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SELF-REPRODUCING KERNELS AND BILINEAR FORMULAS FOR Q-ORTHOGONAL POLYNOMIALS

A. VERMA AND V.K. JAIN

1. Recently Ismail [7] obtained the connection relations and bilinear formulas for the Jacobi and Hahn polynomials by using fractional operators. Later on, Rahman [9, 10] proved some related results by using a completely different approach. Al-Salam and Ismail [1] obtained reproducing kernels for q-Jacobi polynomials which are q-analogues of Ismail's results [7].

In §3 of this paper, following Rahman [10] we prove the following formula for q-Jacobi polynomials

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[q^c;q]_{m+n}[q^d;q]_{m+n}[xq^{1+b};q]_n[yq^{1+b};q]_n(xy)^m z^{m+n}}{[q;q]_m[q;q]_n[q^{1+a};q]_m[q^{1+b};q]_n q^{(1+b)n}}$$
$$(1.1)\qquad = \sum_{r=0}^{\infty} \frac{[q^{1+a};q]_r[q^{1+a+b};q]_r[q^c;q]_r[q^d;q]_r\, z^r q^{r(r-1)}}{[q;q]_r[q^{1+b};q]_r[q^{1+a+b};q]_{2r}}$$
$$\cdot\, {}_2\phi_1\left[\begin{matrix} q^{c+r},\ q^{d+r};\ q: zq^{-1-b} \\ q^{2+a+b+2r} \end{matrix}\right] P_r(x;a,b;q)\, P_r(y;a,b;q),$$

a q-analogue of a result of Feldheim [4].

In §4 we derive some self reproducing kernels and bilinear sums for the q-Hahn polynomials which are q-analogues of results of Rahman [9]. In §5 we also obtain q-analogues of Ismail's [7] connection relations and bilinear formulae for Hahn polynomials.

§6 contains an extension of a result of Andrews and Askey [2] (for q-Jacobi polynomials) to q-Racah polynomials from which we obtain an interesting bilinear formula for q-Racah polynomials.

The formulae obtained in this paper remain valid if we replace q^a, q^b, $\ldots$, by $a, b, \ldots$ respectively.

2. Definitions and notations. Let $|q| < 1$, $[q^a;q]_n = (1-q^a)(1-q^{a+1}) \ldots (1-q^{a+n-1})$, $[q^a;q]_0 = 1$, $[q^a;q]_\infty = \prod_{j=0}^{\infty}(1-q^{a+j})$ and the generalized basic hypergeometric series is defined as

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$$
\begin{aligned}
&{}_{k+1}\phi_{k+r}\left[\begin{matrix} q^{a_1}, \ldots, q^{a_{k+1}}; q; x \\ q^{b_1}, \ldots, q^{b_{k+r}} \end{matrix}\right] \\
&= \sum_{n=0}^{\infty} \frac{[q^{a_1}; q]_n \cdots [q^{a_{k+1}}; q]_n \, x^n (-)^{nr} q^{rn(n-1)/2}}{[q; q]_n [q^{b_1}; q]_n \cdots [q^{b_{k+r}}; q]_n},
\end{aligned} \tag{2.1}
$$

which is convergent for all non-negative values of r except that when $r = 0$, it is convergent only when $|x| < 1|$.

Following Hahn [6], we define the q-Jacobi polynomials as (see also [2])

$$
P_n(x; a, b; q) = {}_2\phi_1\left[\begin{matrix} q^{-n}, q^{1+a+b+n}; q; xq \\ q^{1+a} \end{matrix}\right], \tag{2.2}
$$

which satisfy the orthogonality relation (1; 2.8]

$$
\begin{aligned}
&\frac{1}{(1-q)} \int_0^1 t^a [1 - qt]_b \, P_n(t; a, b; q) \, P_m(t; a, b; q) \, d(t; q) \\
&= \prod \left[\begin{matrix} q, q^{2+a+b}; q \\ q^{1+a}, q^{1+b} \end{matrix}\right] F_n(a, b) \, \delta_{n,m}
\end{aligned} \tag{2.3}
$$

where

$$
\begin{aligned}
&[1 - x]_b = \prod_{j=0}^{\infty} \frac{(1 - xq^j)}{(1 - xq^{b+j})}, \\
&\frac{1}{(1-q)} \int_0^x f(t) d(t; q) = x \sum_{n=0}^{\infty} q^n f(xq^n)
\end{aligned} \tag{2.4}
$$

and

$$
F_n(a, b) = \frac{[q; q]_n [q^{1+b}; q]_n (1 - q^{1+a+b}) \, q^{(1+a)n}}{[q^{1+a}; q]_n \, [q^{1+a+b}; q]_n \, (1 - q)^{1+a+b+2n)}}. \tag{2.5}
$$

