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ON DIOPHANTINE EQUATIONS OF THE FORM $1 + 2^a = p^b q^c + 2^d p^e q^f$

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ABSTRACT. In this paper several Diophantine equations of the form $1 + 2^a = p^b q^c + 2^d p^e q^f$, where p and q are distinct odd primes, and the exponents are non-negative integers are solved. In particular this equation is solved for $(p, q) = (23, 47)$, $(7, 23)$, and $(73,223)$. The related equations $1 + 73^a = 2^b 223^c + 2^d 73^e 223^f$ and $1 + 223^a = 2^b 73^c + 2^d 73^e 223^f$ are also solved. This work extends recent work of the authors and J.L. Brenner.

1. Introduction. In this paper we consider equations of the form

$$(1) \qquad 1 + x = y + z,$$

where the primes dividing xyz are specified. Such equations are exponential Diophantine equations. For example if the primes dividing xyz in (1) are 2, 3, and 5 and $(x, 15) = 1$, then (1) has the form

$$(2) \qquad 1 + 2^a = 3^b 5^c + 2^d 3^e 5^f,$$

where a, b, c, d, e and f are non-negative integers. Thus it is the exponents a, b, c, d, e and f which are to be determined.

These equations (1) and (2) are special cases of the general equation $\sum x_i = 0$, $i = 1, 2, 3, \ldots, m$, where the primes dividing $x_1 x_2 \cdots x_m$ are specified. There has been very little work done in general to solve such equations. For example the equation

$$(3) \qquad 1 + 3^a = 5^b + 3^c 5^d$$

is unsolved. Also it is unknown whether such equations always have a finite number of non-trivial solutions. Such equations always have an infinite number of trivial solutions. For example the equation (3) above has infinitely many solutions of the form $b = d = 0$ and $a = c$.

It follows from work of Dubois and Rhin [5] and Schlickewei [6] that the related equation $p^a \pm q^b \pm r^c \pm s^d = 0$ has only finitely many solutions when p, q, r and s are distinct primes. However, their methods do not seem to apply when the terms in the equation are not powers of distinct primes.

The authors and J.L. Brenner [1], [2], [4] have recently developed

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techniques which solve such equations in some cases. These techniques involve careful consideration of the equation modulo a series of primes and prime powers.

Such equations arise quite naturally in the character theory of finite groups. If G is a finite simple group and p is a prime dividing the order of G to the first power only, then the degrees $x_1, x_2, \ldots, x_m$ of the ordinary irreducible characters in the principal p-block of G satisfy an equation of the form $\sum \delta_i x_i = 0$, $\delta_i = \pm 1$, where the primes dividing $x_1 x_2 \ldots x_m$ are those in $|G|/p$. Much information concerning the group G can be obtained from the solutions to this degree equation. For example, one of the authors in [3] has used solutions to equation (2) above to characterize the simple groups $L(2, 7)$, $U(3, 3)$, $L(3, 4)$ and A_8.

In §2, the equations $1 + 2^a = 23^b 47^c + 2^d 23^e 47^f$, $1 + 2^a = 7^b 23^c + 2^d 7^e 23^f$, and $1 + 2^a = 73^b 223^c + 2^d 73^e 223^f$ are solved.

In §3, the related equations $1 + 73^a = 2^b 223^c + 2^d 73^e 223^f$ and $1 + 223^a = 2^b 73^c + 2^d 73^e 223^f$ are solved.

2. Some equations of the form $\mathbf{1 + 2^a = p^b q^c + 2^d p^e q^f}$. In this section we study equations of the form $1 + 2^a = p^b q^c + 2^d p^e q^f$ where p and q are distinct primes such that the order of 2 is odd mod p and mod q. Hence, if (a, b, c, d, e, f) is a solution of such an equation, $cf = 0$ and $be = 0$. Further, if $b, c = 0$, then $e, f = 0$, $a = d$. We begin by studying the equation

$$(2.1) \qquad 1 + 2^a = 23^b 47^c + 2^d 23^e 47^f.$$

Lemma 2.1. *The equation*

$$(2.2) \qquad 1 + 2^a = 23^b + 2^d 47^f, \qquad b \neq 0$$

has no solution.

