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LOCAL UNIFORM APPROXIMATION BY FUNCTIONS IN A UNIFORM ALGEBRA

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Throughout this paper X will denote a compact Hausdorff space and A will be a function algebra on X. A is *local* if A contains each function $f \in C(X)$ for which there is a collection of closed subsets $\{K_1, \ldots, K_k\}$ of X whose interiors cover X, with $f|_{K_i} \in A|_{K_i}$ for each $i = 1, \ldots, k$. We will call A *strongly local* if the requirement $f|_{K_i} \in A|_{K_i}$ in the above definition can be weakened to $f|_{K_i} \in A_{K_i}$. If A is strongly local it has the following property, not shared by other local function algebras: uniform approximability of a function in $C(X)$ by functions in any dense subalgebra of A is implied by local approximability. Our main result is Theorem 2, which implies that a function algebra with a certain separation property is necessarily strongly local.

Strongly local function algebras are mentioned briefly in [2], where they are referred to as "approximately local". It is known that $R(X)$, the uniform closure in $C(X)$ of the rational functions with no poles on X, is strongly local for any compact plane set X. It is easily shown that $P(X)$, the uniform closure of the polynomials in $C(X)$, is not strongly local for a plane set X which is not polynomially convex. In fact if $x \in X$, λ is in a bounded component of $C \backslash X$ and $f(z) = 1/(z - \lambda)$, then $f \notin P(X)$ but $f|_K \in A_K$ if K is the intersection of X with a sufficiently small closed disc centered at x.

We will be concerned with algebras possessing separation properties. A is *approximately normal* if for any pair of disjoint closed subsets K_1 and K_2 of X and any $\varepsilon > 0$, there is a function $h \in A$ for which $\|1 - h\|_{K_1} < \varepsilon$ and $\|h\|_{K_2} < \varepsilon$. It was shown in [5] that an approximately normal algebra defined on an interval must be local. Approximately normal function algebras in general need not be strongly local. For example, the disc algebra defined on the circle is well-known to be approximately normal, but as a consequence of the preceding paragraph, it is not strongly local. However, approximately normal function algebras on an interval can be shown to possess an intermediate property. We will call A *boundedly strongly local* if A contains each function $f \in C(X)$ for which there is a collection $\{K_1, \ldots, K_k\}$ of closed subsets of X whose interiors

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cover X, and for $i = 1, \ldots, k$ a bounded sequence $\{f_{i,n}\}$ in A such that $\|f_{i,n} - f\|_{K_i} \to 0$. It is clear that strongly local implies boundedly strongly local, which in turn implies local.

THEOREM 1. *If A is an approximately normal function algebra on a compact interval X, then A is boundedly strongly local.*

PROOF. Let $f \in C(X)$ and let $\{K_1, \ldots, K_k\}$ be a collection of closed subsets of X whose interiors cover X, such that for $i = 1, \ldots, k$ there is a bounded sequence of functions in A approaching f uniformly on K_i. Let U_i denote the interior of K_i. We may assume that each K_i is an interval $[a_i, b_i]$ with $a_i < a_{i+1} < b_i < b_{i+1}$. Choose a sequence $\{h_n\}$ in A such that $\|1 - h_n\|_{K_1\setminus U_2} \to 0$ and $\|h_n\|_{X\setminus U_1} \to 0$. Choose sequences $\{f_{1,n}\}$ and $\{f_{2,n}\}$ in A such that $\|f_{1,n}\| \leqq M$ and $\|f_{2,n}\| \leqq M$ for each n and for some $M > 0$. We may also require $\|f_{i,n} - f\|_{K_i} < 2^{-n}\|h_n\|$ for $i = 1, 2$, so that $\|h_n\|\,\|f_{1,n} - f_{2,n}\| \to 0$ on $K_1 \cap K_2$. Now define $g_{2,n} = f_{1,n}h_n + f_{2,n}(1 - h_n)$. Then clearly $\|g_{2,n} - f_{1,n}\|_{K_1\setminus U_2} \to 0$ and $\|g_{2,n} - f_{2,n}\|_{X\setminus U_1} \to 0$. If we write $g_{2,n} = f_{2,n} + h_n(f_{1,n} - f_{2,n})$, then it is easily seen that $\|g_{2,n} - f_{2,n}\|_{K_1\cap K_2} \leqq \|h_n\|_{K_1\cap K_2}\|f_{1,n} - f_{2,n}\|_{K_1\cap K_2} \to 0$. Thus $\{g_{2,n}\}$ is a sequence of functions in A which remains bounded on all of X and approaches f uniformly on $K_1 \cup K_2$. Continuing in this way we can obtain bounded sequences $\{g_{i,n}\}$ in A converging uniformly to f on $K_1 \cup \cdots \cup K_i$. But then $\{g_{k,n}\}$ converges uniformly to f on all of X, so $f \in A$.

