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LOCALLY FINITE GROUPS WHOSE IRREDUCIBLE MODULES ARE FINITE DIMENSIONAL

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1. Introduction. Let G be a group and k be a (commutative) field. The problem of determining when every irreducible kG-module has finite dimension over k has recently been considered by Snider [12], Musson [9] and Wehrfritz [13] [14] [15] for groups which are close to being soluble. In particular, [15] deals with the case when k has characteristic $p > 0$ and G belongs to a certain class of locally finite generalized soluble groups. This has prompted publication of the following result, which is a stronger version of [15] Theorem 1. The terminology is as follows. Let R be a ring with 1, V an irreducible right R-module, and $E = \mathrm{End}_R V$ be the endomorphism ring of V. Then E is a division ring, by Schur's Lemma, and we say that V has *finite endomorphism dimension* if $\dim_E V < \infty$.

THEOREM. *Let G be a locally finite group and k be a field of characteristic $p > 0$. Then every irreducible kG-module has finite endomorphism dimension if and only if $G/0_p(G)$ is almost abelian.*

We recall that a group *almost* has a certain property if it has a subgroup of finite index with the property.

This theorem was proved in 1975 as the result of a stimulating discussion with R.L. Snider. The case when G has no elements of order p had previously been dealt with by Farkas and Snider [1] (see also [3] Theorem B), and this will play an important part in the proof of the present theorem.

If k and G are as given and V is an irreducible kG-module, then as the augmentation ideal of $0_p(G)$ is nil, $0_p(G)$ operates trivially on V. Thus we can view V as an irreducible $G/0_p(G)$-module, and if this group is almost abelian, then V has finite endomorphism dimension (see [3] Lemma 1.1 for example). Thus only the necessity of the condition is at issue in the theorem. A key fact, which is very helpful in working with fields of positive characteristic, is the following, which the author learned from a preliminary version of [1].

LEMMA 1.1. *Let G be a locally finite group, k be a field of positive characteristic, V be an irreducible (right) kG-module, and $E = \mathrm{End}_{kG} V$. Assume that $\dim_E V < \infty$. Then E is a (commutative) field.*

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PROOF. Let $n = \dim_E V$. By the Jacobson Density Theorem, every E-linear transformation of V can be induced by an element of kG, and hence, if A is the annihilator in kG of V, then kG/A is isomorphic to the ring of all $n \times n$ matrices over E. Therefore there exists an idempotent $\bar{e} = e + A \in kG/A = \bar{R}$ such that $\bar{e}\bar{R}\bar{e} \cong E$. Now e lies in kF for some finite subgroup F of G and $e + (A \cap kF)$ is an idempotent in $kF/A \cap kF$. Since kF is artinian, it follows that we may choose e to be an idempotent itself. If L is any finite subgroup of G containing F, then kL is a direct sum of matrix rings over commutative fields (see [**6**, p. 157, (9.7)]). Hence so is $ekLe$. But the image of $ekLe$ in $\bar{R}$ has no zero divisors. Therefore this image is a commutative field. Since $\bar{e}\bar{R}\bar{e}$ is the union of these images, it is commutative.

2. Proof of theorem. The proof of the theorem will involve considering a number of special cases. Throughout, k denotes a fixed field of characteristic $p > 0$, and all groups considered will be assumed locally finite. As in [3], a group H will be called *restricted* if every irreducible kH-module has finite endomorphism dimension.

LEMMA 2.1 [**3**, Lemma 2.2]. *If $B \lhd A \leqq H$ and H is restricted, then A/B is restricted.*

We use this frequently without mention.