Proof. Let (a, b, d, f) be a solution. Clearly $a > 3$ so that $1 \equiv (-1)^b + 2^d(-1)^f \pmod 8$. Thus, either $d \geqq 3$ and b is even, or, $d = 1$, $f \equiv 0$, $b \equiv 1 \pmod 2$. In the latter case, using mod 3, we have a contradiction. Hence b is even. Easily $2^a \equiv 2^{d+8f} \pmod{11}$ so that

$$(2.3) \qquad a \equiv d \equiv 8f \pmod{10}.$$

In particular, $a \equiv d \pmod 2$ so that, using mod 3, f is even. We note that if $b \equiv 0 \pmod 3$, then $b \equiv 0 \pmod 6$ so that, using mod 13, $a \equiv d \pmod 6$. Thus, in the general case, considering our equation mod 63, we have the possibilities listed in Table 2.1. Thus, using mod 13, we have $(a, b, d, f) \equiv (t, 0, t, 0)$, $(t + 6, 0, t, 2)$, $(5, 2, 7, 0)$ or $(5, 2, 1, 2)$ (mod $(12, 6, 12, 4)$), t arbitrary. Hence, using mod 5, we conclude that $(a, b, d, f) \equiv (t, 0, t, 0)$ or $(t + 6, 0, t, 6) \pmod{12}$. Using mod 37,

a	b	d	f
t	0	t	0
5	2	1	2
5	2	3	0
5	2	5	4
1	4	1	2
1	4	3	0
1	4	5	4

Table 2.1. (a, b, d, f) mod 6, t arbitrary.

$a \equiv d \pmod{36}$ so that $(a, b, d, f) \equiv (t, 0, t, 0) \pmod{(36, 12, 36, 12)}$, t arbitrary. Define $m = b/2$. Then

$$1 + 2^b \equiv 2^m + 2^{d-f} \pmod{31}. \tag{2.4}$$

If $m \equiv 0 \pmod 5$, then $a \equiv d - f \pmod 5$, so that, from equation 2.3, $f \equiv 0$, $a \equiv d \pmod 5$. Hence, suppose $m \not\equiv 0 \pmod 5$. Considering (2.4) in base 2 (since $2^5 - 1 = 31$) we conclude that $d \equiv f$, $m \equiv a \pmod 5$. Thus $(a, b, d, f) \equiv (t, 0, t, 0)$ or $(m, 2m, k, k) \pmod{(5, 10, 5, 5)}$. Combining the above results we have the possibilities listed in Table 2.2.

a	b	d	f
t	0	t	0
1	2	9	4
6	2	4	4
2	4	8	8
7	4	3	8
3	6	7	2
8	6	2	2
4	8	6	6
9	8	1	6

Table 2.2. (a, b, d, f) mod 10, t arbitrary.

Using mod 61, we thus conclude that $(a, b, d, f) \equiv (t, 0, t, 0)$ or $(59, 48, 11, 36) \pmod{60}$, t arbitrary. In the latter case we have a contradiction using mod 41. Thus $(a, b, d, f) \equiv (t, 0, t, 0) \pmod{60}$, t arbitrary.

Since $1 + 2^a \equiv 2^d \pmod{23}$, we have $(a, d,) \equiv (0, 1), (1, 8), (3, 5), (8, 2)$ or $(10, 7) \pmod{11}$. Hence after some calculation using mod 89, we have fourteen possibilities listed in Table 2.3.

a	b	d	f
0	52	1	28
0	0	1	16
1	0	8	24
1	4	8	28
1	8	8	40
1	44	8	8
3	0	5	32
3	24	5	36
8	0	2	36
8	64	2	32
10	36	7	8
10	80	7	24
10	84	7	28
10	0	7	40

Table 2.3. (a, b, d, f) mod $(11, 88, 11, 44)$.

