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A PRESERVATION OF INTEGRABILITY CHARACTERIZATION THEOREM

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ABSTRACT. Suppose N is a positive integer and $\mathbf{Q}$ denotes the set to which g belongs if and only if g is a function from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ such that for some $(w_1, \ldots, w_N)$ in $\mathbf{R}^N$ and $d > 0$, $g(w_1, \ldots, w_N)$ is bounded on $[-d; d]$. A characterization is given of those elements f of $\mathbf{Q}$ having the property that if U is a set, $\mathbf{F}$ is a field of subsets of U, each of $\alpha_1, \ldots \alpha_N$ is a function from $\mathbf{F}$ into a collection of subsets of $\mathbf{R}$ with bounded union, ξ is a real-valued, bounded finitely additive function defined on $\mathbf{F}$ and each of the set function integrals $\int_U \alpha_1(I)\xi(I), \ldots, \int_U \alpha_N(I)\xi(I)$ exists, then the integral $\int_U f(\alpha_1(I), \ldots, \alpha_N(I), \xi(I))$ exists, these integrals being limits for subdivision refinement.

1. Introduction. Suppose N is a positive integer. In a previous paper [5] (see [2] for the earlier interval function version) the author showed the following preservation of integrability characterization theorem (see §2 for the notion of integral.

THEOREM 1.A.1. *If f is a function from $\mathbf{R}^N$ into $\mathbf{R}$, then the following two statements are equivalent.*

1) *If $\mathbf{F}$ is a field of subsets of a set U, ξ is a real-valued bounded finitely additive function defined on $\mathbf{F}$, and each of $\alpha_1, \ldots, \alpha_N$ is a function from $\mathbf{F}$ into a collection of subsets of $\mathbf{R}$ with bounded union* (in [5] *the α's were single valued, but the argument carries over for this version with trivial modifications*) *such that each of the integrals $\int_U \alpha_1(I)\xi(I), \ldots, \int_U \alpha_N(I)\xi(I)$ exists, then the integral $\int_U f(\alpha_1(I), \ldots, \alpha_N(I))\xi(I)$ exists.*

2) *The function f is continuous.*

In this paper we extend the above theorem. Notice that if f is given as above and h is a function from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ such that for each $(x_1, \ldots, x_N, z)$ in $\mathbf{R}^{N+1}$, $h(x_1, \ldots, x_N, z) = f(x_1, \ldots, x_N)z$, then the conclusion of statement 1) above has the form "$\int_U h(\alpha_1(I), \ldots, \alpha_N(I), \xi(I))$ exists". The question naturally arises as to whether there exists a class, Q, of functions from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ that includes the functions of the form "$f(x_1, \ldots, x_N)z$", and a subset which has the integrability

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preservation property described in statement 1) with the conclusion of the form given above for h, such that this subset can be characterized in a "sequence vs. convergence" manner similar to continuity. To this end we notice that for h defined as above, trivially there are some $(w_1, \ldots, w_N)$ in $\mathbf{R}^N$ and $d > 0$ such that $h(w_1, \ldots, w_N)$ is bounded on $[-d, d]$. Accordingly, we shall let $\mathbf{Q}$ denote the set to which g belongs if and only if g is a function from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ such that for some $(w_1, \ldots, w_N)$ in $\mathbf{R}^N$ and $d > 0$, $g(w_1, \ldots, w_N)$ is bounded on $[-d, d]$. Our generalization of Theorem 1.A.1 is a characterization of those elements g of $\mathbf{Q}$ such that if $\mathbf{F}$, U, ξ and $\alpha_1, \ldots, \alpha_N$ are as in the hypothesis of statement 1) of Theorem 1.A.1, then the integral $\int_U g(\alpha_1(I), \ldots, \alpha_N(I), \xi(I))$ exists. We shall first describe in general terms what this generalization is. To begin with, it is an assertion of the equivalence of three statements. The first of these statements is the immediately preceding remark about integrability preservation. The second is an analogue of the first statement for bounded interval functions and functions of bounded variation on the interval [0, 1]. The third is a "sequence vs. convergence" condition.

As the reader might guess at this point, even without an explicit rendition of the above three statements, the deduction of the second statement from the first is fairly routine, and we dispose of it (see §3) with relative ease. However, the fact that the first statement follows from the second, even though intuitively plausible, is quite another matter, and its proof involves some fairly intricate considerations, of which one type is the third statement. Thus our arugment for the characterization that we shall state immediately below will proceed as follows: 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1).

THEOREM 3.1. *Suppose g is in $\mathbf{Q}$. The following three statements are equivalent.*

1) *If $\mathbf{F}$ is a field of subsets of a set U, ξ is a real-valued bounded finitely additive function defined on $\mathbf{F}$, an each of $\alpha_1, \ldots, \alpha_N$ is a function from $\mathbf{F}$ into a collection of real number sets with bounded union such that each of the integrals $\int_U \alpha_1(I)\xi(I), \ldots, \int_U \alpha_N(I)\xi(I)$ exists, then the integral $\int_U g(\alpha_1(I), \ldots, \alpha_N(I), \xi(I))$ exists.*

2) *If t is a real-valued function defined and of bounded variation on* $[0; 1]$, *each of $A_1, \ldots, A_N$ is a function from the subintervals of* $[0; 1]$ *into a collection of real number sets with bounded union such that each of the integrals $\int_{[0;1]} A_1(I)dt, \ldots, \int_{[0;1]} A_N(I)dt$ exists, then the integral $\int_{[0;1]} g(A_1(I), \ldots, A_N(I), dt)$ exists.*

3) *The following statements hold.*

a) *The function g is continuous.*

b) *If $0 < \min\{c, M\}$, then there is $d > 0$ such that if $\{(a_1^{(j)}, \ldots, a_N^{(j)},$*

$x_j)\}_{j=1}^{n}$ *is a sequence of elements of* $\mathbf{R}^{N+1}$ *with* $\max\{|a_i^{(j)}|: i = 1, \ldots, N, j = 1, \ldots, n\} \leqq M$ *and* $\sum_{j=1}^{n}|x_j| < d$, *then*

$$\sum_{j=1}^{n} |g(a_1^{(j)}, \ldots, a_N^{(j)}, x_j)| < c\,.$$

c) *Suppose* $\{D(j)\}_{j=1}^{\infty}$ *is a sequence of interval subdivisions of* $[0; 1]$ *such that* $D(n+1) \ll D(n)$ *(see* §2) *for all* n, *and* h *is a real-valued function defined on* $\mathbf{W} = \{x: x \text{ in } \{p, q\}, [p; q] \text{ in } D(n) \text{ for some } n\}$. *Suppose for each* $[p; q]$ *such that* $\{p, q\} \subseteqq \mathbf{W}$, $v[p; q] = \sup\{\sum_{D(m)[p;q]}|h(s) - h(r)|: m$ *a positive integer*, $\{p, q\} \subseteqq \bigcup_{[r;s] \text{ in } D(m)} \{r, s\}$, $D(m)[p; q] = \{[r; s] : [r: s]$ *in* $D(m)$, $p \leqq r < s \leqq q\}\} < \infty$. *Suppose* $M > 0$ *and for each positive integer* n, *each of* $a_1(n), \ldots, a_N(n)$ *is a function from* $D(n)$ *into* $\mathbf{R}$ *such that* $\max\{|a_1(n)(I)|, \ldots, |a_N(n)|\} \leqq M$ *for all* I *in* $D(n)$. *Suppose*

$$\sum_{k=1}^{N}\sum_{j=1}^{\infty}\sum_{D(j)}\left[\sum_{D(j+1)(I)} |a_k(j)(I) - a_k(j+1)(J)|v(J)\right] < \infty$$

(*again, see section* **2** *for notation*). *Then, if* $0 < c$, *there are a real number* $d > 0$ *and a positive integer* N^* *such that for any positive integer* m *greater than or equal to* N^* *and* $D(m)^* = \{I: I \text{ in } D(m), v(I) < d\}$,

$$\left|\sum_{D(m)^*} [g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h) - \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h)]\right| < c\,.$$

The author wishes to thank the referee for his many helpful suggestions for the improvement of this paper, as well as for pointing out at least one forgetful omission. The referee has suggested that certain observations be made about Theorem 3.1. We now complete this introduction with statements of these observations.