LEMMA 2.2. *If H is locally finite and $X/0_p(X)$ is almost abelian for every countable subgroup X of H, then $H/0_p(H)$ is almost abelian.*

PROOF. We may assume that $0_p(H) = 1$, and have to prove that H is almost abelian. If this is not so, then H has a countable subgroup H_1 that is not almost abelian (see [**10**, Chapter 6, Lemma 3.3], or use a simple inverse limit argument as in [7, Chapter 1, §K]). If $1 \neq h \in 0_p(H_1)$, then we know that $\langle h^H \rangle$ is not a p-group and so there exists a finite subgroup F_h of H such that $\langle h^{F_h} \rangle$ is not a p-group. Put $H_2 = \langle H_1, F_h \colon 1 \neq h \in 0_p(H_1) \rangle$. Then H_2 is countable and $0_p(H_2) \cap H_1 = 1$. Constructing H_3 similarly and continuing, we obtain a tower $H_1 \leqq H_2 \leqq \cdots$ of countable subgroups, indexed by the natural numbers. If $K = \bigcup_{i=1}^{\infty} H_i$, then K is countable, $0_p(K) = 1$, and K is not almost abelian. This is a contradiction.

We have to show that if H is any restricted group, then $H/0_p(H)$ is almost abelian. Lemmas 2.1 and 2.2 allow us to restrict to the countable case when convenient.

LEMMA 2.3. *Every p'-section of a restricted group is almost abelian.*

As usual, a section of a group X is a factor H/K, where $K \lhd H \leqq X$. Lemma 2.3 follows from Lemma 2.1 and [**3**, Theorem B], or [**1**].

LEMMA 2.4. *If G is restricted and linear over a field of characteristic p, then $G/0_p(G)$ is almost abelian.*

PROOF. If the assertion is false, then by Lemmas 2.1 and 2.2, we may choose a countable linear group G in characteristic p such that $G/0_p(G)$ is not almost abelian, but $X/0_p(X)$ is almost abelian for every proper Zariski-closed subgroup X of G. (See [16] Chapter 5 for the relevant facts about the Zariski topology).

Let $P = 0_p(G)$. Then P is Zariski-closed in G. For its Zariski-closure $\bar{P}$ is a normal subgroup of G, and if G is linear of degree n, then each element of P has order dividing p^{n-1}, and since this is a polynomial condition, it is inherited by $\bar{P}$. Hence G/P is also linear over a field of characteristic p, it is not almost abelian, but $X/0_p(X)$ is almost abelian for each proper closed subgroup X of G/P. So we may assume that $0_p(G) = 1$.

Let $\Delta(X)$ denote the FC-centre of a group X and $\Delta_{i+1}(X)/\Delta_i(X) = \Delta(X/\Delta_i(X))$. Also let $Z(X) = Z_1(X) \leqq Z_2(X) \leqq \cdots$ denote the upper central series of X.

Now clearly G has no proper closed subgroup of finite index, so G is connected and $\Delta(G) = Z(G)$ [**16**, 5.6]. Considering $G/Z(G)$ similarly and proceeding, we find that $\Delta_i(G) = Z_i(G)$ for all finite i, and these subgroups are closed. Let q be a prime different from p and Q be a Sylow q-subgroup of the nilpotent group $Z_3(G)$. Then Q is Černikov [**16**, 2.6] and so has a divisible characteristic abelian subgroup Q^0 of finite rank and finite index. Then $G/C_G(Q^0)$ is finite [7, 1.F.3], and since G has no proper closed subgroup of finite index, and all centralizers are closed, $Q^0 \leqq Z(G)$. Noting that $0_p(G) = 1$, we see that $Z_3(G)/Z(G)$ is the direct product of its Sylow subgroups, which are finite. The argument just used shows that these are central in $G/Z(G)$. Hence $Z_3(G) = Z_2(G)$ and $\Delta(G/Z_2(G)) = 1$. Also $O_p(G/Z_2(G)) = 1$. Further, as $Z_2(G)$ is closed, $G/Z_2(G)$ is linear in characteristic p.