We now consider equation 2.2 mod 331. (The orders of 2, 23 and 47 are 30, 66, 66 respectively mod 331). Recall that $(a, b, d, f) \equiv (t, 0, t, 0) \pmod{60}$, t arbitrary and write $b = 6k, f = 6m$. Since $23^6 \equiv 111, 47^6 \equiv 74 \pmod{331}$, equation 2.2 becomes $2^t(74^m - 1) \equiv 1 - 111^k \pmod{331}$. Checking cases in table 2.3, we find that $(a, b, d, f) \equiv (23, 48, 41, 6)$ or $(63, 42, 57, 54) \pmod{66}$. Using mod 67, we thus have a contradiction.

LEMMA 2.2. *The equation*

$$(2.5) \qquad 1 + 2^a = 47^c + 2^d 23^e, \qquad c \neq 0$$

has no solution.

PROOF. Suppose that (a, c, d, e) is a solution. Using mod 23, $e = 0$. Hence

$$(2.6) \qquad 1 + 2^a = 47^c + 2^d, \qquad c \neq 0.$$

Since $2^a \equiv 2^d \pmod{23}$, we conclude that $a \equiv d \pmod{11}$ so that $1 \equiv 47^c \pmod{89}$ and $c \equiv 0 \pmod{44}$. Then $2^a \equiv 2^d \pmod{13}$ so that $a \equiv d \pmod{12}$. Hence $1 \equiv 5^c \pmod 7$, $c \equiv 0 \pmod 6$.

Since $1 + 2^a \equiv 2^d \pmod{47}$, there are eleven cases: $(a, d) \equiv (0, 1)$, $(1, 19)$, $(3, 15)$, $(4, 6)$, $(6, 16)$, $(11, 14)$, $(12, 3)$, $(17, 10)$, $(19, 2)$, $(20, 12)$ or $(22, 18) \pmod{23}$. Now consider equation 2.6 mod 139. (The orders of 2 and 47 are 138 and 69 respectively mod 139). After some calculation we have the following nine cases listed in Table 2.4.

a	c	d
23	51	47
69	27	93
93	42	111
3	51	15
49	27	61
50	6	98
29	12	131
88	12	94
22	54	64

Table 2.4. (a, c, d) mod $(138, 69, 138)$.

In each case, using mod 277 we have a contradiction. (The orders of 2, 47 (mod 277) are 92, 138 respectively).

LEMMA 2.3. *The equation*

$$(2.7) \qquad 1 + 2^a = 23^b 47^c + 2^d, \qquad bc \neq 0$$

has no solution.

PROOF. Suppose that (a, b, c, d) is a solution. Clearly $a \geqq 5$. Using mod 32, we have four possiblities:

Case 1. $d = 1$, $b \equiv 2 \pmod 4$, c odd.
Case 2. $d = 3$, $b \equiv 1 \pmod 4$, c odd.
Case 3. $d = 4$, $b \equiv 2 \pmod 4$, c even.
Case 4. $d \geqq 5$, $b \equiv 0 \pmod 4$, c even.

In Case 1 we have an immediate contradiction using mod 3. In Cases 2 and 3 we have contradictions using mod 23. In Case 4, using mod 23, we have the following five possibilities: $(a, d) \equiv (0, 1)$, $(1, 8)$, $(3, 5)$, $(8, 2)$ or $(10, 7) \pmod{11}$. Now $b = 4k$, $c = 2m$ for some integers k, m so that

$$1 + 2^a \equiv 23^{4k} \cdot 47^{2m} + 2^d \equiv 23^{4k+76m} + 2^d \pmod{89}.$$

Thus each of the five possibilities for $(a, d) \pmod{11}$ leads to a contradiction.