Theorem 3.1 remains valid if in statements 1) and 2), respectively, $\alpha_1, \ldots, \alpha_N$ and $A_1, \ldots, A_N$ are singleton-valued, so that we really have five equivalent statements, the first three of which are Theorem 3.1, and the last two of which are the above assertions about singleton-valued functions. We leave to the reader the fairly easy task of modifying the appropriate portions of the proof of Theorem 3.1 for singleton-valued functions.

In Theorem 3.1, statement 1) implies statement 2) independently of whether the function g, from $\mathbf{R}^{N+1}$ into $\mathbf{R}$, is in $\mathbf{Q}$.

Suppose that $a < b$, f is a function with domain the set of all subintervals of $[a; b]$ and range a collection of real number sets with bounded union, and h a function from $[a; b]$ into $\mathbf{R}$ having bounded variation. Let $B = \{(r; s]: a \leqq r < s \leqq b\}$, and $\mathbf{F}$ be the field of subsets of $(a; b]$

which is the collection of all unions of finite subcollections of B. Let ξ denote the bounded, finitely additive function from $\mathbf{F}$ into $\mathbf{R}$ such that if $a \leqq r < s \leqq b$, then $\xi((r;\ s]) = h(s) - h(r)$. Then $\int_{[a;b]} f(I)dh$ exists if and only if $\int_{(a;b]} g(J)\xi(J)$ exists for some g from $\mathbf{F}$ into $\exp(\mathbf{R})$ such that $a \leqq r < s \leqq b$ implies $g((r;s]) = f([r;s])$, in this case equality holds.

$\mathbf{Q} \supseteqq \mathbf{Q}_0 =$ the set of all functions g from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ such that for some f from $\mathbf{R}^N$ into $\mathbf{R}$, $g(x_1, \ldots, x_N, x) = f(x_1, \ldots, x_N)x$ for all $(x_1, \ldots, x_N, x)$ in $\mathbf{R}^{N+1}$. By Theorems 1.A.1 and 3.1, for each such g and f, f is continuous if and only if the third statement of Theorem 3.1 holds. On the other hands, it follows from Theorem 3.1, independently of Theorem 1.A.1, that for each such g and f, f is continuous if and only if the third statement of Theorem 3.1 holds; thus Theorem 1.A.1 is a consequence of Theorem 3.1.

2. Preliminary theorems and definitions. For the notions of subdivision, refinement and integral, we refer the reader to [1] as they apply to real number set-valued interval functions, and to [3] and [6] as they apply to real number set-valued set functions.

In this section we shall state set function theorems that we shall use in §3. Each of these has an interval function version, the stating of which we leave to the reader. Throughout this paper, when in a given discussion, the context of set function versus interval function is clear, we shall refer to such notions as "subdivision", "refinement", "integral", etc., without preamble and with at most minor notational changes. In either setting "$\ll$" shall mean "refinement of". If $E \ll D$ and I is in D, then $E(I)$ denotes $\{J\colon J \text{ in } E,\ J \subseteqq I\}$. In certain computations involving real-valued functions defined on number intervals we shall use the "Δ" notation in the standard way to denote differences; when there is possibility of confusion as to which subdivision the differences arise from, appropriate subscripts will be attached. Finally, in the matter of terminology, if S is a set and α is a function from S into a collection of sets and $T \subseteqq S$, then the statement that a is an α-function on T mean that a is a function with domain T such that if x is in T, then $a(x)$ is in $\alpha(x)$.

The following is one of many well-known characterizations of integral existence, and we shall use both its set function and interval function versions.

THEOREM 2.P.1. *If U is a set, $\mathbf{F}$ is a field of subsets of U, α is a function from $\mathbf{F}$ into a collection of subsets of R, then the following three statements are equivalent.*

1) *$\int_U \alpha(I)$ exists.*

2) *If $0 < c$, then there are K in $\mathbf{R}$ and $D \ll \{U\}$ such that if $E \ll D$ and a is an α-function on E, then $|K - \sum_E a(I)| < c$.*

3) *If* $0 < c$, *then there is* $D \ll \{U\}$ *such that if* $H \ll E \ll D$, a *is an* α-*function on* E, a^* *is an* α-*function on* H *and* $G \subseteqq E$, *then*

$$\sum_G |a(V) - \sum_{H(V)} a^*(I)| < c$$

(*see* [10]).

Suppose U is a set, $\mathbf{F}$ is a field of subsets of U, α is a function from $\mathbf{F}$ into a collection of real number sets with bounded union, and each of ξ and μ is a real-valued bounded finitely aditive function defined on $\mathbf{F}$ with μ nonnegative-valued.

THEOREM 2.A.1. [7] (see [1] for interval function version). $\int_U \alpha(I)\xi(I)$ *exist if and only if* $\int_U \alpha(I) \int_I |\xi(J)|$ *exists.*

We now state two consequences of the Bochner-Radon-Nikodym Theorem.

THEOREM 2.A.2. [6]. *If* $\int_U \alpha(I)\mu(I)$ *exists, then*

$$\int_U \left[\int_V \left|\alpha(V)\mu(I) - \int_I \alpha(J)\mu(J)\right|\right] = 0\,,$$

i.e., if $0 < c$, *then there is* $D \ll \{U\}$ *such that if* $E \ll D$ *and for each* V *in* E, $a(V)$ *is in* $\alpha(V)$, *then*

$$\sum_E \int_V \left|a(V)\mu(I) - \int_I \alpha(J)\mu(J)\right| < c\,.$$

THEOREM 2.A.3. (Michael Keisler, class presentation, also see [8] and [9]). *If* $\int_U \alpha(I)\mu(I)$ *exists and* $0 < c$, *then there is* $D \ll \{U\}$ *such that if* $E \ll D$, *for each* V *in* E, $a^*(V)$ *is in* $\alpha(V)$, $H(V) \ll \{V\}$ *and for each* I *in* $H(V)$, $a(I)$ *is in* $\alpha(I)$, *then*

$$\sum_E \sum_{H(V)} |a^*(V) - a(I)|\mu(I) < c\,.$$

We shall need the following theorem in showing that 3) implies 1) in Theorem 3.1.

THEOREM 2.A.4. [4]. *Suppose that for each* V *in* $\mathbf{F}$, $\mu^*(V) = \inf\{\max\{\mu(I): I \text{ in } D\}: D \ll \{V\}\}$. *Then* $\int_U [\mu(I)^2 - \mu^*(I)^2] = 0$, *which implies that for* $0 < c$, *there is* $D \ll \{U\}$ *such that* $0 \leqq \mu(I) - \mu^*(I) < c$ *if* $E \ll D$ *and* I *is in* E.