The conclusion of Theorem 1 also applies to any approximately normal function algebra defined on an arc. In fact if A is approximately normal on any space X and $K \subseteqq X$ is a closed arc, then A_K is boundedly strongly local. With this observation we can prove the following result.

COROLLARY 1. *If A is approximately normal on X and every connected component of X is either a point or an arc, then A is boundedly strongly local.*

PROOF. Let $f \in C(X)$. Suppose $\{K_1, \cdots, K_k\}$ is a collection of closed subsets of X whose interiors cover X, and $\{f_{i,n}\}$ is a bounded sequence in A with $\|f_{i,n} - f\|_{K_i} \to 0$ for $i = 1, \ldots, k$. Let C be a connected component of X. If C is an arc, then A_C is strongly local and $\|f_{i,n} - f\|_{C\cap K_i} \to 0$ for each i, so $f|_C \in A_C$. If C is a point then $f|_C \in A_C$ trivially. Now let μ be an extreme point of the unit ball of $A^\perp$. Then the closed support of μ is concentrated on some component C_0 by the approximate normality of A. Choose a sequence $\{f_n\}$ in A for which $\|f_n - f\|_{C_0} \to 0$. Then $\int f\, d\mu = \int_{C_0} f d\mu = \int_{C_0} (f - f_n) d\mu \to 0$. This is enough to guarantee that $f \in A$.

COROLLARY 2. *If A is approximately normal on X and each maximal* *set of antisymmetry for X is either a point or an arc, then A is boundedly strongly local.*

PROOF. With f as in Corollary 1, a similar argument establishes that $f|_K \in A_K$ for each maximal antisymmetric set K. But a theorem of Bishop (quoted in [2]) then guarantees that $f \in A$.

COROLLARY 3. *If A is an approximately normal, non-antisymmetric function algebra on a simple closed curve, then A is boundedly strongly local.*

We will next describe conditions on A which will imply strong localness. We seek to avoid the requirement of boundedness in the sequences of approximating functions by imposing requirements of boundedness among the functions which separate the closed sets of A. We will show that the conditions imposed will imply strong localness for algebras defined on any compact Hausdorff space.

It is known (see [3]) that if in the definition of approximate normality we require $\|h\| \leqq M$ with M independent of K_1, K_2 and ε, then $A = C(X)$, Badé and Curtis [1] proved the stronger result that $A = C(X)$ if we require $\|h\| \leqq M$ with M depending on ε, but independent of K_1 and K_2. We will impose bounds that depend on K_1 and K_2, but not on ε. We will call A *boundedly approximately normal* if for any pair of disjoint closed sets K_1 and K_2 in X there is a sequence of functions $\{h_n\}$ in A and some $M > 0$ for which $\|h_n\| \leqq M$ for all n, $\|1 - h_n\|_{K_1} \to 0$ and $\|h_n\|_{K_2} \to 0$. A boundedly approximately normal function algebra need not coincide with $C(X)$. McKissick's algebra [4] is boundedly approximately normal, as is any normal function algebra. On the other hand, we will be able to demonstrate that bounded approximate normality is a strictly stronger property than approximate normality.

For our purposes the essential property of boundedly approximately normal algebras is related to partitions of unity by functions in the algebra. If $\mathscr{U} = \{U_1, \ldots, U_k\}$ is an open cover of X, then an *approximate partition of unity* subordinate to $\mathscr{U}$ is defined to be a sequence of k-tuples of functions $\{(h_{1n}, \ldots, h_{kn})\}$ in $C(X)$ such that $\|h_{in}\| \to 0$ outside U_i for $i = 1, \ldots, k$ and $\sum_{i=1}^{k} h_{in} = 1$ uniformly on X. If there is a constant M such that $\|h_{in}\| \leqq M$ for all choices of i and n, we will say that $\{(h_{1n}, \ldots, h_{kn})\}$ is a *bounded approximate partition of unity* subordinate to $\mathscr{U}$.