Let $\bar{G} = G/Z_2(G)$. By a theorem of Passman [**11**], $k\bar{G}$ is semisimple. It is prime as $\Delta(\bar{G}) = 1$ [**10**, Chapter 4, Theorem 2.10]. Hence, by a theorem of Fisher and Snider [**10**, Chapter 9, Theorem 2.5] $k\bar{G}$ is a primitive ring having a faithful irreducible module V. By hypothesis, V has finite dimension over its endomorphism ring E, and by the Jacobson Density Theorem $k\bar{G}$ is a complete matrix ring over E and so is simple. Consideration of the augmentation ideal gives $\bar{G} = 1$. Hence $G = Z_2(G)$, a nilpotent p'-group, and Lemma 2.2 gives the final contradiction needed to prove Lemma 2.4.

Lemma 2.4 can be expected to play an important part in view of Lemma 1.1. Another important special case is considered in the next lemma.

LEMMA 2.5. *Suppose that G is restricted and $G = QP$, where $Q \lhd G$, Q is a p'-group, and P is a p-group. Then $|P\colon C_P(Q)| < \infty$.*

PROOF. We establish the lemma first in a special case.

(2.5*a*) *If P is abelian, Q is a direct product of elementary abelian groups, and $Q \leqq \Delta(G)$, then $|P: C_p(Q)| < \infty$.*

Suppose that $|P: C_p(Q)|$ is infinite. Since $Q \leqq \Delta(G)$, each finite set of elements of Q lies in a finite P-invariant subgroup, and this, by Maschke's Theorem, is a direct product of minimal normal subgroups of G. Each has centralizer of finite index in P, and so we can choose minimal normal subgroups $Q_1, Q_2, \ldots$ of G, contained in Q, with distinct centralizers. Without loss of generality, $Q = Q_1 \times Q_2 \times \cdots$, and then every normal subgroup of G contained in Q is a direct product of a selection of the Q_i since these are pairwise non-isomorphic P-modules.

For each i, there exists a non trivial homomorphism ϕ_i of Q_i into $\mathbf{Q}/\mathbf{Z}$, the additive group of rationals modulo one. Let ϕ be the homomorphism of Q into $\mathbf{Q}/\mathbf{Z}$ extending all the ϕ_i, and let R be the kernel of ϕ. Then Q/R is locally cyclic, and $\bigcap_{x \in G} R^x = 1$, since this intersection contains no Q_i. Since Q/R is locally cyclic, there exists an irreducible kQ-module U with kernel R (see [**3**, p. 123]). The stabilizer of U lies in $N_G(R)$, so by [**3**, Lemma 2.5], $|P: N_P(R)| < \infty$. Let $P_1 = N_P(R)$, and $P_2 = C_P(Q/R)$. Then $[Q, P_2] \leqq G$ since P is abelian, so $[Q, P_2] \leqq \bigcap_{x \in G} R^x = 1$. Hence $P_2 \leqq C_P(Q)$, so $|P: P_2|$ is infinite, and $|P_1: P_2|$ is also infinite.

Let $H = QP_1/RP_2$. This group is restricted, and it is isomorphic to the semidirect product of the locally cyclic p'-group $\bar{Q} = Q/R$ by the infinite abelian p-group $\bar{P}_1 = P_1/P_2$, which operates faithfully on $\bar{Q}$. We obtain a contradiction by arguing as in the proof of [**3**, Lemma 2.7], ignoring the first two paragraphs.