We now consider the interval [0; 1] and the well-known "standard" associated field of sets. We let $G_{(0;1]} = \{(p;\ q]: 0 \leqq p < q \leqq 1\}$. and $F_{(0;1]}$ denote the collection of all unions of finite subcollections of $G_{(0;1]}$. The collection $F_{(0;1]}$, as is well known, is a field of subsets of (0; 1]. For each V in $F_{(0;1]}$, we shall let C_V denote the collection of all components of V, note that $C_V \subseteq G_{(0;1]}$, and we shall let $C_V(i)$ denote $\{[p; q]: (p; q] \text{ in } C_V\}$.

Once again, we remind the reader that if $0 \leqq p < q \leqq 1$, then a subdivision of $[p; q]$ is a finite collection of nonoverlapping intervals whose union is $[p; q]$, and that if $\mathbf{F}$ is a field of subsets of a set U and W is in $\mathbf{F}$, then a subdivision of W is a finite collection of mutually exclusive sets of $\mathbf{F}$ whose union is W.

Finally, suppose that D is a subdivision of $[0; 1]$ and A is a function from D into $\mathbf{R}$. We shall let A_S denote the function of subintervals of $[0; 1]$ such that if I is a subinterval of $[0; 1]$, then $A_S(I) = A([p; q])$ if $[p; q]$ is in D and $I \subseteq [p; q]$, and $A_S(I) = 0$ otherwse. We note the important fact that if f is a function from $[0; 1]$ into $\mathbf{R}$ and $E \ll D$, then

$$\sum_D A(I)\Delta_I f = \sum_E A_S(J)\Delta_I f = \int_{[0;1]} A_S(J')df\,.$$

3. The preservation of integrability characterization theorem. In this section we prove Theorem 3.1, as stated in the introduction. Throughout this paper we adopt the convention that $a/b = 0$ if $b = 0$, and has the usual meaning otherwise.

PROOF OF THEOREM 3.1. We first show that 1) implies 2). Suppose 1) is true. Suppose that h is a real-valued function defined and having bounded variation on $[0; 1]$, and each of $A_1, \ldots, A_N$ is a function from the subintervals of $[0; 1]$ into a collection of real number sets with bounded union such that each of $\int_{[0;1]}A_1(I)dh, \ldots, \int_{[0;1]}A_N(I)dh$ exists. Let v denote the variation function of h. By Theorem 2.A.1, each of $\int_{[0;1]}A_1(I)dv, \ldots, \int_{[0;1]}A_N(I)dv$ exists.

For each V in $F_{(0;1]}$, we let $h_*(V)$ denote $\sum_{C_V(i)}\Delta h$, and for each W in $F_{(0;1]}$ we let $v_*(W) = \int_W |h_*(I)|$ and note that $v_*(W) = \sum_{C_W(i)} \Delta v$. For each V in $F_{(0;1]}$ and $k = 1, \ldots, N$, we let $\alpha_k(V)$ denote $\{x: x = (\sum_{C_V(i)}a(I)\Delta v/v_*(V)$, such that for each I in $C_V(i)$, $a(I)$ is in $A_k(I)$; or x is in $A_k(I)$ if $C_V(i) = \{I\}\}$.

Suppose $0 < c$ and $k = 1, \ldots, N$. There is $D \ll \{[0; 1]\}$ such that if $E \ll D$ and for each I in E, $a(I)$ is in $A_k(I)$, then

$$\left|\int_{[0;1]} A_k(I)dv - \sum_E a(I)\Delta v\right| < c\,.$$

Let $D_0 = \{(p; q]: [p; q] \text{ in } D\}$. Suppose $H \ll D_0$, and for each V in H, $b(V)$ is in $\alpha_k(V)$. There is a function a from $\bigcup_H C_V(i)$ such that if V is in H, then either $C_V = \{V\}$, $a(I)$ is in $A_k(I)$ for $C_V(i) = \{I\}$ and $b(V) = a(I)$, or $b(V) = (\sum_{C_V(i)}a(I)\Delta v)/v_*(V)$. Note that for each V in H, $b(V)v_*(V) = \sum_{C_V(i)} a(I)\Delta v$. Now, $\bigcup_H C_V(i) \ll D$, so that

$$\left|\int_{[0;1]} A_k(I)dv - \sum_H b(V)v_*(V)\right|$$
$$= \left|\int_{[0;1]} A_k(I)dv - \sum_H \sum_{C_V(i)} a(I)\Delta v\right| < c\,.$$

Therefore $\int_{(0;1]}\alpha_k(I)v_*(I)$ exists, so that $\int_{(0;1]}\alpha_k(I)h_*(I)$ exists.

Therefore $\int_{(0;1]} g(\alpha_1(I), \ldots, \alpha_N(I), h_*(I))$ exists. So suppose $0 < c$. There is $D \ll \{(0; 1]\}$ such that if $E \ll D$ and for each $k = 1, \ldots, N$ and I in D, $b_k(I)$ is in $\alpha_k(I)$, then

$$\left|\int_{(0;1]} g(\alpha_1(J), \ldots, \alpha_N(J), h_*(J)) - \sum_E g(b_1(I), \ldots, b_N(I), h_*(I))\right| < c\,.$$

Let $D(i) = \bigcup_D C_V(i)$. Suppose $H \ll D(i)$, and for each I in H and $k = 1, \ldots, N$, $a_k(I)$ is in $A_k(I)$. Let $H_0 = \{(p; q]: [p: q] \text{ in } H\}$. For each $(p; q]$ in H_0, $a_k([p; q])$ is in $\alpha_k((p; q])$ and $h_*((p; q]) = h(q) - h(p)$. Therefore, since $H_0 \ll D$,

$$\left|\int_{(0;1]} g(\alpha_1(J), \ldots, \alpha_N(J), h_*(J)) - \sum_H g(a_1(I), \ldots, a_N(I), \Delta h)\right|$$
$$= \left|\int_{(0;1]} g(\alpha_1(J), \ldots, \alpha_N(J), h_*(J)) - \sum_{H_0} g(a_1(W), \ldots, a_N(W), h_*(W))\right| < c\,.$$

Therefore $\int_{[0;1]} g(A_1(J), \ldots, A_N(J), dh)$ exists. Therefore 1) implies 2).

We now show that 2) implies 3). Suppose that 2) is true. Let w denote $w_1, \ldots, w_N$. We begin by showing that $g(w, x) \to 0$ as $x \to 0$, and $g(w, 0) = 0$. Suppose not. Then, by some conventional observations, it follows that there are $c > 0$ and a sequence $\{x_v\}_{v=1}^{\infty}$ of numbers such that the $g(w, x_v)$'s are of consistent sign and for each v, $|g(w, x_v)| \geqq c$ and $|x_v| < 1/2^v$.

