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THE RADICAL OF THE RESTRICTED UNIVERSAL ENVELOPING ALGEBRA OF A_1

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ABSTRACT. Let $\mathscr{L}$ be the classical Lie algebra of type A_1 with a basis $\{e, f, h\}$ and $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$ over an algebraically closed field of characteristic $p > 2$. Let $\mathscr{R}$ be the radical of the u-algebra $\mathscr{U}$ of $\mathscr{L}$. Our main result is the obtainment of $(p - 1)/2$ sets of generators of $\mathscr{R}$, and hence $(p - 1)/2$ criteria for complete reducibility of restricted representations of $\mathscr{L}$.

Introduction. For the classical Lie algebra $\mathscr{L}$ with a basic $\{e, f, h\}$ and $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$ over an algebraically closed field $\mathscr{K}$ of characteristic $p > 2$, Jacobson [2] showed that a sufficient condition that a representation ϕ of $\mathscr{L}$ be completely reducible is that $\phi(e)^{p-1} = \phi(f)^{p-1} = 0$. Seligman [4] showed a necessary and sufficient condition for complete reducibility of any restricted representation ϕ of $\mathscr{L}$ to be $\phi(e)^{p-1} + \phi(e)^{p-1}\phi(h) = 0$ and $\phi(f)^{p-1} + \phi(h)\phi(f)^{p-1} = 0$. Using the minimal right ideals in the u-algebra $\mathscr{U}$ constructed by Nielsen [3] and by an approach entirely different from those given by Jacobson and Seligman, we obtained a number of generating sets for the radical $\mathscr{R}$ of $\mathscr{U}$, and hence a number of criteria for complete reducibility of restricted representations of $\mathscr{L}$ including the one obtained by Seligman. Our approach involves only computations within the u-algebra and is easily generalized to give some necessary conditions for complete reducibility of restricted representations of classical Lie algebras of rank $\ell \geqq 1$ as was shown by Wong [6]. Throughout this paper unless otherwise stated $\mathscr{L}$, $\mathscr{K}$, $\mathscr{U}$ and $\mathscr{R}$ will denote the aforesaid Lie algebra, field, u-algebra and radical respectively.

1. The main theorem and its corollaries.

MAIN THEOREM. *Let $\mathscr{L}$ be the classical Lie algebra of type A_1 with a basis $\{e, f, h\}$ and $[e, f] = h$, $[e, h] = 2e$, $[f, h] = -2f$ over an algebraically closed field $\mathscr{K}$ of characteristic $p > 2$. Then the radical $\mathscr{R}$ of the u-algebra $\mathscr{U}$ of $\mathscr{L}$ is generated by any one of the $(p - 1)/2$ sets of elements*

$$\left\{e^{p-\nu}\cdot \prod_{j=1}^{2\nu-1} (h + j),\ \left[\prod_{j=1}^{2\nu-1} (h + j)\right] f^{p-\nu}\right\}, \quad \nu = 1, 2, \ldots (p - 1)/2\,.$$

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COROLLARY 1. *Let ϕ be a restricted representation of $\mathscr{L}$. Then ϕ is completely reducible if and only if*

$$\phi(e)^{p-\nu} \cdot \prod_{j=1}^{2\nu-1} [\phi(h) + jI] = 0$$

and

$$\left[\prod_{j=1}^{2\nu-1} (\phi(h) + jI)\right] \phi(f)^{p-\nu} = 0\,,$$

for any one $\nu \varepsilon \{1, 2, \ldots, (p-1)/2\}$ where I is the identity linear transformation.

The corollary follows since a restricted representation ϕ of $\mathscr{L}$ is completely reducible if and only if ϕ vanishes on the radical $\mathscr{R}$ of $\mathscr{U}$. In case $\nu = 1$, we have the following result.

COROLLARY 2. (SELIGMAN [4]). *Let ϕ be a restricted representation of $\mathscr{L}$. Then ϕ is completely reducible if and only if*

$$\phi(e)^{p-1}\phi(h) = -\phi(e)^{p-1} \text{ and } \phi(h)\phi(f)^{p-1} = -\phi(f)^{p-1}.$$

2. Proofs of main theorem and related lemmas. Our proof of the main theorem is quite a computational one. It could in fact be verified by a computer. First we shall establish a few lemmas which will facilitate the proof.