On the other hand the q-Hahn polynomials $q_n(j; a, b, N; q)$ are defined as

$$
Q_n(j; a, b, N; q) = {}_3\phi_2\left[\begin{matrix} q^{-n}, q^{1+a+b+n}, q^{-j}; q; q \\ q^{1+a}, q^{-N} \end{matrix}\right], \tag{2.6}
$$

These polynomials satisfy, for $q_1 = q$ or p (where $|q| < 1, pq = 1$ throughout the paper), the orthogonality relation (see [17] for details)

$$
\begin{aligned}
&\sum_{j=0}^{N} g(j; q_1) Q_n(j, a, b, N; q_1) Q_m(j; a, b, N; q_1) \\
&= \begin{cases} 0 & \text{if } m \neq n \\ 1/h(n; q_1) & \text{if } m = n \end{cases}
\end{aligned} \tag{2.7}
$$

and the dual orthogonality relation

(2.8)
$$\sum_{n=0}^{N} h(n; q_1)Q_n(j; a, b, N; q_1)Q_n(i; a, b, N; q_1) = \begin{cases} 0 & \text{if } i \neq j \\ 1/g(j; q_1) & \text{if } i = j \end{cases}$$

for $n, m = 0, 1, \ldots, N$, where

(2.9)
$$g(j;q_1) \equiv g(j;a,b;q_1) = \frac{[q_1; q_1]_N[q_1^{1+a}; q_1]_j[q_1^{1+b}; q_1]_{N-j}\, q_1^{(1+a)(N-j)}}{[q_1; q_1]_j[q_1; q_1]_{N-j}[q_1^{2+a+b}; q_1]_N}$$

and

(2.10)
$$h(n; q_1) \equiv h(n; a, b; q_1) = \frac{(-)^n[q_1^{-N}; q_1]_n[q_1^{1+a}; q_1]_n}{[q_1; q_1]_n[q_1^{1+b}; q_1]_n} \cdot \frac{[q_1^{1+a+b}; q_1]_n(1 - q_1^{1+a+b+2n})q_1^{-n/2(n+1)+(N-a)n}}{[q_1^{2+a+b+N}; q_1]_n(1 - q_1^{1+a+b})}.$$

Similarly, following Askey and Wilson [3] we define the q-Racah polynomials by

(2.11)
$$P_n(\mu(x)) = P_n(\mu(x)); a, b, c, d; q) = {}_4\phi_3\left[\begin{matrix} q^{-n}, q^{1+a+b+n}, q^{-x}, q^{1+c+d+x}; q; q \\ q^{1+a}, q^{1+b+d}, q^{1+c} \end{matrix}\right]$$

($\mu(x) = q^{-x} + q^{1+c+d+x}$ and either q^{1+a}, q^{1+b+d} or q^{1+c} is of the form q^{-N}). These polynomials satisfy the orthogonality relation

(2.12)
$$\sum_{x=0}^{N} \omega(x; q)P_n(\mu(x))\, P_m(\mu(x)) = \begin{cases} 0 & \text{if } m \neq n \\ 1/\pi(n; q) & \text{if } m = n \end{cases}$$

and the dual orthogonality relation [17]

(2.13)
$$\sum_{n=0}^{N} \pi(n; q)P_n(\mu(x))\, P_n(\mu(y)) = \begin{cases} 0 & \text{if } y \neq x \\ 1/\omega(x; q) & \text{if } y = x \end{cases}$$

where

(2.14)
$$\omega(x; q) \equiv \omega(x; a, b, c, d; q) = \frac{[q^{1+a};q]_x[q^{1+c};q]_x[q^{1+b+d};q]_x[q^{1+c+d}; q]_x(1 - q^{1+c+d+2x})q^{-x(1+a+b)}}{[q; q]_x[q^{1+d}; q]_x[q^{1+c-b}; q]_x[q^{1+c+d-a}; q]_x(1 - q^{1+c+d})}$$

and

(2.15)
$$\pi(n; q) \equiv \pi(n; a, b, c, d; q) = \prod\left[\begin{matrix} q^{1+d}, q^{1+c-b}, q^{1+c+d-a}, q^{-1-a-b}; q \\ q^{d-a}, q^{c-a-b}, q^{2+c+d}, q^{-b} \end{matrix}\right] \cdot \frac{[q^{1+a}; q]_n\, [q^{1+c}; q]_n\, [q^{1+b+d}; q]_n\, [q^{1+a+b}; q]_n(1 - q^{1+a+b+2n})}{[q; q]_n[q^{1+b}; q]_n[q^{1+a-d}; q]_n[q^{1+a+b-c}; q]_n(1 - q^{1+a+b})q^{(1+c+d)n}}$$

The orthogonality relation (2.12) for q-Racah polynomials reduces for $C \to \infty$ to a relation equivalent to the orthogonality relation (2.7) for the q-Hahn polynomials for $q_1 = q$ whereas for $C \to -\infty$, (2.7) reduces to a result equivalent to (2.7) for $q_1 = p$. Similarly the dual orthogonality relation (2.8) for q-Hahn polynomials could be deduced from the dual orthogonality relation (2.13) satisfied by q-Racah polynomials.