Now we proceed to the general case. Suppose that G is as given, but $|P: C_P(Q)|$ is infinite. We may assume G countable. By Lemma 2.3, Q is almost abelian, and so Q has a characteristic abelian subgroup Q_0 of finite index [**10**, Chapter 12, Lemma 1.2]. Let $C_1 = C_P(Q_0)$ and $C_2 = C_{C_1}(Q/Q_0)$. Then $|C_1: C_2| < \infty$, and $[Q, C_2, C_2] = 1$. Since C_2 is a p-group and Q is a p'-group, $[Q, C_2] = 1$, so $C_2 = C_P(Q)$. Hence $|P: C_1|$ is infinite, so we may replace Q by Q_0 and assume that Q is abelian. Since $P/C_P(Q)$ is an infinite locally finite p-group, it has an infinite abelian subgroup $A/C_P(Q)$ [7, 2.2.5]. The group $QA/C_P(Q)$ is restricted, and is the semidirect product of an abelian p'-group $\bar{Q}$ by an infinite abelian p-group operating faithfully on $\bar{Q}$. So without loss of generality, Q and P are both abelian. Letting Q_1 be the subgroup generated by the elements of prime order in Q, we have $C_P(Q) = C_P(Q_1)$, so replacing Q by Q_1, we may even assume that Q is a direct product of elementary abelian groups.

Let $D_1 = Q \cap \Delta(G)$. Applying (2.5*a*) to D_1P, we find that $|P: C_P(D_1)| < \infty$. Similarly, if $D_2/D_1 = Q/D_1 \cap \Delta(G/D_1)$, we have $|P: C_P(D_2/D_1)| < \infty$.

Now $C_P(D_1) \cap C_P(D_2/D_1) = C_P(D_2)$, whence $|P: C_P(D_2)| < \infty$ and $D_1 = D_2$. Let $C = C_P(Q/D_1)$. Then $C_C(D_1) = C_P(Q)$, so $|C: C_P(Q)| < \infty$, and $|P: C|$ is infinite. Consider the restricted group $\bar{G} = QP/D_1C$. It is the semidirect product of the normal abelian q-subgroup $\bar{Q} \cong Q/D_1$ by an infinite abelian p-group $\bar{P}$ operating faithfully on $\bar{Q}$. Further, $\bar{Q} \cap \Delta(\bar{G}) = 1$. Hence $\Delta(\bar{G}) \leqq C_{\bar{G}}(\bar{Q}) = \bar{Q}$, so $\Delta(\bar{G}) = 1$ and $k\bar{G}$ is prime. Furthermore, if $D_{\bar{G}}(\bar{Q}) = \{x \in \bar{G}: |\bar{Q}: C_{\bar{Q}}(x)| < \infty\}$ is the near centralizer of $\bar{Q}$ in $\bar{G}$, then we easily see that $\bar{P} \cap D_{\bar{G}}(\bar{Q}) \leqq \Delta(\bar{G})$, so it follows that $\bar{Q} = D_{\bar{G}}(\bar{Q})$. By [**10**, Chapter 7, Lemma 2.10], $k\bar{G}$ is semisimple. The theorem of Fisher and Snider [**10**, Chapter 9, Theorem 2.5] shows that $k\bar{G}$ is primitive, and a faithful irreducible module for $k\bar{G}$ is of finite endomorphism dimension, whence the Jacobson Density Theorem shows that $k\bar{G}$ is simple and we obtain the contradiction $\bar{G} = 1$ by considering the augmentation ideal. This concludes the proof of Lemma 2.5.

LEMMA 2.6. *Suppose that G is restricted and has a series of finite length in which each factor is either a p-group or a p'-group. Then $G/0_p(G)$ is almost abelian.*

PROOF. We may suppose that G is countable (Lemma 2.2) and that $0_p(G) = 1$ as usual. Let $Q = 0_{p'}(G)$ and $Q_1 = 0_{p'p}(G)$. Then $Q_1 = QP$ for some p-group P, as Q_1 is countable. Now $Q \geqq C_G(Q)$ [**4**, Lemma 5.4], and by Lemma 2.5, $|P: C_P(Q)| < \infty$. Hence P is finite. Also $Q_1/Q \geqq C_G(Q_1/Q)$, so now $|G: Q_1| < \infty$. Since Q is almost abelian by Lemma 2.3, the claim is established.