LEMMA 1. *Let $A(h)$ be a polynomial in h over $\mathscr{K}$ and let n be any positive integer. Then*

(1) $A(h)e^n = e^n A(h - 2n)$,

(2) $f^n A(h) = A(h - 2n) f^n$,

(3) $fe^n = e^n f - ne^{n-1}[h - (n - 1)]$, *and*

(4) $f^n e = ef^n - n[h - (n - 1)]f^{n-1}$.

LEMMA 2. *Let n be a positive integer such that $0 \leqq n \leqq p - 2$. Then*

(5) $e^n(h - n) = (e^{n+1}f - fe^{n+1})/(n + 1)$, *and*

(6) $(h - n)f^n = (ef^{n+1} - f^{n+1}e)/(n + 1)$.

Lemmas 1 and 2 are proved by induction on n.

LEMMA 3. *Let m and n be any two positive integers less than p. Then*

$$f^n e^m = \sum_{j=0}^{\mathrm{Min}(m,n)} (-1)^j j! \binom{m}{j}\binom{n}{j} e^{m-j} \left\{\prod_{i=1}^{j} (h - m - n + j + i)\right\} f^{n-j}.$$

PROOF. Using formula (4) we prove the lemma by induction on m. A complete proof is given by Wong in [5].

Next we shall need a theorem obtained by Nielsen to construct the irreducible $\mathscr{U}$-modules with which we can easily carry out the computations in our proofs.

THEOREM 4. (NIELSEN [3, p. 17]). *Let $\mathscr{L}_\ell$ be a classical Lie algebra of rank ℓ with a basis $\{e_1, \ldots, e_m, h_1, \ldots, h_\ell, e_{-1}, \ldots, e_{-m}\}$ over an algebraically closed field of characteristic $p > 7$ and let $\mathscr{U}_\ell$ be the u-algebra of $\mathscr{L}_\ell$. Let $E^{p-1} = e_1^{p-1} \cdots e_m^{p-1}$, $F^{p-1} = e_{-1}^{p-1} \cdots e_{-m}^{p-1}$ and $H(c) = \prod_{i=1}^{\ell} H(h_i, c_i)$ for $c = (c_1, \ldots, c_\ell) \in (\mathscr{Z}_p)^\ell$, where $H(h_i, 0) = 1 - h_i^{p-1}$ and $H(h_i, c_i) = \sum_{j=1}^{p-1}(h_i/c_i)^j$, if $c_i \neq 0$. Then the p^ℓ right ideals $E^{p-1}H(c)F^{p-1}\mathscr{U}_\ell$ in $\mathscr{U}_\ell$ form a complete set of representatives of all isomorphic classes of irreducible $\mathscr{U}_\ell$-modules.*

From Nielsen's theorem when setting $\ell = 1$ we have $H(0) = 1 - h^{p-1}$, $H(i) = \sum_{j=1}^{p-1}(h/i)^j$ for $i = 1, 2, \ldots, p-1$, and that $\{e^{p-1}H(i)f^{p-1}\mathscr{U} | i = 0, 1, \ldots, p-1\}$ is a complete set of nonisomorphic irreducible $\mathscr{U}$-modules. This leads us to the following result.

PROPOSITION 5. *Let $m_{p-1} = e^{p-1}H(p-1)f^{p-1}$, and $m_i = e^{p-1}H(p-2-i)f^{p-1}$ for $i = 0, 1, \ldots, p-2$. Then for each $i = 0, 1, \ldots, p-1$, $L(i) = m_i\mathscr{U}$ is an irreducible $\mathscr{U}$-module having a minimal vector m_i with weight $-i$ and a maximal vector m_ie^i with weight i. $\{m_i, m_ie, \ldots, m_ie^i\}$ and $\{m_ie^i, m_ie^i f, \ldots, m_ie^i f^i\}$ are two bases of $L(i)$ and $m_ie^if^i = \delta_i m_i$ for some $0 \neq \delta_i \in \mathscr{K}$.*