3. We begin this section by proving a bilinear formula for q-Jacobi polynomials, viz.,

$$\begin{aligned}\phi(x, y) &\equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[xq^{1+b}; q]_n[yq^{1+b}; q]_n(xy)^m z^{m+n} A_{m+n}}{[q, q]_m[q; q]_n q^{1+a}; q]_m[q^{1+b}; q]_n\, q^{(1+b)n}} \\ &= \sum_{r=0}^{\infty} \frac{[q^{1+a}; q]_r[q^{1+a+b}; q]_r(1 - q^{1+a+b+2r})z^r\, q^{r(r-1)}}{[q; q]_r[q^{1+b}; q]_r(1 - q^{1+a+b})} \\ &\quad \cdot \sum_{m=0}^{\infty} \frac{z^m A_{m+r}\, q^{-m(1+b)}}{[q; q]_m\, [q^{2+a+b}; q]_{2r+m}} P_r(x; a, b; q)\, P_r(y; a, b; q)\end{aligned} \tag{3.1}$$

where

$$A_r = \frac{[q^{(\alpha_t+2}; q]_r}{[q^{(\beta_t)}; q]_r}, a > 0,\ b > 0,\ \alpha_i > 0,\ \beta_j > 0,\ |z| < 1.$$

Before proving (3.1) we prove the following connection relation

$$\frac{1}{(1-q)}\int_0^1 y^a[1-qy]_b\, \phi x, y)P_r(y;\, a,\, b;\, q)d(y; q) = \lambda_r P_r(x; a, b; q) \tag{3.2}$$

where

$$\lambda_r = \prod\begin{bmatrix} q,\ q^{2+a+b};\ q \\ q^{1+a},\ q^{1+b} \end{bmatrix} \sum_{m=0}^{\infty} \frac{A_{m+r} z^{m+r}\, q^{r^2-m(1+b)-ar}}{[q; q]_m\, [q^{2+a+b}; q]_{2r+m}}. \tag{3.3}$$

PROOF OF (3.2). The left hand side of (3.2) (say S) may be rewritten as

$$\begin{aligned} S &\equiv \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[x\, q^{1+b}; q]_n x^m z^{m+n} A_{m+n} q^{-n(1+b)}}{[q; q]_m[q; q]_n[q^{1+a}; q]_m[q^{1+b}; q]_n} \\ &\quad \cdot \sum_{s=0}^{r} \frac{[q^{-r}; q]_s[q^{1+a+b+r}; q]_s q^s}{[q; q]_s[q^{1+a}; q]_s} \cdot \frac{1}{(1-q)}\int_0^1 y^{a+s+m}[1-qy]_{b+n} d(y; q) \\ &= \prod\begin{bmatrix} q,\ q^{2+a+b};\ q \\ q^{1+a},\ q^{1+b} \end{bmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{[x\, q^{1+b}; q]_n x^m z^{m+n} q^{-n(1+b)} A_{m+n}}{[q; q]_m[q; q]_n[q^{2+a+b}; q]_{m+n}} \\ &\quad \cdot {}_3\phi_2\begin{bmatrix} q^{-r},\ q^{1+a+b+r},\ q^{1+a+m};\ q;\ q \\ q^{1+a},\ q^{2+a+b+m+n} \end{bmatrix}. \end{aligned} \tag{3.4}$$

However, in the transformation [11]

$$\begin{aligned} &{}_4\phi_3\begin{bmatrix} q^a,\ q^b,\ q^c,\ q^{-n};\ q;\ q \\ q^e,\ q^g,\ q^h \end{bmatrix} \\ &= \frac{[q^{g-c}; q]_n[q^{e+g-a-b}; q]_n}{[q^g; q]_n[q^{e+g-a-b-c}; q]_n}\, {}_4\phi_3\begin{bmatrix} q^{e-a},\ q^{e-b},\ q^c,\ q^{-n};\ q;\ q \\ q^e,\ q^{c-g+1-n},\ q^{c-h+1-n} \end{bmatrix}, \end{aligned} \tag{3.5}$$

with $a + b + c + 1 - n = e + g + h$; substituting for $h = a + b + c + 1 - n - e - g$ and then letting $a \to -\infty$, we get

$$(3.6) \quad {}_3\phi_2\begin{bmatrix} q^b, q^c, q^{-n}; q; q^{e+g-a-b-c} \\ q^a, q^g \end{bmatrix} = \frac{[q^{g-c}; q]_n}{[q^g; q]_n} \, {}_3\phi_2\begin{bmatrix} q^{e-b}, q^c, q^{-n}; q; q \\ q^e, q^{b-g+1-n} \end{bmatrix}$$

