, each point of T' is isolated, and a typical neighborhood of t_0 is $V \cup (V' - \{t_0'\})$, where V is a neighborhood of t_0 in T. $A(T)$ is again a zero-dimensional compact Hausdorff space. There is a natural retraction $\pi: A(T) \to T$ which sends any t_0 and t_0' to t_0. A convenient summary of facts about $A(T)$ (in a more general setting) appears in [**16**] (see also Mathematical Reviews 55 # 6358).

THEOREM 8.1. *If T is infinite, then the algebra $\mathscr{D}$ of clopen subsets of $A(T)$ fails both (G) and (N).*

PROOF. Choose any sequence (t_0) of distinct points of T, and let $\mu_n = \delta(t_n') - \delta(t_n)$. If $f \in C(A(T))$, then $g = f - f \circ \pi$ is continuous on $A(T)$ and vanishes on T. Thus for any $\varepsilon > 0$, $\{p \in A(T): |g(p)| \geqq \varepsilon\}$ is a compact, hence finite, subset of T'. It follows that $\mu_n(f) = f(t_n') - f(t_n) = g(t_n') \to 0$. Hence $\mu_n \to 0$ in $\sigma(M(T), C(T))$, but clearly $\mu_n \nrightarrow 0$ in $\sigma(M(T), M(T)')$, so $\mathscr{D}$ fails (G).

Similarly, let $\mu_n = n(\delta(t_n') - \delta(t_n))$. If $D \in \mathscr{D}$, then $C = D \cap T$ is clopen in T. We claim that $D\Delta(C \cup C')$ is finite. If $D - (C \cup C')$ is infinite, then $D - C'$ is infinite and has no cluster point in D, a contradiction. If $(C \cup C') - D$ is infinite, then $C' - D$ is infinite, hence has a cluster point in $C \subset D$. This contradicts the fact that D is open in $A(T)$ and establishes the claim. Thus for any $D \in \mathscr{D}$, $\mu_n(D)$ is eventually 0, so (μ_n) is $\mathscr{D}$-bounded, but not uniformly bounded.

The map $f \to (f|T, f - f \circ \pi)$ is a topological isomorphism of $C(A(T))$ onto $C(T) \oplus c_0(T')$, yielding an alternate proof that $\mathscr{D}$ fails (G) (see [**29**, VIII. 2]). $M(A(T))$ is isometrically isomorphic to the ℓ^1-sum of $M(T)$ and $\ell^1(T')$.

The next result establishes an interesting relationship between $A(T)$ and a Jordan algebra of Borel sets of T, in the case where T is perfect. Note that the derived set of any zero-dimensional F-space is such a T (an argument is easily constructed using the fact that F-spaces have no non-trival convergent sequences).

THEOREM 8.2. *If T has no isolated points, then the Stone space of J_Δ(Bo) is homeomorphic to $A(T)$.*

PROOF. Let $S = \mathrm{St}(J_\Delta(\mathrm{Bo}))$, $\rho: S \to T$ the continuous surjection induced by the embedding of $\mathscr{C}$ as a sub-algebra of J_Δ(Bo). We will find a homeomorphism ψ of $A(T)$ onto S such that $\rho \circ \psi = \pi$, the natural retraction of $A(T)$ onto T.

Define $\psi: A(T) \to S$ by $\psi(t) = \{B \in J_\Delta(\mathrm{Bo})\colon B$ contains a deleted neighborhood of t in $T\}$, $\psi(t') = \{B \in J_\Delta(\mathrm{Bo})\colon t \in B\}$. Then $\psi(t)$ and $\psi(t')$ are filters in J_Δ(Bo), and $\psi(t')$ is clearly maximal. To see that $\psi(t)$ is an ultrafilter, suppose $B \in J_\Delta$(Bo) and $B\Delta C$ is finite for some clopen set C in T. If B contains no deleted neighborhood of t, then neither does C. Since t is not isolated, $\tilde{C}$ is a neighborhood of t, and so $\tilde{B}$ contains a deleted neighbordhood of T, i.e., $\tilde{B} \in \psi(t)$.

For each $t \in T$, $\{t\} \in \psi(t')$, but $\{t\} \notin \psi(t)$; it follows easily that ψ is one-to-one. ψ is automatically continuous at any point of T'. Now fix $t_0 \in T$, and let B_0 be a fixed member of $\psi(t_0)$. Then B_0 contains some $C - \{t_0\}$, where C is a clopen neighborhood of t_0 in T. Form $D = (C \cup C') - \{t_0'\}$, a clopen neighborhood of t_0 in $A(T)$. If $t \in C$, then $C - \{t_0\} \in \psi(t')$, and so $B_0 \in \psi(t')$. If $t' \in C' - \{t_0'\}$, then $C - \{t_0\} \in \psi(t')$, and so $B_0 \in \psi(t')$. Thus ψ maps D into the clopen subset of S determined by B_0, so ψ is continuous at t_0.