PROOF. Since $f^p = 0$, $m_if = e^{p-1}H(p-2-i)f^p = 0$. Hence m_i is a minimal vector. By formula (2) we have $m_ih = -im_i$, hence $-i$ is the minimal weight of $L(i)$. By Lemma 3, $m_ie^{i+1} = 0$, and $m_ie^j \neq 0$ for $j = 0, 1, \ldots, i$. Hence m_ie^i is a maximal vector and $\{m_i, m_ie, \ldots, m_ie^i\}$ forms a basis of $L(i)$. Again by Lemma 3 we have $m_ie^if^{i+1} = 0$ and $m_ie^if^j \neq 0$ for $j = 0, 1, \ldots, i$. Hence $\{m_ie^i, m_ie^if, \ldots, m_ie^if^i\}$ also forms a basis of $L(i)$. Since

$$m_ie^if^i = (-1)^i i! \binom{p-1}{i} \left\{\prod_{j=1}^{i} [j - (i+1)]\right\} m_i,$$

the last assertion of our proposition is proved.

LEMMA 6. *Let m and n be any two elements in $\mathscr{Z}_p$ with $m \neq n$. Then*

$$-1 = (h+m)(h+n)g(h) + \prod_{j \in \mathscr{Z}_p - \{m\}} (h+j) + \prod_{j \in \mathscr{Z}_p - \{n\}} (h+j),$$

where $g(h)$ is some polynomial in h over $\mathscr{K}$.

PROOF. Let x be an indeterminate. Since $\mathscr{K}$ is of characteristic $p > 2$, $x^p - x = \prod_{j \in \mathscr{Z}_p}(x+j)$. Computing derivatives of both sides we have

$$-1 = \sum_{i \in \mathscr{Z}_p} \prod_{j \in \mathscr{Z}_p - \{i\}} (x+j) = (x+m)(x+n)g(x) + \prod_{j \in \mathscr{Z}_p - \{m\}} (x+j) + \prod_{j \in \mathscr{Z}_p - \{n\}} (x+j),$$

where

$$g(x) = \sum_{i \in \mathscr{X}_p - \{m, n\}} \prod_{j \in \mathscr{X}_p - \{i, m, n\}} (x + j)$$

is a polynomial in x over $\mathscr{K}$. Replacing x by h we have the lemma.

LEMMA 7. *For each $\nu = 1, 2, \ldots, 1/2(p - 1)$, let $\mathscr{N}_\nu$ be the two-sided ideal in $\mathscr{U}$ generated by the two elements $e^{p-\nu}\prod_{j=1}^{2\nu-1}(h + j)$ and $[\prod_{j=1}^{2\nu-1}(h + j)] f^{p-\nu}$, and let $\mathscr{R}$ be the radical of $\mathscr{U}$. Then for $\nu = 2, 3, \ldots, 1/2(p - 1)$, $\mathscr{N}_{\nu-1}$ is contained in $\mathscr{N}_\nu$, and $\mathscr{N}_{(1/2)(p-1)}$ is contained in $\mathscr{R}$.*

PROOF. By Lemma 6,

$$\begin{aligned}
&e^{p-1}(h + 1) \\
&\quad = -e^{p-1}(h + 1)[(h + 2)(h + 3)g(h) + \sum_{i=2}^{3} \prod_{j \in \mathscr{X}_p - \{i\}} (h + j)] \\
&\quad \equiv -e^{p-2}e(h + 1)\sum_{i=2}^{3} \prod_{j \in \mathscr{X}_p - \{i\}} (h + j) \quad (\text{mod } \mathscr{N}_2), \text{ by formula (1)}, \\
&\quad = -e^{p-2}(h + 3)\left[\sum_{i=2}^{3} \prod_{j \in \mathscr{X}_p - \{i+2\}} (h + j)\right]e \\
&\quad = -e^{p-2}(h + 3)(h + 2)(h + 1)\left[\sum_{i=2}^{3} \prod_{j \in \mathscr{X}_p - \{i+2, 1, 2\}} (h + j)\right]e \\
&\quad \equiv 0 \quad (\text{mod } \mathscr{N}_2)\,.
\end{aligned}$$

Similarly, by Lemma 6 and formula (2), we prove $(h + 1)f^{p-1} \equiv 0$ $(\text{mod}\mathscr{N}_2)$. Hence $\mathscr{N}_1$ is contained in $\mathscr{N}_2$.