Next, transforming the ${}_3\phi_2$ on the left hand side of (3.6) by using (3.6) again with $e \to g, g \to e$, we get

$$(3.7) \quad {}_3\phi_2\begin{bmatrix} q^{g-b}, q^c, q^{-n}; q; q \\ q^g, q^{c-e+1-n} \end{bmatrix} = \frac{[q^e; q]_n[q^{g-c}; q]_n}{[q^g; q]_n[q^{e-c}; q]_n} \, {}_3\phi_2\begin{bmatrix} q^{e-b}, q^c, q^{-n}; q; q \\ q^e, q^{c-g+1-n} \end{bmatrix}.$$

Now, using (3.7) with $n = r$, $c = 1 + a + b + r$, $b = -m$, $g = 1 + a$, $e = -m - n$, to transform the ${}_3\phi_2$ on the right hand side of (3.4), we get

$$S = \prod\begin{bmatrix} q, q^{2+a+b}; q \\ q^{1+a}, q^{1+b} \end{bmatrix} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{A_{m+n}x^m z^{m+n}[xq^{1+b}; q]_n[q^{1+b}; q]_r}{[q; q]_m[q; q]_n[q^{1+a}; q]_r[q^{2+a+b}; q]_{m+n}}$$

$$\cdot \frac{[q^{-m-n}; q]_r \, q^{r(1+a+m+n)-n(1+b)}}{[q^{2+a+b+m+n}; q]_r} \, {}_3\phi_2\begin{bmatrix} q^{-r}, q^{1+a+b+r}, q^{-n}; q; q \\ q^{1+b}, q^{-m-n} \end{bmatrix}$$

$$= \prod\begin{bmatrix} q, q^{2+a+b}; q \\ q^{1+a}, q^{1+b} \end{bmatrix} \sum_{s=0}^{r} \frac{[q^{-r}; q]_s[q^{1+a+b+r}; q]_s \, q^s}{[q; q]_s[q^{1+b}; q]_s}$$

$$\cdot \sum_{m=r}^{\infty} \frac{A_m(xz)^m[q^{-m}; q]_r[xq^{1+b}; q]_s[q^{1+b}; q]_r \, q^{r(1+a+m)+sm}}{[q; q]_m[q^{2+a+b}; q]_{m+s}[q^{1+a}; q]_r \, x^s \, q^{s(s+1+b)}}$$

$$\cdot \, {}_2\phi_0\begin{bmatrix} xq^{1+b+s}, q^{-m+s}, q; q/x^{m-2s-1-b} \\ — \end{bmatrix},$$

Summing the ${}_2\phi_0$ by a limiting case of the Gauss' summation theorem [**12**; 3.3.2.6.]

$${}_2\phi_1\begin{bmatrix} q^a, q^{-n}; q; q^{n+c-a} \\ q^c \end{bmatrix} = \frac{[q^{c-a}; q]_n}{[q^c; q]_n},$$

we get

$$S = \lambda_r \frac{[q^{1+b}; q]_r(-)^r \, q^{-r(r+1)/2}}{[q^{1+a}; q]_r \, q^{br}} \, {}_3\phi_2\begin{bmatrix} q^{-r}, q^{1+a+b+r}, xq^{1+b}; q; q \\ q^{1+b}, 0. \end{bmatrix}.$$

Transforming the ${}_3\phi_2$ of the above expression by the formula [**16**]

$$(3.8) \quad {}_2\phi_1\begin{bmatrix} q^a, q^b; q; q^{e-a-b+x} \\ q^e \end{bmatrix} = \prod\begin{bmatrix} q^{e-a}, q^{e-b}; q \\ q^e, q^{e-a-b} \end{bmatrix} {}_3\phi_2\begin{bmatrix} q^a, q^b, q^x; q; q \\ q^{a+b-e+1}, 0 \end{bmatrix}$$

(where a, b or x is a non-negative integer, if only $x = -N$ then), $|q^{e-a-b+x}| < 1$ we get (3.2).