COROLLARY 2.7. *If G is restricted and has a normal subgroup G_1 such that $G_1/0_p(G_1)$ and $(G/G_1)/0_p(G/G_1)$ are almost abelian, then $G/0_p(G)$ is almost abelian.*

PROOF. We may suppose as usual that $0_p(G) = 1$. Then G_1 is almost abelian and so has a characteristic abelian subgroup A of finite index; also there is a subgroup H of finite index in G, containing G_1, such that $(H/G_1)/0_p(H/G_1)$ is abelian. Let $C = C_H(G_1/A)$. Then $|G: C| < \infty$ and C satisfies the hypotheses of Lemma 2.6, from which the claimed result follows.

We are almost ready to deal with the locally soluble case of the main theorem. First we require a very crude result on finite p-soluble groups.

LEMMA 2.8. *Let H be a finite p-soluble group, S be a Hall p'-subgroup of H, and suppose that S has an abelian subgroup of index n. Let r be the number of prime divisors of n. Then the p-length of H is at most $r+1$.*

PROOF. We carry this out by induction on r. Let $P = 0_p(H)$, $Q = 0_{p,p'}(H)$ and let A be an abelian subgroup of index n in S. We know that $Q/P \geqq C_{G/P}(Q/P)$, hence if $Q/P \leqq AP/P$ then we deduce that $Q = AP$. In that case, G/Q has a Hall p'-subgroup of order n and its p-length is trivially at most r. So the result is established in this case, and in particular this deals with the case $r = 1$. If however, $Q/P \nleqq AP/P$ then as $Q = (S \cap Q)P$, we have $S \cap Q > S \cap A$, so $|SQ: AQ| = |S: A(S \cap Q)|$ divides $|S: A| = n$ properly. Thus the Hall p'-subgroup SQ/Q of H/Q has an abelian subgroup of index dividing n properly. By induction the p-length of H/Q is at most $(r - 1) + 1 = r$, and that of H is at most $r + 1$.

LEMMA 2.9. *If G is restricted and locally p-soluble, then $G/0_p(G)$ is almost abelian.*

PROOF. As usual we may assume that G is countable. Let $G_1 \leqq G_2 \leqq \cdots$ be a tower of finite subgroups of G such that $\bigcup_{i=1}^{\infty} G_i = G$, and choose a tower $S_1 \leqq S_2 \leqq \cdots$ such that S_i is a Hall p'-subgroup of G_i for each i. Then $S = \bigcup_{i=1}^{\infty} S_i$ is almost abelian by Lemma 2.3, and so there exists an integer n such that each S_i has an abelian subgroup of index dividing n. By Lemma 2.9, the groups G_i have bounded p-lengths. This is well known to imply that G has a finite series whose factors are p-groups or p'-groups. The assertion now follows from Lemma 2.6.

In view of this result, non-abelian composition factors can be expected to play an important part in the remainder of the proof. The following two lemmas will be useful. The first is rather stronger than we need, but seems quite interesting.

LEMMA 2.10. *Let H be a finite group and F be a field of characteristic $q \geqq 0$. Suppose that $0_q(H) = 1$. (We put $0_0(H) = 1$). Then there exists an irreducible representation of H over F with nilpotent kernel.*

PROOF. Suppose the result is false, and let H be a counterexample of minimal order. Then the Frattini subgroup $\Phi(H)$ of H is trivial (see [5] for elementary properties of the Frattini subgroup). For otherwise, since $0_q(H/\Phi(H)) = 1$, we find that $H/\Phi(H)$ has an irreducible F-representation with nilpotent kernel $K/\Phi(H)$, and this lifts to an irreducible F-representation of H with nilpotent kernel K.

Hence the Fitting subgroup T of H is a direct product of minimal normal subgroups of H. Let T_1 be the direct product of a set of minimal normal subgroups of H consisting of one representative from each H-isomorphism type of minimal normal subgroups, and T_2 be a normal subgroup of H complementing T_1 in T. If $T_2 \neq 1$, then H/T_2 has an irreducible F-representation with nilpotent kernel L/T_2. Now T/T_2 is a direct product of minimal normal subgroups of H/T_2 and so is centralized by L/T_2. Hence L centralizes T_1, and therefore also T_2 and T. So L is nilpotent, a contradiction.