Assuming that $\mathscr{N}_{\nu-1}$ is contained in $\mathscr{N}_\nu$ for $\nu \in \{2, 3, \ldots, (p - 3)/2\}$, we shall infer that $\mathscr{N}_\nu$ is contained in $\mathscr{N}_{\nu+1}$. By Lemma 6,

$$\begin{aligned}
&e^{p-\nu} \cdot \prod_{j=1}^{2\nu-1} (h + j) \\
&\quad = -e^{p-\nu}\left[\prod_{j=1}^{2\nu-1} (h + j)\right]\left[(h + 2\nu)(h + 2\nu + 1)g(h) + \sum_{i=2\nu}^{2\nu+1} \prod_{j \in \mathscr{X}_p - \{i\}} (h + j)\right] \\
&\quad \equiv -e^{p-(\nu+1)}e\left[\prod_{j=1}^{2\nu-1} (h + j)\right]\sum_{i=2\nu}^{2\nu+1} \prod_{j \in \mathscr{X}_p - \{i\}} (h + j) (\text{mod } \mathscr{N}_{\nu+1}), \text{ by formula (1)} \\
&\quad = -e^{p-(\nu+1)}\left[\prod_{j=3}^{2\nu+1} (h + j)\right]\left[\sum_{i=2\nu}^{2\nu+1} \prod_{j \in \mathscr{X}_p - \{i+2\}} (h + j)\right]e \\
&\quad = -e^{p-(\nu+1)}\left[\prod_{j=1}^{2\nu+1} (h + j)\right]\left[\sum_{i=2\nu}^{2\nu+1} \prod_{j \in \mathscr{X}_p - \{i+2, 1, 2\}} (h + j)\right]e \\
&\quad \equiv 0 \quad (\text{mod } \mathscr{N}_{\nu+1})\,.
\end{aligned}$$

Similarly, by Lemma 6 and formula (2) we prove that $[\prod_{j=1}^{2\nu-1}(h + j)]f^{p-\nu} \equiv 0\,(\text{mod } \mathscr{N}_{\nu+1})$. Hence $\mathscr{N}_\nu$ is contained in $\mathscr{N}_{\nu+1}$. This proves that for $\nu = 2, 3, \ldots, (p - 1)/2$, $\mathscr{N}_{\nu-1}$ is contained in $\mathscr{N}_\nu$. Next we establish that

$\mathscr{N}_{(p-1)/2}$ is contained in $\mathscr{R}$ by showing that the generators of $\mathscr{N}_{(p-1)/2}$ annihilate all the irreducible $\mathscr{U}$-modules $L(i)$.

Since by Proposition 5, $L(i)$ has a basis $\{m_i e^n | n = 0, 1, \ldots, i\}$, and $m_i e^j = 0$ for $j > i$, $(m_i e^n) e^{(p+1)/2} \cdot \prod_{j=1}^{p-2}(h + j) = 0$ for $i = 0, 1, \ldots, (1/2)(p-1)$ and $n = 0, 1, \ldots, i$. Hence for $i = 0, 1, \ldots, (1/2)(p-1)$, $L(i)$ is annihilated by $e^{(p+1)/2} \prod_{j=1}^{p-2}(h + j)$. For $i = (p+1)/2, \ldots, p-1$, since $\prod_{j=1}^{p-2}(h + j) = \prod_{j=2}^{p-1}(h - j)$,

$$\begin{aligned}(m_i e^n) e^{(p+1)/2} \prod_{j=1}^{p-2}(h + j) &= m_i e^{n+(p+1)/2} \cdot \prod_{j=2}^{p-1}(h - j), \quad \text{by formula (1)},\\ &= m_i \left[\prod_{j=2}^{p-1}(h - j + 2n + p + 1)\right] e^{n+(p+1)/2}\\ &= \left[\prod_{j=2}^{p-1}(-i - j + 2n + p + 1)\right] m_i e^{n+(p+1)/2} = 0,\end{aligned}$$

because if $n + (p+1)/2 > i$, $m_i e^{n+(p+1)/2} = 0$. If $n + (p+1)/2 \leqq i$, $p + 1 \leqq 2n + p + 1 \leqq 2i \leqq 2(p-1)$ which then implies $2 = (p+1) - (p-1) \leqq 2n + (p+1) - i \leqq i \leqq p - 1$. Hence $2 \leqq 2n + p + 1 - i \leqq p - 1$. Since j ranges from 2 to $p - 1$, $\prod_{j=2}^{p-1}(2n + p + 1 - i - j) = 0$. Therefor $e^{(p+1)/2} \cdot \prod_{j=1}^{p-2}(h + j)$ annihilates all $L(i)$ for $i = 0, 1, \ldots, p-1$, and is in $\mathscr{R}$. Similarly, by Proposition 5 and formula (2) we prove that $[\prod_{j=1}^{p-2}(h + j)] f^{(p+1)/2}$ is in $\mathscr{R}$. Hence $\mathscr{N}_{(p-1)/2}$ is contained in $\mathscr{R}$. This proves the lemma.