However (3.2) may be rewritten in the form

$$(3.9) \quad \frac{1}{(1-q)}\int_0^1 G(x, y)f_r(y)d(y; q) = \lambda_r f_r(x)$$

where

$$G(x, y) = \{x^a y^a [xq; q]_b [yq; q]_b\}^{1/2} \phi(x, y)$$

and

$$f_r(y) = \left\{ \frac{[q^{1+a}; q]_\infty [q^{1+b}; q]_\infty [q^{1+a}; q]_r [q^{1+a+b}; q]_r (1-q^{1+a+b+2r})}{[q; q]_\infty [q^{2+a+b}; q]_\infty [q; q]_r [q^{1+b}; q]_r (1-q^{1+a+b})} \right.$$

$$\left. \cdot \frac{y^a [yq; q]_b}{q^{(1+a)r}} \right\}^{1/2} \cdot P_r(y; a, b; q).$$

Clearly (3.9) is an integral equation satisfied by $P_n(x; a, b; q)$. The kernel $G(x, y)$ in (3.9) is a symmetric kernel and λ_r are the eigenvalues. The completeness of the system of orthogonal polynomials $\{P_n(x; a, b; q)\}_{n=0}^{\infty}$ follows on using [14; Th. 3.1.5]. The eigenvalues λ_r for

$$A_s = \frac{[q((\alpha_t+2); q]_s}{[q^{(\beta_t)}; q]_s}$$

are positive for $a, b, \alpha_i, \beta_j > 0$ and $|z| < 1|$.

Next, we observe that the symmetric kernel $G(x, y)$ belongs to $L^2(d\mu)$ where μ is a measure defined as $\mu(x, y) = \nu(x)\nu(y)$ and

$$d\nu(x) = \begin{cases} 1 & \text{if } x = q^i,\ i = 0, 1, \ldots \\ 0 & \text{if otherwise.} \end{cases}$$

Indeed if $\|G\|_\mu$ be the norm of $L^2(d\mu)$, we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \{G(q^i, q^j)\}^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}$$

$$\cdot \frac{[q^{(\alpha_t+2)}; q]_{m+n} [q^{(\alpha_t+2)}; q]_{r+s} [q^{1+i}; q]_{b+n} [q^{1+j}; q]_{b+n} [q^{1+i+b}; q]_s}{[q; q]_m [q; q]_n [q^{(\beta_t)}; q]_{m+n} [q^{(\beta_t)}; q]_{r+s} [q^{1+a}; q]_m [q^{1+b}; q]_n [q; q]_r}$$

$$\cdot \frac{[q^{1+j+b}; q]_s\, z^{m+n+r+s}\, q^{(r+m+a)(i+j)}}{[q^{1+b}; q]_s [q; q]_s [q^{1+a}; q]_r}$$

$$\leqq \left\{ [q; q]_b\, {}_2\phi_1\left[\begin{matrix} 0, 0; q; q \\ q^{1+b} \end{matrix}\right] {}_{t+1}\phi_t\left[\begin{matrix} 0, 0, \ldots, 0, q^{1+h}; q; z \\ q^{(\beta_t)} \end{matrix}\right] {}_{t+2}\phi_{t+1}\left[\begin{matrix} q^{(\alpha_t+2)}; q; z \\ q^{1+a}, q^{(\beta_t)} \end{matrix}\right] \right\}^2$$

(see [15] for details). Hence the right hand side is convergent and positive under the stated conditions.

Thus we are in a position to apply the extension of Mercer's theorem (see Al-Sslam and Ismail [1] for details) for the connection relation (3.9) to get the bilinear formula

$$\sum_{r=0}^{\infty} f_r(x) f_r(y) \lambda_r = G(x, y). \tag{3.10}$$

Proof of (3.1) is completed by substituting for $f_r(x)$, $f_r(y)$, λ_r and $G(r, y)$ in (3.10)

SPECIAL CASES. (i) (3.1) for $A_s = [q^c; q]_s[q^d; q]_s$ yields (1.1).

(ii) In (1.1) setting $z = q^{a+2b-c-d+3}$ and summing the inner ${}_2\phi_1$ on the right hand side by the q-analogue of Gauss' summation theorem, we get the following interesting bilinear formula for q-Jacobi polynomials:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q^c;q]_{m+n}[q^d;q]_{m+n}[xq^{1+b};q]_n[yq^{1+b};q]_n(xy)^m\, q^{(a+2b-c-d+3)(m+n)}}{[q;q]_m[q;q]_n[q^{1+a};q]_m[q^{1+b};q]_n\, q^{n(1+b)}}$$

$$(3.11)\qquad = \prod\begin{bmatrix} q^{2+a+b-c},\ q^{2+a+b-d};\ q \\ q^{2+a+b},\ q^{2+a+b-c-d} \end{bmatrix}\sum_{r=0}^{\infty}\frac{[q^{1+a};q]_r[^{1+a+b};q]_r}{[q;q]_r[q^{1+b};q]_r}$$

$$\cdot\ \frac{[q^c;q]_r[q^d;q]_r(1-q^{1+a+b+2r})\, q^{(r+2+a+2b-c-d)}}{[q^{2+a+b+c};q]_r[q^{2+a+b-d};q]_r(1-q^{1+a+b})}$$

$$P_r(x;a,b;q)P_r(y;a,b;q)$$

provided $|q^{2+2b-c-d+3}\ 1| < 1$. This is different from the bilinear generating function for the q-Jacobi polynomials due to Stanton **[13]**.