ψ has dense range: if $B \in J_\Delta$(Bo), $B \neq \varnothing$, choose $t_0 \in B$. Then $\psi(t_0')$ is in the clopen subset of S determined by B. Now we have ψ a homeomorphism of $A(T)$ onto S. Finally, $\rho \circ \psi(t) = \{C \in \mathscr{C}\colon C \in \psi(t)\} = \{C \in \mathscr{C}\colon C$ contains a deleted neighborhood of $t\} = \{C \in \mathscr{C}\colon t \in \mathscr{C}\} =$ (the ultrafilter on $\mathscr{C}$ determined by) $t = \pi(t)$, while $\rho \circ \psi'(t') = \{C \in \mathscr{C}: C \in \psi(t')\} = \{C \in \mathscr{C}: t \in \mathscr{C}\} = t = \pi(t')$. This completes the proof.

COROLLARY 8.3. *If T has no isolated points, then $J_\Delta(\mathrm{Bo})$ fails both* (G) *and* (N).

Theorem 8.5 shows that $J_\Delta(\mathrm{Ba})$ can have both (G) and (N), but this is not always true (Proposition 6.1).

The following example should be compared with the results of **[26]**.

EXAMPLE 8.4. Suppose T contains an uncountable, nowhere-dense zero-set Z. Then $Y = Z \cup Z' = \pi^{-1}(Z)$ is a zero-set in $A(T)$ whose interior is not a Baire set in $A(T)$. Indeed $Y^\circ = Z'$ is uncountable while every clopen subset of T' is finite. Thus Y° is not a cozero set in $A(T)$ and therefore not a Baire set. Note that if T is metrizable, then $A(T)$ is first countable.

The Alexandroff duplicate of T is perhaps most interesting in our context when T is an F-space **[13, 31]**. Here the basic diagram of inclusions (§6) reduces to three distinct algebras:

$$\mathscr{C} = J_{na}(\mathrm{Ba}) \to J_\Delta(\mathrm{Bo}) \to J_{na}(\mathrm{Bo})\,.$$

THEOREM 8.5. *If T is an F-space, then* (a) $J_{na}(\mathrm{Ba}) = \mathscr{C}$ *has* (G) *and* (N), *and* (b) $J_f(\mathrm{Bo}) = J_{na}(U)$.

PROOF. Since an F-space has no non-trivial convergent sequences, every closed scattered subset is finite, and so $J_{na}(\mathscr{B}) = J_f(\mathscr{B})$ for $\mathscr{B} = \mathrm{Ba}(T)$, $\mathrm{Bo}(T)$, or $U(T)$. This proves (b), since a subset of T with finite boundary is a Borel set. Now suppose that B is a Baire set in T with boundary $\{t_1, \ldots, t_n\}$. Choose disjoint clopen sets $(C_i)_{i=1}^n$ with $t_i \in C_i$ and $\bigcup_{i=1}^n C_i = T$. Then each $B_i = B \cap C_i$ is a Baire set whose boundary is a single point, so we may assume that $\partial B = \{t_0\}$. Thus $B = B^\circ$ or $B = \bar{B}$. If $B = B^\circ$, then B is an open Baire set, hence a cozero-set, and so B is a disjoint union of non-empty clopen sets (D_n). Then $\bigcup D_{2n}$ and $\bigcup D_{2n+1}$ are disjoint cozero-sets, each containing t_0 in their closure, and this contradicts the fact that T is an F-space. If $B = \bar{B}$, then $\tilde{B}$ is an open Baire set with boundary $\{t_0\}$, and we reach a contradiction as before. Hence B must be clopen.

If T is extremally disconnected **[13]**, the inclusion diagram reduces to two distinct algebras.

THEOREM 8.6. *If T is extremally disconnected, then $J_\Delta(\mathrm{Bo}) = J_f(\mathrm{Bo}) = J_{na}(\mathrm{Bo})$.*

PROOF. It suffices to consider a Borel set B with one-point boundary $\{t_0\}$. If $B = B^\circ$, then $\bar{B} = \bar{B}^\circ$ is clopen, by the extremally disconnected property, and $B \Delta \bar{B} = \{t_0\}$. If $B = \bar{B}$, then $B^\circ = (\bar{B})^\circ$ is clopen, and $B \Delta B^\circ = \{t_0\}$. Hence $B \in J_\Delta(\text{Bo})$.