Our theorem will be proved if we show that $\mathscr{R}$ is contained in $\mathscr{N}_1$. For this we need the concept of extent vectors defined by Curtis [1]. The extent of a standard monomial $e^m h^k f^n$ in $\mathscr{U}$ is defined as the integer $m - n$. A nonzero element $u \in \mathscr{U}$ is called an extent vector if u is a linear combination of standard monomials of the same extent, and the common extent is defined as the extent of u and is denoted by $\mathscr{E}(u)$.

PROPOSITION 8. (NIELSEN [3, p. 11]). *If u and v are two extent vectors in $\mathscr{U}$, then each standard monomial of uv has extent equal to $\mathscr{E}(u) + \mathscr{E}(v)$.*

LEMMA 9. *If $x = u_1 + \cdots + u_n \in \mathscr{R}$, where u_j is an extent vector of extent $\mathscr{E}(u_j)$, and $\mathscr{E}(u_i) \neq \mathscr{E}(u_j)$ for $i \neq j$, then $u_j \in \mathscr{R}$ for $j = 1, \ldots, n$.*

PROOF. By Proposition 5, for each $i = 0, 1, \ldots, p-1$, $\{m_i e^\nu | \nu = 0, 1, \ldots, i\}$ is a basis for the irreducible $\mathscr{U}$-module $L(i)$, and since $x \in \mathscr{R}$, $0 = m_i e^\nu x = \sum_{j=1}^{n} m_i e^\nu u_j$. If $\mathscr{E}(m_i e^\nu u_j) > p - 1 =$ maximal extent, $m_i e^\nu u_j = 0$, If $\mathscr{E}(m_i e^\nu u_j) \leqq p - 1$, then since $\mathscr{E}(m_i) = 0$ and by proposition 8, $\mathscr{E}(m_i e^\nu u_j) = \nu + \mathscr{E}(u_j) \neq \nu + \mathscr{E}(u_k) = \mathscr{E}(m_i e^\nu u_k)$ for $j \neq k$. Since elements of $\mathscr{U}$ which are of different extents are linearly independent and the $m_i e^\nu u_j$'s are either zero or of different extents, $m_i e^\nu u_j = 0$

for $\nu = 0, 1, \ldots, i$. Hence u_j annihilates $L(i)$ for $i = 0, 1, \ldots, p-1$, and $u_j \in \mathscr{R}$ for $j = 1, \ldots, n$.

LEMMA 10. *Let $x \in \mathscr{R}$ be an extent vector. If $\mathscr{E}(x) \leqq 0$, then $x \in \langle\!\langle (1+h)f^{p-1} \rangle\!\rangle$, the two-sided ideal in $\mathscr{U}$ generated by $(1+h)f^{p-1}$. If $\mathscr{E}(x) \geqq 0$, then $x \in \langle\!\langle e^{p-1}(1+h) \rangle\!\rangle$, the two-sided ideal in $\mathscr{U}$ generated by $e^{p-1}(1+h)$.*

PROOF. Let $\mathscr{N} = \langle\!\langle (1+h)f^{p-1} \rangle\!\rangle$, and let $\mathscr{E}(x) = -d$ for some $d \in \{0, 1, \ldots, p-1\}$. For each $j \in \{d, d+1, \ldots, p-1\}$, let $\mathscr{S}_j = \{e^{j-d}A(h)f^j \mid A(h)$ is a polynomial in h over $\mathscr{K}\}$. $\mathscr{S}_j$ is a vector space over $\mathscr{K}$ and $\sum_{j=d}^{p-1} \mathscr{S}_j$ is the set of all extent vectors in $\mathscr{U}$ of extent $-d$. Our proof is carried out by induction in the following manner: first we show that $x \in \mathscr{R} \cap \mathscr{S}_{p-1}$ implies $x \in \mathscr{N}$. Our next step is to assume that $x \in \mathscr{R} \cap \sum_{j=k+1}^{p-1} \mathscr{S}_k$ for $k \geqq d$ implies $x \in \mathscr{N}$, and then to infer that $x \in \mathscr{R} \cap \sum_{j=k}^{p-1} \mathscr{S}_j$ implies $x \in \mathscr{N}$.