(iii) (1.1) for $c, d \to \infty$, yields the following bilinear formula;

$$[q;q]_a[q;q]_b\, j_a([xyz/-x^2yzq^{1+b}];q)\, j_b([zq^{-1-b}/-zy];q)$$

$$(3.12)\qquad = \sum_{r=0}^{\infty}\frac{[q^{1+a};q]_r[q^{1+a+b};q]_r[q;q]_{1+a+b+2r}\, z^r q^{r(r-1)}}{[q;q]_r[q^{1+b};q]_r[q^{1+a+b};q]_{2r}}$$

$$\cdot\ (zq^{-1-b})^{(1+a+b+2r)/2},\ J_{1+a+b+2r}(\sqrt{zq^{-1-b}};q)\ P_r(x;a,b;q)$$

$$P_r(y;a,b;q),$$

where, following Jackson **[8]**, the two Bessel functions are defined as

$$J_\alpha(x;q) = \sum_{r=0}^{\infty}\frac{x^{\alpha+2r}}{[q;q]_r[q;q]_{\alpha+r}},$$

$$j_\alpha([x/+y];q) = \sum_{r=0}^{\infty}\frac{[-y/x;q]_r\, x^r}{[q;q]_r\,[q;q]_{\alpha+r}}.$$

(3.12) is a q-analogue of a result of Bateman **[18**; p. 370] to which it reduces on replacing z by $(1-q)^2z$ and then letting $q \to 1$. Other special cases of (3.1) may also be discussed.

4. In this section we obtain the following reproducing kernel of q-Hahn polynomials, which is q-analogue of a result of Rahman **[9]**;

$$K_N(i,j;a,b,c,d,g;p) = \frac{[p;p]_i[p;p]_{N-i}\, p^{N(d+g-c)+i(a+c)}}{[p^{a+c};p]_i[p^{b+d+g-c};p]_{N-i}}$$

$$(4.1)\qquad \cdot\sum_{k_1=0}^{\min(i,j)}\sum_{k_2=\max(i,j)}^{N}\frac{[p^d;p]_{j-k_1}[p^g;p]_{k_2-j}[p^b;p]_{N-k_2}[p;p]_{k_2-k_1}}{[p;p]_{j-k_1}[p;p]_{k_2-j}[p^{d+g};p]_{k_2-k_1}}$$

$$\cdot\ \frac{[p^{d+g-c};p]_{k_2-i}[p^a;p]_{k_1}[p^c;p]_{i-k_1}\, p^{k_2(c-g)-ak_1-dj-ic}}{[p;p]_{k_2-i}[p;p]_{k_1}[p;p]_{i-k_1}[p;p]_{N-k_2}}.$$

PROOF OF (4.1). Let

$$P_n(j) = {}_3\phi_2\left[\begin{matrix} q^{-n},\ q^{a+b+d+g+n-1},\ q^{b+g+N-j};\ q;\ q^{1+j} \\ q^{a+b+d+g+N},\ q^{1+b} \end{matrix}\right], \tag{4.2}$$

Multiplying both the sides of (4.2) by

$$\frac{[q^d; q]_{j-k_1}[q^g; q]_{k_2-j}\, q^{g(j-k_1)}}{[q; q]_{j-k_1}[q; q]_{k_2-j}}$$

and summing with respect to j from k_1 to k_2, we get

$$S_1 \equiv \sum_{j=k_1}^{k_2} \frac{[q^d; q]_{j-k_1}[q^g; q]_{k_2-j}\, q^{g(j-k_1)}}{[q; q]_{k-k_1}[q; q]_{k_2-j}} P_n(j)$$

$$= \sum_{j=k_1}^{k_2}\sum_{r=0}^{n} \frac{[q^{-n}; q]_r[q^{a+b+d+g+n-1}; q]_r[q^{b+g+N-j}; q]_r[q^d; q]_{j-k_1}}{[q; q]_r[q^{a+b+d+g+N}; q]_r[q^{b+g}; q]_r[q; q]_{j-k_1}} \cdot \frac{[q^g; q]_{k_2-j}\, q^{r(1+j)+(j-k_1)g}}{[q; q]_{k_2-j}}. \tag{4.3}$$