Therefore $T_2 = 1$, and the abelian part of the socle of H is a direct product of minimal normal subgroups that are pairwise non isomorphic H-groups. This is well known to imply, since $0_q(H) = 1$, that H has a faithful irreducible representation over F (see **[8]**).

LEMMA 2.11. *Let H be a finite group and F be a field. Suppose that $H = AB$, $A \lhd H$, $A \cap B = 1$, and let U be an irreducible FA-module and V be an irreducible FB-module. Then there exists an irreducible FH-module W such that the restriction W_A of W to A is a direct sum of conjugates of U, and W_B contains a submodule isomorphic to V.*

PROOF. Let W_1 be the induced module $U^H = \bigoplus_{b \in B} U \otimes b$. Each of the summands $U \otimes b$ is an irreducible FA-module isomorphic to a conjugate of U. Hence W_1, and therefore any section S/T of it, where S and T are H-submodules of W_1, is a direct sum of conjugates of U (up to isomorphism).

Also $(W_1)_B$ is a free FB-module, and so it contains a submodule X isomorphic to V **[10**, Chapter 2, pp. 61–62]. Let W_2 be an FH-submodule of W_1 maximal subject to $W_2 \cap X = 0$, and W_3/W_2 be a minimal FH-submodule of W_1/W_2. Then $W = W_3/W_2$ has the required properties.

The next special case is the final one needed for the proof of the main theorem.

LEMMA 2.12. *Let G be restricted and residually finite. Then $G/0_p(G)$ is almost abelian.*

PROOF. If G is almost locally soluble the assertion follows from Lemma 2.9. Otherwise we obtain a contradiction as follows. First we build up a sequence $F_1 \leqq F_2 \leqq \cdots$ of finite subgroups of G such that

(2.12a) $F_{i+1} = E_{i+1}F_i$, $E_{i+1} \lhd F_{i+1}$, $E_{i+1} \cap F_i = 1$,

and

(2.12b) The product of the orders of the non-abelian composition factors of E_{i+1} is at least $i + 1$.

We may begin by putting $F_1 = 1$. Suppose F_i has been obtained. Choose a normal subgroup L of finite index in G such that $F_i \cap L = 1$. Since G is residually finite but not almost locally soluble, it is easy to see that there exists a normal subgroup M of G contained in L such that $|L/M|$ is finite and the product of the orders of the non abelian composition factors of L/M is at least $i+1$. Now let E_{i+1} be any finite F_i-invariant subgroup of L such that $L = ME_{i+1}$, and put $F_{i+1} = E_{i+1}F_i$.

Now suppose we have an irreducible kF_i-module V_i. By Lemma 2.10, applied to $E_{i+1}/0_p(E_{i+1})$, there exists an irreducible kE_{i+1}-module W_{i+1} such that if K_{i+1} is the kernel of the representation of E_{i+1} on W_{i+1}, then K_{i+1} is soluble, and in particular the product of the orders of the non-abelian composition factors of E_{i+1}/K_{i+1} is at least $i+1$, by (2.12b). By Lemma 2.11, we can choose an embedding of V_i in an irreducible kF_{i+1}-module V_{i+1} whose restriction to E_{i+1} is a direct sum of conjugates of W_{i+1}. In this way we construct a direct system $V_1 \to V_2 \to \cdots$ such that V_i is an irreducible kF_i-module, and if L_i is the kernel of the corresponding representation of F_i, then

(2.12c) The product of the orders of the non-abelian composition factors of F_i/L_i is at least i.