There is a function h defined on $[0; 1]$ such that $h(0) = 0$ and if n is a positive integer such that $1/(n + 1) < x \leqq 1/n$, then $h(x) = \sum_{v=n}^{\infty} x_v$. Clearly $h(1/n) - h(1/(n + 1)) = x_n$ for all n, h is of bounded variation on $[0; 1]$, $h(x) \to 0$ as $x \to 0$, and trivially, each of the integrals $\int_{[0;1]} w_1 dh, \ldots, \int_{[0;1]} w_N dh$ exists. Therefore the integral $\int_{[0;1]} g(w, dh)$ exists. From the afrorementioned properties of h and given boundedness conditions on g, it follows that there are $D \ll \{[0; 1]\}$ and $M \geqq 0$ such that if $E \ll D$ and $[0; s]$ is in E, then

$$\max\{|g(w, h(s) - h(0))|, |\sum_E g(w, \Delta h)|\} \leqq M\,.$$

There are $[0; q]$ in D and a positive integer v such that $1/v < q$. From the conditions on g at the end of the previous paragraph and on h at the beginning of this one, it follows that there are a positive integer $t > v$ and $Q \ll \{[1/t; 1/v]\}$ such that

$$\begin{aligned}\left|\sum_Q g(w, \Delta h)\right| &= \sum_Q \left|g(w, \Delta h)\right| \\ &> 2M + \left|g(w, h(q) - h(1/v))\right| + \left|\sum_{D-\{[0;q]\}} g(w, \Delta h)\right|,\end{aligned}$$

so that

$$\begin{aligned}M &< -M - |g(w, h(q) - h(1/v))| - \left|\sum_{D-\{[0;q]\}} g(w, \Delta h)\right| + \left|\sum_Q g(w, \Delta h)\right| \\ &\leqq -|g(w, h(1/t) - h(0))| + \left|g(w, h(q) - h(1/v)) \right.\\ &\qquad \left. + \sum_{D-\{[0;q]\}} g(w, \Delta h) + \sum_Q g(w, \Delta h)\right| \\ &\leqq \left|g(w, h(1/t) - h(0)) + \sum_Q g(w, \Delta h) \right.\\ &\qquad \left. + g(w, h(q) - h(1/v)) + \sum_{D-\{[0;q]\}} g(w, \Delta h)\right|,\end{aligned}$$

a contradiction, inasmuch as

$$\{[0; 1/t]\} \cup Q \cup \{[1/v; q]\} \cup [D - \{[0; q]\}] \ll \mathrm{D}.$$

Therefore $g(w, x) \to 0$ as $x \to 0$, and $g(w, 0) = 0$.

We now show part a) of 3), i.e., that g is continuous. Suppose not. Then there are $(a_1, \ldots, a_N, b)$ in $\mathbf{R}^{N+1}$, $c > 0$ and for each odd positive integer v, $(a_1(v), \ldots, a_N(v), b_v)$ in $\mathbf{R}^{N+1}$ such that $|g(a_1, \ldots, a_N, b) - g(a_1(v), \ldots, a_N(v), b_v)| \geqq c$ and $|b - b_v| + \sum_{k=1}^{N}|a_k(v) - a_k| < 1/2^v$.

There is a function h defined on $[0; 1]$ such that $h(0) = 0$ and if n is a positive integer such that $1/(n + 1) < x \leqq 1/n$, then $h(x) = b_n$ if n is odd, and $h(x) = b$ if n is even. From the second of the immediately preceding inequalities it is clear that h is of bounded variation on $[0; 1]$ and that $h(0+) = b$.

For each $k = 1, \ldots, N$, there is a function P_k of the subintervals of $[0; 1]$ such that if $0 \leqq p < q \leqq 1$, then $P_k[p; q]) = w_k$ when $0 < p$, and if n is a positive integer such that $1/(n + 1) < q \leqq 1/n$, then $P_k([0; q]) = a_k(n)$ when n is odd and $P_k([0; q]) = a_k$ when n is even. If $D \ll \{[0; 1]\}$, $k = 1, \ldots, N$ and $[0; q]$ is in D, then

$$\sum_D P_k(I)\Delta h = P_k([0; q])[h(q) - h(0)] + w_k[h(1) - h(q)],$$

which easily implies that $\int_{[0;1]} P_k(I)dh$ exists and is $a_k b + w_k[b_1 - b]$.

Therefore $\int_{[0;1]} g(P_1(I), \ldots, P_N(I), dh)$ exists, so that there is $D \ll \{[0; 1]\}$ such that if $E \ll D$ and $E^* \ll D$, then

$$\left|\sum_E g(P_1(I), \ldots, P_N(I), \Delta h) - \sum_{E^*} g(P_1(I), \ldots, P_N(I), \Delta h)\right| < c/4 .$$

There is $d > 0$ such that if $|x| < d$, then $|g(w_1, \ldots, w_N, x)| < c/4$. There are t and q with $0 < t < q$ and $[0; q]$ in D such that if $0 < r < s \leqq t$, then $|h(s) - h(r)| < d$. There is an odd positive integer z such that $1/z \leqq t$. Let

$$E = [\{0; 1/(z+1)]\} \cup \{[1/(z+1); 1/z]\} \cup \{[1/z; q]\} \cup (D - \{[0; q]\})$$

and

$$E^* = \{[0; 1/z]\} \cup \{[1/z; q]\} \cup (D - \{[0; q]\}) .$$

Clearly $E \ll D$ and $E^* \ll D$. Therefore

$$\begin{aligned} c/4 &> \left|\sum_E g(P_1(I)), \ldots, P_N(I), \Delta h) - \sum_{E^*} g(P_1(I), \ldots, P_N(I), \Delta h)\right| \\ &= |g(a_1, \ldots, a_N, b) + g(w_1, \ldots, w_N, h(1/z) - h(1/(z+1)) \\ &\quad - g(a_1(z), \ldots, a_N(z), b_z)| \\ &\geqq |g(a_1, \ldots, a_N, b) - g(a_1(z), \ldots, a_N(z), b_z)| \\ &\quad - |g(w_1, \ldots, w_N, h(1/z) - h(1/(z+1))| \\ &\geqq c - c/4 = 3c/4 , \end{aligned}$$

a contradiction. Therefore g is continuous.

We now show that part b) of 3) is true. Suppose not. Then there are $M > 0$ and $c > 0$ such that for $0 < d$ there are a positive integer m and a sequence $\{(a_1(s), \ldots, a_N(s), x_s)\}_{s=1}^m$ of elements of $\mathbf{R}^{N+1}$ with $\max\{|a_k(s)|: k = 1, \ldots, N;\ s = 1, \ldots, m\} \leqq M$ and $\sum_{s=1}^m |x_s| < d$, but $\sum_{s=1}^m |g(a_1(s), \ldots, a_N(s), x_s)| \geqq 2c$. It follows that for each $d > 0$ there is a sequence $\{(a_1(s), \ldots, a_N(s), x_s)\}_{s=1}^m$ of elements of $\mathbf{R}^{N+1}$ such that $\max\{|a_k(s)|: k = 1, \ldots, N;\ s = 1, \ldots, m\} \leqq M$, $\sum_{s=1}^m |x_s| < d$, the values of $\{g(a_1(s), \ldots, a_N(s), x_s)\}_{s=1}^m$ are either all nonpositive or all nonnegative, and $\sum_{s=1}^m |g(a_1(s), \ldots, a_N(s), x_s)| \geqq c$. It therefore follows by some routine observations that for each positive integer p there is a sequence $\{(a_1(p, s), \ldots, a_N(p, s), x_s(p))\}_{s=1}^{m(p)}$ of elements of $\mathbf{R}^{N+1}$ such that $\max\{|a_k(p, s)|: k = 1, \ldots, N;\ s = 1, \ldots, m(p)\} \leqq M$, $\sum_{s=1}^{m(p)} |g(a_1(p, s), \ldots, a_N(p, s), x_s(p))| \geqq c$, and such that $\sum_{p=1}^{\infty} \sum_{s=1}^{m(p)} |x_s(p)| < \infty$ and either the values of $\{g(a_1(p, s), \ldots, a_N(p, s), x_s(p))\}_{s=1}^{m(p)}$ are all nonpositive for all p or are all nonnegative for all p. Now, let $t_1, \ldots, t_w, \ldots$ denote $x_1(1), \ldots, x_{m(1)}(1), x_1(2), \ldots, x_{m(2)}(2), \ldots$, and for each $k = 1, \ldots, N$, let $b_k(1), \ldots, b_k(w), \ldots$ denote $a_k(1, 1), \ldots, a_k(1, m(1)), a_k(2, 1), \ldots, a_k(2, m(2)), \ldots$. From the preceding statements we clearly see that $\sum_{w=1}^{\infty} |t_w| < \infty$ and that if each of n and n' is a positive integer, then

$$\left|\sum_{w=n'}^{n+n'} g(b_1(w), \ldots, b_N(w), t_w)\right| = \sum_{w=n'}^{n+n'} |g(b_1(w), \ldots, b_N(w), t_w)| \to \infty$$

as $n \to \infty$.