On the right hand side of (4.3) replacing $q^{jr}[q^{b+g+N-j}; q]_r$ by

$$[q^{b+N+k_2}; q]_r\, q^{r(k_2+g)} \sum_{m=0}^{r} \frac{[q^{-r}; q]_m[q^{g+k_2-j}; q]_m\, q^{m(1-N+j)}}{[q; q]_m[q^{1-b-N-r+k_2}; q]_m\, q^{m(b+g)}}$$

(q-analogue of Gauss' summation theorem [**13**; 3.3.2.6]).
Rearranging the series, and summing the resulting innermost ${}_2\phi_1$ by the q-analogue of Vandermonde's thereom [**12**; 3.3.2.7] and then multiply by

$$\frac{[q^b; q]_{N-k_2}[q; q]_{k_2-k_1}[q^{d+g-0}; q]_{k_2-i}\, q^{b(k_2-i)}}{[q; q]_{N-k_2}[q^{d+g}; q]_{k_2-k_1}[q; q]_{k_2-i}},$$

summing with respect to k_2 from i to N, we get

$$S_2 \equiv \sum_{k_2=i}^{N}\sum_{j=k_1}^{k_2} \frac{[q^d; q]_{j-k_1}[q^g; q]_{k_2-j}[q^b; q]_{N-k_2}[q; q]_{k_2-k_1}[q^{d+g-c}; q]_{k_2-i}}{[q; q]_{j-k_1}[q; q]_{j_2-j}[q; q]_{N-k_2}[q^{d+g}; q]_{k_2-k_1}[q; q]_{k_2-i}}$$

$$\cdot\, q^{g(j-k_1)+b(k_2-i)} P_n(j) = \sum_{r=0}^{n}\sum_{m=0}^{r} \frac{[q^{-n}; q]_r}{[q; q]_r} \tag{4.4}$$

$$\cdot\, \frac{[q^{a+b+d+g+n-1}; q]_r[q^{-r}; q]_m[q^g; q]_m\, q^{r(1+g)+m(1-N+k_2-b-g)}}{[q^{a+b+d+g+N}; q]_r[q^{b+g}; q]_r[q; q]_m[q^{d+g}; q]_m}$$

$$\cdot \sum_{k_2=i}^{N} \frac{[q^b; q]_{N-k_2+r}[q^{d+g-c}; q]_{k_2-i}[q^{d+\sigma+k_2-k_1}; q]_m\, q^{k_2r-b(k_2-i)}}{[q; q]_{N-k_2}[q; q]_{k_2-i}[q^{1-b-N-r+k_2}; q]_m}$$

Once again in the right hand expression of (4.4) replacing $[q^{d+g+1k_2-k_1}; q]_m$ by

$$[q^{c+i-k_1}; q]_m \sum_{h=0}^{m} \frac{[q^{-m}; q]_h[q^{d+g-c+k_2-i}; q]_h\, q^h}{[q; q]_h\, [q^{1-c-i+k_1-m}; q]_h}$$

(q-analogue of Vandermonde's theorem [**12**; 3.3.2.7]) and rearranging the series, and summing the resulting innermost ${}_2\phi_1$ by the q-analogue of Vandermonde's theorem [12; 3.3.27] and then multiplying by

$$\frac{[q^a;q]_{k_1}[q^c;q]_{i-k_1}\,q^{ck_1}}{[q;q]_{k_1}[q;q]_{i-k_1}}$$

and summing with respect to k_1 from 0 to i, we get

$$S_3 \equiv \sum_{k_1=0}^{i}\sum_{k_2=i}^{N}\sum_{j=k_1}^{k_2} \frac{[q^d;q]_{j-k_1}[q^g;q]_{k_2-j}[q^b;q]_{N-k_2}[q;q]_{k_2-k_1}}{[q;q]_{j-k_1}[q;q]_{k_2-j}[q;q]_{N-k_2}[q^{d+g};q]_{k_2-k_1}}$$
$$\cdot \frac{[q^{d+g-c};q]_{k_2-i}[q^a;q]_{k_1}[q^c;q]_{i-k_1}\,q^{k_1(c-g)+jg+b(k_2-i)}}{[q;q]_{k_2-i}[q;q]_{k_1}[q;q]_{i-k_1}}\,P_n(j)$$

$$(4.5)\qquad = \sum_{r=0}^{n}\sum_{m=0}^{r}\sum_{h=0}^{m} \frac{[q^{-n};q]_r[q^{a+b+d+g+n-1};q]_r[q^{-r};q]_m[q^g;q]_m[q^b;q]_{r-m}}{[q\,;q]_r[q^{a+b+d+g+N};q]_r[q^{b+g};q]_r[q;q]_m[q^{d+g};q]_m}$$
$$\cdot \frac{[q^{-m};q]_h[q^{d+g-c};q]_h[q^{b+d+g-c+r-m+h};q]_{N-i}[q^c;q]_{i+m}(-)^m}{[q;q]_i[q;q]_h[q;q]_{N-i}[q^{1-c-i-m};q]_h}$$
$$\cdot\, q^{r(1+g+i+m)+m(1-g-i)+h-m(m+1)/2}\,{}_2\phi_1\left[\begin{matrix}q^a,\,q^{-i};\,q\\ q^{1-c-i-m+h}\end{matrix}\right].$$