COROLLARY 8.7. *If T is extremally disconnected and has no isolated points, then* (a) $J_{na}(\text{Ba})$ *always has* (G) *and* (N), *and* (b) $J_{na}(\text{Bo}) = J_{na}(U)$ *never has* (G) *nor* (N).

PROOF. (a) This follows from Theorem 8.5 (a) and [31]; (b) Apply 8.3 and 8.6.

EXAMPLE 8.8. The conclusion of 8.6 need not hold for basically disconnected T. Let S be the non-discrete P-space of cardinal $\aleph_1$ described in [13, 4N]. Thus $S = [0, \omega_1]$ as a set, but all $\alpha < \omega_1$ are isolated points of S. Let $T = \beta S$. Then T is basically disconnected [13, 4N.3, 6M.1], hence an F-space, and may be identified with the one-point compactification of $\bigcup_{\alpha<\omega_1} \beta[0, \alpha]$ (again, each $[0, \alpha]$ is a countable discrete space). Let $D = \{\alpha < \omega_1 \colon \alpha \text{ is an even ordinal}\}$,

$$A = \bigcup_{\alpha<\omega_1} \text{Cl}(D \cap [0, \alpha]),$$

where the closure is taken in T. Then $A \in J_f(\text{Bo})$, but $A \notin J_\Delta(\text{Bo})$. Indeed $\partial A = \{\omega_1\}$, while any clopen set in T either contains or completely misses a neighborhood of ω_1.

Now we turn to the important class of quasi-F-spaces.

DEFINITION 8.9. [5, 6]. T is a quasi-F-space if every dense cozero-set is C^*-embedded in T.

In contrast, T is an F-space if every cozero-set is C^*-embedded in T. If $\mathscr{A}$ is a Boolean algebra, then its Stone space T is a quasi-F-space if and only if $\mathscr{A}$ is *up-down semi-complete* [5, 29], i.e., if a disjoint sequence (A_n) in $\mathscr{A}$ has a supremum in $\mathscr{A}$, then so does every subsequence. It is equivalent to require that if $A_1 \leqq A_2 \leqq \cdots \leqq B_2 \leqq B_1$ and $\wedge (B_n - A_n) = 0$ in $\mathscr{A}$, then there is a member C of $\mathscr{A}$ with $A_n \leqq C \leqq B_n \forall n$ (necessarily $C = \bigvee A_n = \bigwedge B_n$). This should be compared with property (I) of [31]. Schachermayer [29, V.1.5] shows that the classical Jordan algebra of subsets of [0, 1] is up-down semi-complete. His proof adapts easily to show the following proposition.

PROPOSITION 8.10. *The algebras* $J_c(\text{Bo})$, $J_{na}(\text{Bo})$, *and* $J_{na}(U)$ *are up-down semi-complete for every* T.

Note that these algebras contain all singleton sets, while the corresponding algebras of Baire sets do not, unless T is first countable. None of the other Jordan algebras are up-down semi-complete for all T. If $T = 2^A$, then T is not a quasi-F-space. Thus if A is uncountable, Example 6.4

shows that $\mathscr{C} = J_\Delta(\text{Ba}) = J_f(\text{Ba}) = J_c(\text{Ba}) = J_{na}(\text{Ba})$ is not up-down semi-complete. If T is first-countable, scattered, or an F-space, then $J_{na}(\text{Ba})$ is up-down semi-complete. If $T = 2^N$, then Theorem 8.2 can be used to show that $J_\Delta(\text{Bo})$ is not up-down semi-complete. $J_f(\text{Bo})$ is not up-down semi-complete for $T = [0, \omega_0^2]$.

We mention now an important special class of quasi-F-spaces.

DEFNINTION 8.11. **[22]**. T is an almost-P-space if every non-empty zero-set of T has non-empty interior.

The equivalent condition on $\mathscr{C}$ is that no non-trivial suprema of countably infinite subcollections of $\mathscr{C}$ exist. This is called property $(n - \sigma)$ by Schachermayer, and Cantor separability by Walker **[39]**. Any almost-P-space is clearly a quasi-F-space. Note that any $\mu \in \text{ba}(\mathscr{C})$ is trivially countably additive in the Boolean sense. See **[6, 38]** for additional information.

Our final result shows that, for T an infinite F-space, passage to the Alexandroff duplicate produces an almost-P-space on which $J_{na}(\text{Ba})$ fails both (G) and (N).

THEOREM 8.12. *Let T be an infinite F-space. Then* (a) *$A(T)$ is not an F-space*; (b) *$A(T)$ is an almost-P-space, hence is a quasi-F-space*; *and* (c) *the algebra of Baire sets in $A(T)$ with scattered boundary fails* (G) *and* (N).