Combining the above results, we have the following theorem.

THEOREM 2.4. *Equation* 2.1 *has only the solutions* $(a, b, c, d, e, f) = (t, 0, 0, t, 0, 0)$, t arbitrary.

Secondly, we find all solutions to

$$1 + 2^a = 7^b 23^c + 2^d 7^e 23^f. \tag{2.8}$$

LEMMA 2.5. *The equation*

$$1 + 2^a = 7^b + 2^d, \qquad b \neq 0 \tag{2.9}$$

has only the solutions $(a, b, d) = (6, 2, 4)$ *or* $(3, 1, 1)$.

PROOF. Let (a, b, d) be another solution. Then $2^a \equiv 2^d \pmod 3$ so that $a \equiv d \pmod 2$. Clearly $a \geqq 7$ so that $1 \equiv 7^b + 2^d \pmod{128}$. Easily, $d \geqq 4$. We consider three cases.

Case 1 is $d = 4$. Then $2^b = 7^b + 15$ so that $7^b \equiv 113 \pmod{128}$, $b \equiv 10 \pmod{16}$.

Hence $2^a \equiv 7^{10} + 15 \equiv 0 \pmod{17}$, a contradiction.

Case 2 is $d = 5$. Then $2^a = 7^b + 31$, $2^a \equiv 3 \pmod 7$, again a contradiction.

Case 3 is $d \geqq 6$. Then $1 \equiv 7^b \pmod{64}$ so that $b \equiv 0 \pmod 8$. Thus, using mod 7, 9 successively, $(a, b, d) \equiv (0, 2, 4)$ or $(3, 1, 1) \pmod{(6, 3, 6)}$. Therefore, using mod 19, $(a, b, d) \equiv (6, 2, 4)$ or $(3, 1, 1) \pmod{(18, 3, 18)}$. Since $b \equiv 0 \pmod 8$, we immediately have a contradiction in each case using mod 73.

LEMMA 2.6. *The equation*

$$1 + 2^a = 23^c + 2^d 7^e, \qquad c \neq 0 \tag{2.10}$$

has no solutions.

PROOF. Let (a, c, d, e) be a solution, $c \neq 0$. By Lemma 2.1, $e \neq 0$. Using mod 7, $(a, c) \equiv (0, 1) \pmod 3$. Clearly $a > 4$ so that using mod 16, c is even and $c \equiv 4 \pmod 6$. Further, using mod 9, $(a, d, e) \equiv (0, 0, 1), (0, 2, 2), (0, 4, 0), (3, 1, 1), (3, 3, 2)$ or $(3, 5, 0) \pmod{(6, 6, 3)}$. Thus in particular $a \equiv d \pmod 2$ so that, using mod 11, e is even. Therefore, using mod 13, $(a, d, e) \equiv (0, 2, 8), (0, 8, 2), (3, 5, 0), (3, 11, 6), (6, 4, 6)$ or $(6, 10, 0) \pmod{12}$. Hence, using mod 5, $(a, c, d, e) + (6, 4, 4, 6)$ or $(6, 4, 10, 0) \pmod{12}$. In both cases we have a contradiction mod 37.

LEMMA 2.7. *The equation*

$$1 + 2^a = 7^b + 2^d 23^f, \qquad bf \neq 0 \tag{2.11}$$

has no solution.

PROOF. Let (a, b, d, f) be a solution, $bf \neq 0$. Clearly $a \geqq 6$. Using mod 64, we conclude that $d \geqq 3$. Thus $1 \equiv 7^b \pmod 8$, b even. Now $1 + 2^a \equiv 2^{d+f} \pmod 7$ so that $(a, d, f) \equiv (0, 0, 1), (0, 1, 0)$ or $(0, 2, 2) \pmod 3$. Each case yields four cases, considering (2.11) mod 9. These are listed in Table 2.5.

a	b	d	f	a	b	d	f
0	4	3	1	3	4	1	0
0	4	0	4	3	4	4	3
3	2	0	1	0	0	2	2
3	2	3	4	0	0	5	5
0	2	4	0	3	0	2	5
0	2	1	3	3	0	5	2

Table 2.5. (a, b, d, f) mod 6.