When $x \in \mathscr{R} \cap \mathscr{S}_{p-1}$, $x = e^{p-1-d}A(h)f^{p-1}$. Let m_{p-1} be as defined in Proposition 5. Then $0 = m_{p-1}e^d x = \delta_{p-1}A(-1)m_{p-1}$ where $0 \neq \delta_{p-1} \in \mathscr{K}$. Hence $A(-1) = 0$ and $x = e^{p-1-d}B(h)(h+1)f^{p-1}$ where $B(h)$ is some polynomial in h over $\mathscr{K}$. Hence $x \in \mathscr{N}$.

When $x \in \mathscr{R} \cap \sum_{j=k}^{p-1} \mathscr{S}_j$ for some $k \in \{d, d+1, \ldots, p-1\}$, $x = \sum_{j=k}^{p-1} e^{j-d}A_j(h)f^j$ where the $A_j(h)$'s are polynomials in h over $\mathscr{K}$. For $i \in \{k, k+1, \ldots, p-1\}$, let m_i be as defined in Proposition 5. Then since $e^{i-(k-d)}x \in \mathscr{R}$ and by Proposition 5, $0 = m_i e^{i-(k-d)}x = A_k(i)m_i e^i f^k$. Since $i \geqq k$, $m_i e^j f^k \neq 0$. Hence $A_k(i) = 0$ for $i \in \{k, k+1, \ldots, p-1\}$ and $A_k(h) = B(h)\prod_{j=k}^{p-1}(h-j)$ where $B(h)$ is some polynomial in h over $\mathscr{K}$. Hence

$$x = e^{k-d}B(h)\left[\prod_{j=k}^{p-1}(h-j)\right]f^k + \sum_{j=k+1}^{p-1} e^{j-d}A_j(h)f^j .$$

Now for $\nu = 1, 2, \ldots, p-k-1$, we claim that

$$(*) \qquad x \equiv (-1)^\nu e^{k-d}B(h)\left[\prod_{j=k+\nu}^{p-1} \cdot (h-j)\right]f^{k+\nu}e^\nu/[(k+1)\cdots(p-1)] \left(\operatorname{mod} \sum_{j=k+1}^{p-1} \mathscr{S}_j\right).$$

Using formulas (1) and (6) we prove (*) easily by induction on ν. Setting $\nu = p-1-k$ in (*) we have $x \equiv y \pmod{\sum_{j=k+1}^{p-1}\mathscr{S}_j}$, where $y = (-1)^{p-1-k}e^{k-d}B(h)(h+1)f^{p-1}e^{p-1-k}/[(k+1)\cdots(p-1)]$. Since y is in $\mathscr{N}_1$ defined in Lemma 7, $y \in \mathscr{R}$. Hence $x - y \in \mathscr{R} \cap \sum_{j=k+1}^{p-1}\mathscr{S}_j$ and by induction hypothesis $x - y \in \mathscr{N}$. Since $y \in \mathscr{N}$, we have $x \in \mathscr{N}$. This proves the first part of the lemma. Similarly, by Proposition 5, formulas (2) and (5), and Lemma 7 we prove the second part of the lemma.

PROOF OF THE MAIN THEOREM. Since Lemma 7 affirms that $\mathscr{N}_1 \subseteqq \cdots \subseteqq \mathscr{N}_{(p-1)/2} \subseteqq \mathscr{R}$. It remains for us to show that $\mathscr{R} \subseteqq \mathscr{N}_1$. Let $0 \neq x \in \mathscr{R}$. Since each element in $\mathscr{U}$ is a finite sum of extent vectors, $x = u_1 + \cdots + u_n$, where the u_j's are extent vectors and $\mathscr{E}(u_j) \neq \mathscr{E}(u_k)$ for $j \neq k$. By Lemma 9, each $u_j \in \mathscr{R}$. By Lemma 10, $u_j \in \mathscr{N}_1$ for $j = 1, \ldots, n$. Hence $\mathscr{R} \subseteqq \mathscr{N}_1$. This completes the proof of the main theorem.

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