There is a function h defined on $[0; 1]$ such that $h(0) = 0$ and if $0 < x \leqq 1$, then $h(x) = \sum_{w=v}^{\infty} t_w$, where $1/(v + 1) < x \leqq 1/v$. Clearly h is of bounded variation on $[0; 1]$. For each $k = 1, \ldots, N$, there is a function A_k of the subintervals of $[0; 1]$ such that if I is a subinterval of $[0; 1]$, then $A_k(I) = b_k(w)$ when $I = [1/(w + 1), 1/w]$, and $A_k(I) = 0$ otherwise.

Suppose $k = 1, \ldots, N$. Let A denote A_k. We show that $\int_{[0;1]} A(I)dh$ exists. Suppose $0 < c$. There is a positive integer X such that $\sum_{w=X}^{\infty}|t_w| < c/(1 + M)$. Let D denote $\{[0; 1/(X + 1)], [1/(X + 1); (1/2)(1/(X + 1) + 1/X)], [(1/2)(1/(X + 1) + 1/X); 1/X], \ldots, [1/2; (1/2)(1/2 + 1)], [(1/2)(1/2 + 1); 1]\}$. Suppose $E \ll D$. Clearly, if I is in E and for some positive integer w, $I = [1/(w + 1); 1/w]$, then $1/w \leqq 1/(X + 1)$, so that for $E' = \{I\colon I \text{ in } E, A(I) \neq 0\}$,

$$\left|\sum_{E} A(I)\Delta h\right| \leqq \sum_{E'} |A(I)||\Delta h| \leqq M \sum_{w=X+1}^{\infty} |t_w| < Mc/(1 + M) < c\,.$$

Therefore $\int_{[0;1]} A(I)dh$ exists and is 0.

Therefore $\int_{[0;1]} g(A_1(I), \ldots, A_N(I), dh)$ exists. However, suppose $D \ll \{[0; 1]\}$ and $0 < P$. We shall show that there is $E \ll D$ such that $P < |\sum_E g(A_1(I), \ldots, A_N(I), \Delta h)|$. Because g is continuous, there is $S \geqq 0$ such that if $(b_1, \ldots, b_N, t)$ is in $\mathbf{R}^{N+1}$ and $\max\{|b_1|, \ldots, |b_N|\} \leqq M$ and $|t| \leqq 1 + \sum_{w=1}^{\infty}|t_w|$, then $|g(b_1, \ldots, b_N, t)| \leqq S$. There are $H \ll D$ and $q > 0$ such that $[0; q]$ is in H and $q < 1$. Let L denote the number of elements in H. There is a positive integer v such that $1/v < q$. There is a positive integer v such that $u > v$ such that

$$\left|\sum_{w=v}^{u} g(b_1(w), \ldots, b_N(w), t_w)\right| > (L + 1)S + P\,.$$

Let E denote $(H - \{[0; q]\}) \cup \{[0; 1/(u + 1)], [1/(u + 1); 1/u], \ldots, [1/(v + 1); 1/v], [1/v; q]\}$. Then

$$\begin{aligned}
&\left|\sum_{E} g(A_1(I), \ldots, A_N(I), \Delta h)\right| \\
&\quad = |g(A_1([0; 1/(u + 1)]), \ldots, A_N([0; 1/(u + 1)]), \Delta h) \\
&\qquad + \sum_{w=v}^{u} g(A_1([1/(w + 1); 1/w]), \ldots, A_N([1/(w + 1); 1/w]), \Delta h) \\
&\qquad + g(A_1([1/v; q]), \ldots, A_N([1/v; q]), \Delta h) \\
&\qquad + \sum_{H-\{[0;q]\}} g(A_1(I), \ldots, A_N(I), \Delta h)| \\
&\quad \geqq -S + \left|\sum_{w=v}^{u} g(b_1(w), \ldots, b_N(w), t_w)\right| - S - (L - 1)S \\
&\quad > -S + (L + 1)S + P - S - (L - 1)S = P.
\end{aligned}$$

Therefore $\int_{[0;1]} g(A_1(I), \ldots, A_N(I), dh)$ does not exist, a contradiction. Therefore b) of 3) is true.

We now show that c) of 3) is true. Suppose that the hypothesis is satisfied, but that the conclusion is not. Then, if $d > 0$ and N^* is a positive integer, there is a positive integer $m \geqq N^*$ such that for $D(m)^* = \{I: I$ in $D(m)$, $v(I) < d\}$,

$$\Big| \sum_{D(m)^*} [g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h)$$
$$- \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h)] \Big| \geqq c\,.$$

Now, for each x in $[0; 1]$, let $\phi(x) = \inf\{z: x \leqq z, z \text{ in } \mathbf{W}\}$. In a fashion similar to showing that a function having bounded variation on an interval is quasi-continuous on that interval, it follows that if x is in $[0; 1]$ and $\phi(x)$ is not in $\mathbf{W}$, then there is a number $r(\phi(x))$ such that $h(z) \to r(\phi(x))$ as $z \to \phi(x)$ for z in $\mathbf{W}$ and $\phi(x) < z$.

There is a function t from $[0; 1]$ into $\mathbf{R}$ such that if x is in $[0; 1]$, then $t(x) = h(\phi(x))$ when $\phi(x)$ is in $\mathbf{W}$, and $t(x) = r(\phi)x))$ when $\phi(x)$ is not in $\mathbf{W}$.

We shall now show that t has bounded variation on $[0; 1]$, and that $\int_{[p;q]} |dt| = v([p; q])$ for all $[p; q]$ such that $\{p; q\} \subseteqq \mathbf{W}$. Suppose $D \ll \{[0; 1]\}$. Let Z denote the number of elements of D. Suppose $0 < c$. Beginning at the right-most interval of D, we see that there is a non-decreasing function β from $\bigcup_D \{p, q\}$ into $\mathbf{W}$ such that if x is in $\bigcup_D \{p, q\}$, then $x \leqq \beta(x)$ and $|t(x) - h(\beta(x))| < c/(2Z)$, so that

$$\begin{aligned}\sum_D |t(q) - t(p)| &= \sum_D |t(q) - h(\beta(q)) + h(\beta(q)) - h(\beta(p)) + h(\beta(p)) - t(p)| \\ &\leqq Zc/(2Z) + \sum_D |h(\beta(q)) - h(\beta(p))| + Zc/(2Z) \\ &\leqq c + v([0; 1])\,.\end{aligned}$$

Therefore t has bounded variation on $[0; 1]$ and $\int_{[0;1]} |dt| \leqq v([0; 1])$. Since $h \subseteqq t$, it follows that $v([p; q) \leqq \int_{[p;q]} |dt|$ for all $[p; q]$ such that $\{p, q\} \subseteqq \mathbf{W}$. Therefore $v([p; q]) = \int_{[p;q]} |dt|$ for all $[p; q]$ such that $\{p, q\} \subseteqq \mathbf{W}$.