Using mod 13, mod 5 successively, we have $(a, b, d, f) \equiv (6, 2, 4, 0)$, $(6, 8, 10, 0)$, $(9, 8, 6, 1)$ or $(9, 4, 4, 3) \pmod{12}$. Using mod 19, there is a contradiction in the last two cases. In the first two cases we conclude $a \equiv 6 \pmod{18}$ so that $a \equiv 6 \pmod{36}$. Hence, using mod 37, $(a, b, d, f) \equiv (6, 2, 4, 0) \pmod{12}$.

Since $1 + 2^a \equiv 7^b \pmod{23}$, $(a, b) \equiv (0.14), (1, 2), (3, 4), (8, 6)$ or $(10, 10) \pmod{(11, 12)}$. Combining results, in each case we have a contradiction using mod 67.

LEMMA 2.8. *The equation*

$$1 + 2^a \equiv 7^b 23^c + 2^d, \qquad bc \neq 0 \tag{2.12}$$

has no solution.

PROOF. Let (a, b, c, d) be a solution, $bc \neq 0$. Clearly $a \geqq 5$. Thus $1 \equiv 7^{b-c} + 2^d \pmod{32}$ so that $d \geqq 4$. If $d = 4$, then $2^a \equiv 15 \pmod{23}$, a contradiction. Hence $d \geqq 5$ so that $1 \equiv 7^{b-c} \pmod{32}$ and $b \equiv c \pmod 4$. Thus using mod 5, $a \equiv d \pmod 4$. Since $1 + 2^a \equiv 2^d \pmod 7$, $(a, d) \equiv (0, 1) \pmod 3$. Thus, using mod 9, we have the possibilites listed in Table 2.6.

a	b	c	d
0	0	4	4
0	4	2	4
0	2	0	4
3	0	2	1
3	4	0	1
3	2	4	1

Table 2.6. (a, b, c, d) mod 6.

Thus using mod 13 we have the cases listed in Table 2.7.

a	b	c	d
6	0	4	10
6	4	8	10
6	8	0	10
9	6	2	1
9	10	6	1
9	2	10	1

Table 2. 7. (a, b, c, d) mod 12.

Using mod 19, 27 successively and combining results once again, we have the cases listed in Table 2.8.

a	b	c	d
18	0	28	10
18	28	32	22
18	28	8	34
6	20	0	22
30	20	24	34
9	18	2	13
9	18	14	1
9	18	26	25
21	10	18	1
23	10	30	1
21	14	10	13

Table 2.8. (a, b, c, d) mod 36.

In all cases, we have a contradiction using mod 37.

Combining the results of the four previous lemmas, we have the following theorem.

THEOREM 2.9. *Equation* 2.8 *has only the solutions* $(a, b, c, d, e, f) = (t, 0, 0, 0, t, 0, 0), (6, 2, 0, 4, 0, 0)$ *or* $(3, 1, 0, 1, 0, 0)$, *t arbitrary.*

Lastly, we study the equation

$$1 + 2^a = 73^b 223^c + 2^d 73^e 223^f . \tag{2.13}$$

LEMMA 2.10. *The equation*

$$1+2^a=73^b+2^d 223^f, \qquad b \neq 0 \tag{2.14}$$

has no solution.

PROOF. Suppose that (a, b, d, f) is a solution, $b \neq 0$. Since $1 + 2^a \equiv 2^{d+2f} \pmod{73}$, $a \equiv 0$, $d + 2f \equiv 1 \pmod 9$. If b is odd, using mod 16, we deduce that $d = 3$ so that $f \equiv 8 \pmod 9$. Immediately, using mod 9, we have a contradiction. Hence b is even so that $2^a \equiv 2^d \pmod{37}$, $a \equiv d \pmod{36}$, $a \equiv d \equiv 0 \pmod 9)$, $f \equiv 5 \pmod 9$. Again we have a contradiction using mod 9.