LEMMA. *If A is boundedly approximately normal on any compact Hausdorff space X, then A contains a bounded approximate partition of unity subordinate to any finite open cover of X.*

PROOF. Let $\{U_1, \cdots, U_k\}$ be a finite open cover of X, and choose an open cover $\{V_1, \cdots, V_k\}$ with $\bar{V}_i \subseteqq U_i$ for $i = 1, \cdots, k$. For each i, choose a sequence of functions $\{g_{in}\}$ in A such that $\|1 - g_{in}\|_{\bar{V}_i} \to 0$, $\|g_{in}\|_{X \setminus U_i} \to 0$ and $\|g_{in}\| \leqq M_i$ for all n. Let $M = \max\{M_i\}$. Define $h_{in} = g_{in} \prod_{j=1}^{i-1}(1 - g_{jn})$ for $i = 1, \cdots, k$. Then $\|h_{in}\|_{X \setminus U_i} \to 0$ since $\|g_{in}\|_{X \setminus U_i} \to 0$ and $\|1 - g_{jn}\| \leqq M + 1$ for $j \neq i$. To show that $\|1 - \sum_{j=1}^{k} h_{jn}\| \to 0$, let $x \in X$ and choose a closed neighborhood K of x such that $K \subseteqq V_i$ for some i. Then $\|1 - \sum_{j=1}^{k} h_{jn}\|_K = \|\prod_{j=1}^{k}(1 - g_{jn})\|_K \to 0$ since $\|1 - g_{in}\|_K \to 0$ and $\|1 - g_{jn}\|_K \leqq M + 1$ for $j \neq i$. Since X can be covered by finitely many such sets K, the result follows.

THEOREM 2. *Let A be a function algebra on X containing a bounded approximate partition of unity subordinate to each finite open cover of X. Then A is strongly local.*

PROOF. Suppose $f \in C(X)$ and $\{U_1, \cdots, U_k\}$ is an open cover of X such that $f|_{\bar{U}_i} \in A_{\bar{U}_i}$ for $i = 1, \cdots, k$. Choose sequences $\{f_{in}\}$ in A such that $\|f_{in} - f\|_{\bar{U}_i} \to 0$ for each i. Choose an open cover $\{V_1, \cdots, V_k\}$ of X with $\bar{V}_i \subseteqq U_i$ for $i = 1, \cdots, k$, and let $\{(h_{1n}, \cdots, h_{kn})\}$ be an approximate partition of unity subordinate to $\{V_1, \cdots, V_k\}$ with $\|h_{in}\| \leqq M$ for all choices of i and n. Choosing subsequences of $\{h_{in}\}$ if necessary, we may also assume that $\|h_{in} f_{in}\|_{X \setminus V_i} \to 0$ for $i = 1, \cdots, k$. Define $f_n = \sum_{i=1}^{k} h_{in} f_{in}$. Then $f_n \in A$ for each n, and we will show that $\|f_n - f\| \to 0$.

Let $x \in X$. Renumbering if necessary, suppose $x \in (U_1 \cap \cdots \cap U_j)$ and $x \notin (U_{j+1} \cup \cdots \cup U_k)$. Choose a closed neighborhood K of x such that $K \subseteqq (U_1 \cap \cdots \cap U_j)$ and $K \cap (V_{j+1} \cup \cdots \cup V_k) = \emptyset$. Then

$$\begin{aligned} \|f_n - f\|_K &\leqq \Big\| \sum_{i=1}^{j} h_{in}(f_{in} - f) \Big\|_K + \Big\| f\Big(\sum_{i=1}^{j} h_{in} - 1\Big) \Big\|_K + \Big\| \sum_{i=j+1}^{k} h_{in} f_{in} \Big\|_K \\ &\leqq M \sum_{i=1}^{j} \big\| f_{in} - f \big\|_K + \|f\| \Big\| \sum_{i=1}^{j} h_{in} - 1 \Big\|_K + \sum_{i=j+1}^{k} \big\| h_{in} f_{in} \big\|_K . \end{aligned}$$

But $\|f_{in} - f\|_K \to 0$ for $i = 1, \cdots, j$ since $K \subseteqq U_i$. Furthermore $\|\sum_{i=1}^{j} h_{in} - 1\|_K \to 0$ since $K \cap (V_{j+1} \cup \cdots \cup V_k) = \emptyset$, and $\|h_{in} f_{in}\|_K \to 0$ for $i = j + 1, \cdots, k$ for the same reason. This is enough to establish that $\|f_n - f\| \to 0$ on all of X.

COROLLARY. *Every boundedly approximately normal function algebra is strongly local.*

The last corollary now permits a straightforward demonstration that bounded approximate normality is a strictly stronger condition that approximate normality. Let A be the disc algebra defined on the circle. As previously observed, A is approximately normal but not strongly local. By the corollary A is thus not boundedly approximately normal.

In connection with the corollary it would be interesting to know if there exist any boundedly approximately normal function algebras which are not normal.

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