Now, for each $[p; q] \subseteqq [0; 1]$, let $\gamma[p; q]$ denote the smallest positive integer n such that some interval of $D(n) \subseteqq [p; q]$, provided there is such; otherwise, let $\gamma[p; q]$ denote the smallest positive integer $> 1/(q - p)$.

For each $[p; q] \subseteqq [0; 1]$ and $k = 1, \ldots, N$, let

$$A_k([p; q]) = \{x : x = \Big\{\sum_{D(n)} \Big[a_k(n)(I) \int_{I \cap [p;q]} |dt|\Big]\Big\} \Big/ \int_{[p;q]} |dt|$$

for some $n \geqq \gamma[p; q]$, or $x = a_k(n)([p; q])$ for some $n \geqq \gamma[p; q]$ such that $[p; q]$ is in $D(n)\}$.

Suppose $k = 1, \ldots, N$ and $0 < c'$. Let A denote A_k and a denote a_k. There is a positive integer T such that

$$\sum_{j=T}^{\infty} \sum_{D(j)} \left[\sum_{D(j+1)(I)} |a(j)(I) - a(j+1)(J)| v(J) \right] < c' .$$

There is $D \ll D(T)$ such that the length of each interval of D is less than the length of each interval of $D(n)$ for which $n \leqq T$, and such that the reciprocal of the length of each interval of $D > T$.

Now suppose $E \ll D$ and for each $[p; q]$ in E, $b[p; q]$ is in $A([p; q))$. For each $[p; q]$ in E, there is a positive integer $n^*[p; q] \geqq \gamma[p; q] > T$ such that either

$$b[p; q] = \left\{ \sum_{D(n^*[p;q])} \left[a(n^*[p; q])(I) \int_{I \cap [p;q]} |dt| \right] \right\} \Big/ \int_{[p;q]} |dt| ,$$

or $b[p; q] = a(n^*[p; q])([p; q])$ with $[p; q]$ in $D(n^*[p; q])$; note that in either case

$$b[p; q] \Delta t = \left[\left\{ \sum_{D(n^*[p;q])} a(n^*[p; q])(I) \int_{I \cap [p;q]} |dt| \right\} \Big/ \int_{[p;q]} |dt| \right] \Delta t .$$

Thus, for $[r; s]$ in $D(T)$ and $[p; q]$ in E,

$$\begin{aligned}
&\left| \sum_{D(T)} a(T)([r; s]) h \Big|_r^s - \sum_E b[p; q] \Delta t \right| = \Bigg| \sum_{D(T)} a(T)([r; s]) h \Big|_r^s \\
&\quad - \sum_E \left[\left\{ \sum_{D(n^*[p;q])} a(n^*[p; q])(I) \int_{I \cap [p;q]} |dt| \right\} \Big/ \int_{[p;q]} |dt| \right] \Delta t \Bigg| \\
&= \Bigg| \sum_{D(T)} \Bigg\{ a(T)([r; s]) t \Big|_r^s - \sum_{E[r;s]} \Bigg[\Bigg\{ \sum_{D(n^*[p;q])} a(n^*[p; q])(I) \cdot \\
&\qquad \int_{I \cap [p;q]} |dt| \Bigg\} \Big/ \int_{[p;q]} |dt| \Bigg] \Delta t \Bigg\} \Bigg| \\
&= \Bigg| \sum_{D(T)} \sum_{E[r;s]} \Bigg\{ a(T)([r; s]) - \Bigg[\Bigg\{ \sum_{D(n^*[p;q])} a(n^*[p; q])(I) \cdot \\
&\qquad \int_{I \cap [p;q]} |dt| \Bigg\} \Big/ \int_{[p;q]} |dt| \Bigg\} \Delta t \Bigg| \\
&= \Bigg| \sum_{D(T)} \sum_{E[r;s]} \Bigg\{ a(T)([r; s]) \int_{[p;q]} |dt| - \Bigg\{ \sum_{D(n^*[p;q])} a(n^*[p; q])(I) \cdot \\
&\qquad \int_{I \cap [p;q]} |dt| \Bigg\} \Bigg\} \left(\Delta t \Big/ \int_{[p;q]} |dt| \right) \Bigg| \\
&= \Bigg| \sum_{D(T)} \sum_{E[r;s]} \Bigg\{ \sum_{D(n^*[p;q])} [a(T)([r; s]) - a(n^*[p; q])(I)] \cdot \\
&\qquad \int_{I \cap [p;q]} |dt| \Bigg\} \left(\Delta t \Big/ \int_{[p;q]} |dt| \right) \Bigg|
\end{aligned}$$

$$\leqq \sum_{D(T)} \sum_{E[r;s]} \left| \sum_{D(n^*[p;q])} [a(T)([r;s]) - a(n^*[p;q])(I) \cdot \int_{I\cap[p;q]} |dt| \right| \cdot 1$$

$$\leqq \sum_{D(T)} \sum_{E[r;s]} \sum_{D(n^*[p;q])} |a(T)([r;s]) - a(n^*[p;q](I)| \int_{I\cap[p;q]} |dt| \,.$$

For each positive integer n, we shall let $\delta(n)$ denote $a(n)_S$ (see §2). We see that the preceding sum equals

$$\begin{aligned} &\sum_{D(T)} \sum_{E[r;s]} \int_{[p;q]} |\delta(T)(J) - \delta(n^*[p;q])(J)| \int_J |dt| \\ &\quad = \sum_E \int_{[p;q]} |\delta(T)(J) - \delta(n^*[p;q])(J)| \int_J |dt| \\ &\quad \leqq \sum_E \int_{[p;q]} \sum_{H=T}^{Q} |\delta(H)(J) - \delta(H+1)(J)| \int_J |dt| \,, \end{aligned}$$

where $Q = \max\{n^*[p;q], [p;q] \text{ in } E\}$, so that the preceding sum equals

$$\begin{aligned} &\int_{[0;1]} \sum_{H=T}^{Q} |\delta(H)(J) - \delta(H+1)(J)| \int_J |dt| \\ &\quad = \sum_{H=T}^{Q} \int_{[0;1]} |\delta(H)(I) - \delta(H+1)(J)| \int_J |dt| \\ &\quad = \sum_{H=T}^{Q} \sum_{D(H)} \left[\sum_{D(H+1)(I)} |a(H)(I) - a(H+1)(J)| v(J) \right] < c' \,. \end{aligned}$$

Therefore $\int_{[0;1]} A_k(J) dt$ exists for $k = 1, \ldots, N$.