LEMMA 2.11. *The equation*

$$1+2^a= 73^b 223^c + 2^d 73^e, \qquad c \neq 0, \tag{2.15}$$

has no solution

PROOF. Let (a, b, c, d, e) be a solution, $c \neq 0$. Suppose that $e \neq 0$. Then $b = 0$ and $1 + 2^a \equiv 2^{2c} \pmod{73}$, $a \equiv 0$, $c \equiv 5 \pmod 9$. Since $2^a \equiv 2^d(-1)^e \pmod{37}$, $a \equiv d$ or $d + 18 \pmod{36}$. In any event, $a \equiv d \pmod 6$ so that using mod 9, we have an immediate contradiction.

Hence $e = 0$. If $b \neq 0$. then using mod 73 once again, $a \equiv 0$, $d \equiv 1 \pmod 9$. Using mod 7, we have an immediate contradiction.

Thus $b = 0$ so that

$$1 + 2^a = 223^c + 2^d, \qquad c \neq 0. \tag{2.16}$$

Using mod 37, $a \equiv d \pmod{36}$. Thus $1 \equiv 3^c \pmod 5$ so that $c \equiv 0 \pmod 4$. Further $a > 9$. Also $1 + 2^a \equiv 2^d \pmod{223}$. Thus there are nine pairs (a, d) (mod 37). Since $a \equiv d \pmod 2$, we have eighteen possibilities: $(a, d) \equiv (0, 38), (32, 40), (4, 60), (19, 41), (5, 45), (8, 24), (33, 19), (29, 53), (18, 22), (37, 1), (69, 3), (41, 23), (56, 4), (42, 8), (45, 61), (70, 56), (66, 16)$ or $(55, 59) \pmod{74}$. Consider the prime $P = 1777$ (The orders of 2, 223 are 74, $16 \cdot 37$, respectively mod P). We have three cases.

Case 1 is $d \geqq 9$. Then $1 \equiv (223)^c$ (mod (512) so that $c \equiv 0 \pmod{16}$. Using mod P and considering the eighteen possibilities for $(a, d) \pmod{74}$, we conclude after some calculation that $(a, c, d) \equiv (0, 368, 38)$, or $(37, 480, 1) \pmod{(74, 592, 74)}$. Hence, since $a \equiv d \pmod 4$, we have four possibilities: $(a, c, d) = (0, 368, 112), (74, 368, 38), (37, 480, 1)$ or $(111, 480, 75) \pmod{(148, 592, 148)}$. Now consider the prime $Q = 149$. (2 and 223 are primitive roots mod Q and $Q - 1 \mid 592$). Hence each of the four triples (a, c, d) above yields a contradiction considering (2. 16) mod Q.

Case 2 is $d = 8$. Then examining the eighteen possibilities for $(a, b) \pmod{74}$ above, we conclude that $a \equiv 42 \pmod{74}$ and in fact $a \equiv 116 \pmod{148}$. Thus $110 \equiv 223^c \pmod{149}$ so that $c \equiv 82 \pmod{148}$, $c \not\equiv 0 \pmod 4$, a contradiction.

Case 3 is $d < 8$. Then examining $(a, d) \pmod{74}$, we have $d = 1, 3$ or 4. Easily, using mod 64, we have a contradiction.

Thus we have proven the following theorem.