The direct limit V of the V_i is an irreducible kF-module, where $F = \bigcup_{i=1}^{\infty} F_i$, and by Lemmas 2.1 and 1.1, V is finite dimensional over some field. Hence if L is the kernel of the corresponding representation of F, then $\bar{F} = F/L$ is linear over a field of characteristic p, and by Lemma 2.4, $\bar{F}/0_p(\bar{F})$ is almost abelian. Therefore there exists a number n such that the product of the orders of the nonabelian composition factors of a finite subgroup of $\bar{F}$ cannot exceed n. But each F_i/L_i is a homomorphic image of such a subgroup, so (2.12c) is contradicted.

Proof of theorem. Let G be an arbitrary restricted locally finite group. The Jacobson radical $J(kG)$ of kG is nil [**10**, Chapter 7, Lemma 4.2] so if $g \in G \cap (1 + J(kG))$, then there exists an integer n such that

$$0 = (g - 1)^{p^n} = g^{p^n} - 1 .$$

Therefore $G \cap (1 + J(kG)) = 0_p(G)$ Hence $0_p(G)$ is the intersection of the kernels of the irreducible representations of G. By Lemma 1.1, each of these representations is finite dimensional over some field, and if K is a corresponding kernel, then G/K is linear over a field of characteristic p. By Lemma 2.4 there exist normal subgroups $L \leqq P_K \leqq A_K \leqq G$ of G such that P_K/K is a p-group, A_K/P_K is abelian and G/A_K is finite. Let $P = \bigcap P_K$, $A = \bigcap A_K$, over all kernels K of irreducible representations of G. Clearly, as $\bigcap K = 0_p(G)$, we also have $P = 0_p(G)$, and A/P is abelian and G/A is residually finite. By Lemma 2.12, $(G/A)/0_p(G/A)$ is almost abelian. Finally, $G/0_p(G)$ is almost abelian by Corollary 2.7. This concludes the proof.

References

1. D. Farkas and R.L. Snider, *On group algebras whose simple modules are injective*, Trans. Amer. Math. Soc. **194** (1974), 241–248.

2. J.W. Fisher and R.L. Snider, *Prime von Neumann regular rings and primitive group algebras*, Proc. Amer. Math. Soc. **44** (1974), 244–250.

3. B. Hartley, *Injective modules over group rings*, Quarterly J. Math. (2) **28** (1977), 1-29.

4. ———, *Sylow subgroups of locally finite groups*, Proc. London Math. Soc. (3) **23** (1971), 159–192.

5. B. Huppert, *Endliche Gruppen* I, Springer-Verlag, Berlin 1967.

6. I.M. Isaacs, *Character theory of finite groups*, Interscience, New York, 1976.

7. O.H. Kegel and B.A.F. Wehrfritz, *Locally finite groups*, North Holland, Amsterdam, 1973.

8. R. Kochendörffer, *Uber treue irreduzible Darstellungen endlicher Gruppen*, Math. Nachr. **1** (1948), 25–39.

9. I.M. Musson, *Representations of infinite soluble groups*, preprint, Univ. of Alberta.

10. D.S. Passman, *The algebraic structure of group rings*, Interscience, New York, 1977.

11. ———, *On the semisimplicity of group rings of linear groups*, II, Pacific J. Math. **48** (1973), 215–234.

12. R.L. Snider, *Solvable groups whose irreducible modules are finite dimensional*, Comm. Algebra **10** (1982), 1477–1485.

13. B.A.F. Wehrfritz, *Groups whose irreducible representations have finite degree*, I, Math. Proc. Cambridge Philos. Soc. **90** (1981), 411–421.

14. ———, *Groups whose irreducible representations have finite degree*, II, Proc. Edinburgh Math. Soc.

15. ———, *Groups whose irreducible representations have finite degree*, III, Math. Proc. Cambridge Philos. Soc. **91** (1982), 397–406.

16. ———, *Infinite linear groups*, Springer Verlag, Berlin, 1973.

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