Therefore $\int_{[0;1]} g(A_1(J), \ldots, A_N(J), dt)$ exists. However, suppose $H \ll \{[0;1]\}$. By b) there is $d' > 0$ such that if $\{(z_1(j), \ldots, z_N(j), x_j)\}_{j=1}^n$ is a sequence of elements in $\mathbf{R}^{N+1}$ with $\max\{|z_k(j)| : k = 1, \ldots, N; j = 1, \ldots, n\} \leqq M$ and $\sum_{j=1}^n |x_j| < d'$, then

$$\sum_{j=1}^{n} |g(z_1(j), \ldots, z_N(j), x_j)| < c/(8W) \,,$$

where W equals the number of intervals in H. There are a positive integer m, and an I an $D(m)$ such that $v(I) < d'$ and such that $D(m)^* = \{I : I \text{ in } D(m), v(I) < d'\}$,

$$\begin{aligned} &\Bigg| \sum_{D(m)^*} \Bigg[g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h) \\ &\quad - \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h) \Bigg] \Bigg| \geqq c \,. \end{aligned}$$

Suppose $D(m)^{**} = \{I : I \text{ in } D(m)^*, I \text{ a subset of no interval of } H\}$. $D(m)^{**}$ contains not more than W elements. Furthermore,

$$\left| \sum_{D(m)^{**}} \left[g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h) \right.\right.$$

$$\left.\left. - \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h) \right] \right|$$

$$\leqq \sum_{D(m)^{**}} |g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h)|$$

$$+ \sum_{D(m)^{**}} \sum_{D(m+1)(I)} |g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h)|$$

$$\leqq Wc/(8W) + Wc/(8W) = c/4 .$$

Therefore $D(m)^{**}$ is a proper subset of $D(m)^*$. Let $E(m)^* = D(m)^* - D(m)^{**}$. Then

$$\left| \sum_{E(m)^*} \left[g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I t) \right.\right.$$

$$\left.\left. - \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J t) \right] \right|$$

$$= \left| \sum_{D(m)^*} \left[g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h) \right.\right.$$

$$\left. - \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h) \right]$$

$$- \sum_{D(m)^{**}} \left[g(a_1(m)(I), \ldots, a_N(m)(I), \Delta_I h) \right.$$

$$\left.\left. - \sum_{D(m+1)(I)} g(a_1(m+1)(J), \ldots, a_N(m+1)(J), \Delta_J h) \right] \right|$$

$$\geqq c - c/4 = 3c/4 .$$

Therefore $\int_{[0;1]} g(A_1(J), \ldots, A_N(J), dt)$ does not exist, a contradiction. Therefore 2) implies 3).

We now show that 3) implies 1). Suppose 3) is true. Suppose the hypothesis of 1) is satisfied, but that $\int_U (g(\alpha_1(I), \ldots \alpha_N(I), \xi(I))$ does not exist. Then there is $c > 0$ such that for $D \ll \{U\}$ and K in $\mathbf{R}$ there is $E \ll D$ and for $k = 1, \ldots, N$ there is an α_k-function a_k on E such that

$$\left| K - \sum_E g(a_1(I), \ldots, a_N(I), \xi(I)) \right| \geqq a .$$

Let $\eta = \int|\xi|$. By Theorem 2.A.1 each of $\int_U \alpha_1(I)\eta(I), \ldots, \int_U \alpha_N(I)\eta(I)$ exists.

By routine considerations involving common refinements, there is a sequence $\{H(n)\}_{n=1}^{\infty}$ of subdivisions of U such that if n is a positive integer, then

i′) $\eta(U) - \sum_{H(n)} |\xi(I)| < 1/n$, ii′) if I in $E \ll H(n)$, then $\eta(I) - \eta^*(I) < 1/2^n$, and

iii′) if $E \ll D \ll H(n)$, and for $k = 1, \ldots, N$, each of $b(k)$ and $a(k)$ is an α_k-function on D and E respectively, then

$$\sum_{k=1}^{N} \sum_{D} \sum_{E(I)} |b(k)(I) - a(k)(J)|\eta(J) < 1/2^n .$$

By induction there is a sequence $\{E(n)\}_{n=1}^{\infty}$ such that $E(1) \ll H(1)$, for all n, $E(n+1) \ll E(n)$, $E(n+1) \ll H(n+1)$, and for each $k = 1, \ldots, N$ there is an α_k-function $a_k(n)$ on $E(n)$ such that

$$\Big| \sum_{E(n)} g(a_1(n)(I), \ldots, a_N(n)(I), \xi(I))$$

$$- \sum_{E(n+1)} g(a_1(n+1)(J), \ldots, a_N(n+1)(J), \xi(J)) \geqq c .$$

By induction there is a sequence $\{D(n)\}_{n=1}^{\infty}$ of interval subdivisions of $[0; 1]$ and a sequence $\{X(n)\}_{n=1}^{\infty}$ of functions such that for all n,

i) $D(n+1) \ll D(n)$,

ii) $X(n)$ is a reversible function from $D(n)$ onto $E(n)$, and

iii) for each $[p; q]$ in $D(n)$, $\{X(n+1)([r; s]) : [r; s]$ in $D(n+1)$, $[r; s] \subseteqq [p; q]\} = \{I : I$ in $E(n+1), I \subseteqq X(n)([p; q])\}$.

This implies that

$$\sum_{D(n+1)([p;q])} \xi(X(n+1)([r; s])) = \xi(X(n)([p; q])) .$$

For each positive integer n, $[p; q]$ in $D(n)$ and $k = 1, \ldots, N$, let $b_k(n)([p; q]) = a_k(n)(X(n)([p; q]))$. Let $W = \{x : x$ in $\{p, q\}$ for some $[p; q]$ in $D(n)$ for some $n\}$. Suppose $0 < x$ in $\mathbf{W}$. For some u and v, $[v; x]$ is in $D(u)$. Suppose n is a positive integer such that for some p, $[p; x]$ is in $D(n)$. Then for some p^*, $[p^*; x]$ is in $D(n+1)$. Now, for each m such that $[w; x]$ is in $D(m)$ for some w, let $D(m)(x)$ denote $\{[r; s] : [r; s]$ in $D(m)$, $s \leqq x\}$. We see that $D(n+1)(x) = \bigcup_{D(n)(x)} D(n+1)(I)$. Therefore

$$\sum_{D(n+1)(x)} \xi(X(n+1)([r; s])) = \sum_{D(n)(x)} \sum_{D(n+1)(I)} \xi(X(n+1)([r; s]))$$
$$= \sum_{D(n)(x)} \xi(X(n)(I)) .$$

It therefore follows that there is a function h defined on $\mathbf{W}$ such that $h(0) = 0$, and if $0 < x$ in $\mathbf{W}$, then $h(x) = \sum_{D(m)(x)} \xi(X(m)(I))$ for all m such that for some z, $[z; x]$ is is $D(m)$; note that if $[r; s]$ is in $D(m)$, then $h(s) - h(r) = \xi(X(m)([r; s]))$.