PROOF. (a) Let (U_n) be a disjoint sequence of non-empty clopen subsets of T (with the relative topology). Choose a point t_n in U_n, and let $V = \{t_n'\}_{n=1}^\infty$, $W = \bigcup_{n=1}^\infty (U_n \cup (U_n' - \{t_n'\}))$. Then V and W are disjoint cozero sets in $A(T)$. If t_0 is a cluster point of (t_n) in T, then $t_0 \in \bar{V} \cap \bar{W}$; hence $A(T)$ is not an F-space.

(b) Let U be a dense cozero set in $A(T)$; it suffices to show that $U = A(T)$ Clearly $T' \subset U$. Let $U = \bigcup_1^\infty D_n$, where each D_n is a clopen subset of $A(T)$. Let $C_n = D_n \cap T$; then (as in Theorem 8.1) $D_n \Delta (C_n \cup C_n')$ is finite for all n. Since $T' \subset \bigcup_1^\infty D_n$, $T' - \bigcup_1^\infty \text{m}_n'$ is at most countable. Thus $T - \bigcup_1^\infty C_n$ is countable, so $A(T) - U$ is a countable zero-set of $A(T)$ contained in T. By **[13**, 14N**]**, this zero-set is empty, so $U = A(T)$.

(c) Using 8.1, it suffices to show that a Baire set B of $A(T)$ with scattered boundary must be clopen. Since $A(T)$ admits no non-trivial convergent sequences (for T an F-space), ∂B is finite. As in 8.5, it is enough to show that B cannot have a one-point boundary $\{t_0\}$. If so, then necessarily $t_0 \in T$. Either $B = B^\circ$ or $B = \bar{B} = B^\circ \cup \{t_0\}$. In the first case, $B \cap T$ is a Baire set in T, hence cannot have a one-point boundary, by 8.5. Thus $B \cap T$ is clopen in T and misses t_0, so t_0 has a clopen neighborhood V in $A(T)$ with $B \cap V \subset T'$. Now $B \cap V$ is an open Baire set in $A(T)$, hence cozero, and a subset of T', hence countable. Let $B \cap V = \{t_n'\}$. Then t_0 is the unique cluster point of (t_n') in $A(T)$, so $t_n \to t_0$. Since $A(T)$ has no non-trivial convergent sequences, this is a contradiction.

The other case ($B = \bar{B}$) is handled similarly by considering $\tilde{B}$.

9. Some open questions. (1) *A* family $\mathscr{B}$ of subsets of T is called a *bounding class* if every $\mathscr{B}$-bounded sequence in $M(T)$ is uniformly bounded; $\mathscr{B}$ is called a *converging class* if every $\mathscr{B}$-Cauchy sequence is $\sigma(M(T), M(T)')$-convergent. It is known, for example, that the cozero sets or the regular open sets in T form both a bounding and a converging class (see, for example, Gänssler **[12]** and Wells **[40]**).

Say that an algebra $\mathscr{B}$ of subsets of T, $\mathscr{C} \subset \mathscr{B} \subset U(T)$, is a *bounding or converging algebra* if the above conditions hold. If $\mathscr{B}$ is Nikodym, then it is a bounding algebra; if $\mathscr{B}$ has (G′), then it is a converging algebra. Do the converses hold? Note that, for example, the "bounding property" relates to $M(T)$, while the Nikodym property relates to $M(\mathrm{St}(\mathscr{B}))$, a much larger collection of measures in general.

(2) Corollary 5.9 and Theorem 8.5 show that $J_{na}(\mathrm{Ba})$ has the Nikodym property if T is either first countable or an F-space. However, the proofs are totally dissimilar. Is there a single argument which can unify these two cases?

(3) Let $\mathscr{C}$ = clopen subsets of T. If $\mathscr{C}$ has the Nikodym property, then every $\mathscr{C}$-Cauchy sequence in $M(T)$ is weak*-Cauchy (hence weak*-convergent). In turn, this implies that every $\mathscr{C}$-Cauchy sequence is $\mathscr{C}$-convergent. Are these three properties equivalent? Note that all three fail if T has a non-trivial convergent sequence (use the example in §3).

(4) When is the Stone space of a Boolean algebra supercompact **[36]**? What measure-theoretic properties do supercompact spaces have?

(5) Do $J_{na}(\mathrm{Bo})$ and $J_{na}(U)$ have the Nikodym property for $T = 2^A$, A uncountable?

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Department of Mathematics, University of North Carolina, Chapel Hill, NC 27514

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115