THEOREM 2.12. *The only solutions to* (2.13) *are* $(a, b, c, d, e, f) = (t, 0, 0, t, 0, 0)$, *$t$ arbitrary.*

3. Related equations. In this section we study two equations related to (2.13).

THEOREM 3.1. *The equation*

$$1 + 73^a = 2^b 223^c + 2^d 73^e 223^f \tag{3.1}$$

has only the solutions $(t, 0, 0, 0, t, 0)$, *t arbitrary.*

PROOF. Let (a, b, c, d, e, f) be another solution. Using mod 3, b and d are even. Thus, using mod 8, $b = d = 0$. Further, since 73 has order $3 \cdot 37$ mod 223, $cf = 0$ so that $f = 0$. Hence we have

$$1 + 73^a = 223^c + 73^e, \qquad ace \neq 0\,. \tag{3.2}$$

Thus $1 \equiv 4^c \pmod{73}$ so that $c \equiv 0 \pmod 9$. Using mod 4, we conclude that $c \equiv 0 \pmod{18}$. Hence $(-8)^a \equiv (-8)^e \pmod{27}$ so that $a \equiv e \pmod 3$. Further $9^a \equiv 9^e \pmod{32}$ so that $a \equiv e \pmod 4$. Hence $1 \equiv 3^c \pmod 5$ so that $c \equiv 0 \pmod 4$. Therefore $73^a \equiv 73^e \pmod{128}$ so that $a \equiv e \pmod{16}$. Thus $1 \equiv 2^c \pmod{17}$ so that $c \equiv 0 \pmod 8$. Hence $1 + 73^a \equiv 73^e \pmod{223}$. Since $a \equiv e \pmod 3$, after considerable calculation, we find that there are eighteen possibilities: $(a, e) \equiv (0, 24), (1, 64), (5, 74), (8, 2), (9, 81), (12, 96), (15, 12), (26, 59), (30, 51), (31, 40), (34, 100), (55, 1), (61, 79), (81, 21), (89,14), (96, 108), (99, 84)$, or $(102, 72) \pmod{111}$. Hence, using mod 149 (73 and 223 have orders 37 and 148 respectively), we eliminate all but five cases. Thus $(a, c, e) \equiv (5, 88, 0), (15, 84, 12), (31, 64, 3), (15, 32, 14)$, or $(22, 116, 34) \pmod{(37, 148, 37)}$. Now consider (3.2) mod 37^2. (73, 223 have orders 74, 37 respectively). Since $a \equiv e \pmod 2$, we have ten possibilities. Only one of these, $(a, e, c) \equiv (5, 14, 37) \pmod{(74, 37, 74)}$, does not yield a contradiction. Finally consider (3.2) mod the prime 593. (73, 223) have orders 8.37 and 592 respectively). Since $c \equiv 0$, $a \equiv e$ (mod 8), there are eight cases. In each of these we have a contradiction.

THEROEM 3.2. *The only solutions to*

$$1 + 223^a \equiv 2^b 73^c + 2^d 73^e 223^f \tag{3.3}$$

are $(a, b, c, d, e, f) \equiv (t, 0, 0, 0, 0, t)$, *t arbitrary.*

PROOF. Let (a, b, c, d, e, f) be another solution. Using mod 73, $ce = 0$. Immediately, using mod 64, $\min\{b, d\} \leqq 5$. Also, using mod 3, b and d are even.

Suppsoe $c = 0$ so that

$$1 + 223^a = 2^b + 2^d 73^e 223^f . \tag{3.4}$$

Easily $b, d > 0$. Further $2 \equiv 2^b \pm 2^d$ (mod 37). Since b and d are even, in each of the four cases b or $d = 2$ or 4 and we easily have a contradiction.

Hence $c > 0$, $e = 0$ so that

$$1 + 223^a = 2^b 73^c + 2^d 223^f . \tag{3.5}$$

Suppose $bd = 0$ so that $b = d = 0$, $1 + 223^a = 73^c + 223^f$. Immediately $7^a \equiv 7^f$ (mod 9) so that $a \equiv f$ (mod 3), yielding a contradiction mod 73. Hence $bd > 0$, $2 \equiv \pm 2^b + 2^d$ (mod 37). As in the previous case we have a contradiction.

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