Suppose $\{p, q]\} \subseteqq \mathbf{W}$ and $p < q$. There are positive integers n' and n'' such that p is in $\bigcup_{[r;s] \text{ in } D(n')}\{r, s\}$ and q is in $\bigcup_{[r;s] \text{ in } D(n'')}\{r, s\}$. From i) it follows that if n is a positive integer greater than or equal to $\max\{n', n''\}$, then $\{p, q\} \subseteqq \bigcup_{[r;s] \text{ in } D(n)} \{r, s\}$. If m is a positive integer and $\{p, q\} \subseteq \bigcup_{[r;s] \text{ in } D(m)}\{r, s\}$, we shall let $D(m)[p;q]$ denote $\{[r;s]:[r;s]$ in $D(m)$, $p \leqq r < s \leqq q\}$. Suppose n is a positive integer, $p < q$ and $\{p, q\} \subseteq \bigcup_{[r;s] \text{ in } D(n)} \{r, s\}$. Then

$$\sum_{D(n)[p;q]} |h(s) - h(r)| = \sum_{D(n)[p;q]} |\xi(X(n)([r;s]))|$$

$$\leqq \eta\Big(\bigcup_{D(n)[p;q]} X(n)([r;s])\Big) \leqq 1/n + \sum_{D(n)[p;q]} \Big|\xi(X(n)([r;s])\Big)\Big| .$$

Furthermore,

$$\bigcup_{D(n+1)[p;q]} X(n+1)([v, w]) = \bigcup_{D(n)[p;q]} X(n)([r;s]) ,$$

and we note that if z is a positive integer such that $V = \bigcup_{D(m)[p;q]}X(m)([t;u])$ for all $m \geqq z$, then $\eta(V) = v[p;q] = \sup\{\sum_{D(m)[p;q]}|h(s) - h(r)| : m$ a positive integer, $\{p, q\} \subseteq \bigcup_{[r;s] \text{ in } D(m)}\{r, s\}\}$.

Now, if n is a positive integer, then

$$\sum_{k=1}^{N} \sum_{D(n)} \sum_{D(n+1)[p;q]} |a_k(n)(X(n)([p;q])) - a_k(n+1)(X(n+1)([r;s]))|v[r;s]$$

$$= \sum_{k=1}^{N} \sum_{D(n)} \sum_{D(n+1)[p;q]} |a_k(n)(X(n)([p;q]))$$

$$- a_k(n+1)(X(n+1)([r;s]))|\eta(X(n+1)([r;s]))$$

$$= \sum_{k=1}^{N} \sum_{E(n)} \sum_{E(n+1)(I)} |a_k(n)(I) - a_k(n+1)(J)|\eta(J) < 1/2^n ,$$

which implies that

$$\sum_{k=1}^{N} \sum_{n=1}^{\infty} \sum_{D(n)} \sum_{D(n+1)[p;q]} |b_k(n)[p;q] - b_k(n+1)[r;s]|v[r;s] < \infty .$$

It follows that there are $d > 0$ and a positive interger n^* such that if m is a positive integer greater than or equal to n^* and $D(m)^* = \{[p;q] : [p;q]$ is in $D(m)$, $v[p;q] < d\}$, then

$$\Big| \sum_{D(m)^*}\Big[g\Big(b_1(m)[p;q], \ldots, b_N(m)[p;q\Big], h\Big|_q^p\Big)$$

$$- \sum_{D(m+1)[p;q]} g(b_1(m+1)[r;s], \ldots, b_N(m+1)[r;s], h|_r^s)]| < c/4 .$$

so that if $E(m)^* = \{V : V$ in $E(m)$, $\eta(V) < d\}$, then $E(m)^* = \{X(m)([p;q] : [p;q]$ is in $D(m)$, $v(p;q] < d\}$, which implies that

$$\Big| \sum_{E(m)^*}\Big[g\Big(a_1(m)(V), \ldots, a_N(m)(V), \xi(V)\Big)$$

$$- \sum_{E(m+1)(V)} g(a_1(m+1)(I), \ldots, a_N(m+1)(I), \xi(I)\Big)\Big]\Big| < c/4 .$$

There is a positive integer T such that $Td > \eta(U)$. There is $d^* > 0$ such that if $\max\{|x_1 - x_1'|, \ldots, |x_N - x_N'|, |z - z'|\} < d^*$, then $|g(x_1, \ldots, x_N, z) - g(x_1', \ldots, x_N', z')| < c/(4T)$. There is $d^{**} > 0$ such that if $\{(a_1(i), \ldots, a_N(i), z_i)\}_{i=1}^{w}$ is a sequence such that for each $i = 1, \ldots, w$ and $k = 1, \ldots, N$, $a_k(i)$ is in the range union of α_k and $\sum_{i=1}^{w}|z_i| < d^{**}$, then $\sum_{i=1}^{w}|g(a_1(i), \ldots, a_N(i), z_i)| < c/(4T)$. Note that if $H \subseteqq D \ll \{U\}$ and $\eta(I) \geqq d$ for all I in H, then there are not more than T elements in H.

There is a positive integer $Q \geqq n^*$ such that $1/2^Q < \min\{d^*d/4, d/4, d^{**}, d^*\}$. Let $E(Q)^* = \{V : V \text{ in } E(Q), \eta(V) < d\}$. There are not more than T elements in $E(Q) - E(Q)^*$. Also,

$$d^*d/4 > 1/2^Q > \sum_{k=1}^{N} \sum_{E(Q)-E(Q)^*} \left[\sum_{E(Q+1)(V)} |a_k(Q)(V) - a_k(Q+1)(I)|\eta(I)\right],$$

and $\eta(V) - \eta^*(V) < 1/2^Q$ for all V in $E(Q)$. Suppose V is in $E(Q) - E(Q)^*$. For some I_V in $E(Q+1)(V)$,

$$\eta(V) - \eta(I_V) \leqq \eta(V) - \eta^*(V) < 1/2^Q.$$

This implies that

$$\begin{aligned} d^*d/4 &> \sum_{k=1}^{N} |a_k(Q)(V) - a_k(Q+1)(I_V)|\eta(I_V) \\ &\geqq \left(\sum_{k=1}^{N} |a_k(Q)(V) - a_k(Q+1)(I_V)|\right)(\eta V) - 1/2^Q) \\ &\geqq \left(\sum_{k=1}^{N} |a_k(Q)(V) - a_k(Q+1)(I_V)|\right)(d - d/4), \end{aligned}$$

so that $\sum_{k=1}^{N}|a_k(P)(V) - a_k(Q+1)(I_V)| < d^*/3$; furthermore, $|\xi(V) - \xi(I_V)| \leqq \eta(V - I_V) = \eta(V) - \eta(I_V) < d^*$, so that $|g(a_1(Q)(V), \ldots, a_N(Q)(V), \xi(V)) - g(a_1(Q+1)(I_V), \ldots, a_N(Q+1)(I_V), \xi(I_V))| < c/(4T)$. Moreover,

$$\sum_{E(Q+1)(V)-\{I_V\}} |\xi(I)| \leqq \eta(V - I_V) = \eta(V) - \eta(I_V) < d^{**},$$

so that

$$\sum_{E(Q+1)(V)-\{I_V\}} |g(a_1(Q+1)(I), \ldots, a_N(Q+1)(I), \xi(I))| < c/(4T).$$

Therefore

$$\begin{aligned} &\left|\sum_{E(Q)}\left[g\Big(a_1(Q)(V), \ldots, a_N(Q)(V), \xi(V)\Big)\right.\right. \\ &\qquad \left.\left. - \sum_{E(Q+1)(V)} g\Big(a_1(Q+1)(I), \ldots, a_N(Q+1)(I), \xi(I)\Big)\right]\right| \\ &< c/4 + \sum_{E(Q)-E(Q)^*} [|g(a_1(Q)(V), \ldots, a_N(Q)(V), \xi(V)) \end{aligned}$$

$$- g(a_1(Q + 1)(I_V), \ldots, a_N(Q + 1)(I_V), \xi(I_V))|$$
$$+ \sum_{E(Q+1)(V)-\{I_V\}} |g(a_1(Q + 1)(I), \ldots, a_N(Q + 1)(I), \xi(I))|]$$
$$< c/4 + Tc/(4T) + Tc/(4T) < c,$$

a contradiction. Therefore 3) implies 1), and therefore 1), 2) and 3) are equivalent.

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