In (4.5) summing the inner ${}_2\phi_1$ by the q-analogue of Vandermonde's theorem and once again replacing $[q^{a+c+i};q]_h$ by

$$\sum_{s=0}^{h}\frac{[q^{-h};q]_s[q^{b+d+g-c+N-i+r-h};q]_s[q^{a+b+d+g+N+r-h+s};q]_{h-s}}{[q;q]_s\,q^{-s(i+h+a+c)}},$$

we get on rearranging the series

$$S_3 = \frac{[q^{b+d+g-c};q]_{N-i}[q^{a+c};q]_i}{[q;q]_{N-i}\,[q;q]_i}\sum_{t=0}^{n}\sum_{r=0}^{n-t}\sum_{m=0}^{t}\frac{[q^{-n};q]_{r+t}[q^{-t};q]_m}{[q;q]_t[q;q]_r[q;q]_m}$$
$$(4.6)\qquad \cdot \frac{[q^{a+b+d+g+n-1};q]_{r+t}[q^g;q]_{m+r}[q^b;q]_{t-m}[q^{b+d+g-c+N-i};q]_t}{[q^{b+g};q]_{r+t}[q^{a+b+d+g+N};q]_t[q^{d+g};q]_{m+r}[q^{a+c};q]_r}$$
$$\cdot \frac{[q^c;q]_r[q^{d+g-c};q]_m}{[q^{b+d+g-c};q]_t}(-)^m q^{t(g+1)+m(c-g+1)+r+t(m+i)-m(m+1)/2}$$
$$\cdot\,{}_3\phi_2\left[\begin{matrix}q^{-m},\,q^{c+r},\,q^{c-b-d-g+1-t};\,q;\,q^{a+b+t}\\ q^{a+c+r},\,q^{c-d-g+1-m}\end{matrix}\right]$$

Transforming the ${}_3\phi_2$ in (4.6) by (3.6) and then rearranging the series and summing the resulting innermost ${}_2\phi_1$ by the q-analogue of Gauss' theorem [19; 3.3.2.6], we get

$$S_3 = \frac{[q^{b+d+g-c};q]_{N-i}[q^{a+c};q]_i}{[q;q]_{N-i}[q;q]_i}\sum_{t=0}^{n}\sum_{r=0}^{n-t}\frac{[q^{-n};q]_{r+t}[q^{a+b+d+g+n-1};t]_{r+t}}{[q;q]_t[q;q]_r[q^{a+b+d+g+N};q]_t}$$
$$(4.7)\qquad \cdot \frac{[q^g;q]_r[q^c;q]_r[q^b;q]_t[q^{b+d+g-c+N-i};q]_t[q^d;q]_t\,q^{t(1+i+g)+r}}{[q^{b+g};q]_{r+t}[q^{d+g};q]_{r+t}[q^{a+c};q]_r[q^{b+d+g-c};q]_t}$$
$$\cdot\,{}_4\phi_3\left[\begin{matrix}q^{-t},\,q^{g+r},\,q^a,\,q^{1+c-b-d-g-t};\,q;\,q\\ q^{1-d-t};\,q^{a+c+r},\,q^{1-b-t}\end{matrix}\right]$$

Transforming the inner ${}_4\phi_3$ in (4.7) by (3.5), rearranging the series and summing the resulting innermost ${}_3\phi_2$ by the q-analogue of Saalschiitz summation theorem [**12**; 3.3.2.2], we obtain on using (3.5) and replacing q by $1/p$ on both the sides

(4.8)
$$\sum_{j=0}^{N}\sum_{k_1=0}^{\min(i,j)}\sum_{k_2=\max(i,j)}^{N}\frac{[p^d:p]_{j-k_1}[p^g;p]_{k_2-j}[p^b;p]_{N-k_2}[p;p]_{k_2-k_1}}{[p;p]_{j-k_1}[p;p]_{k_2-j}[p;p]_{N-k_2}[p^{d+g};p]_{k_2-k_1}}$$
$$\cdot\frac{[p^{d+g-c};p]_{k_2-i}[p^a;p]_{k_1}[p^c;p]_{i-k_1}p^{k_2(c-g)-ak_1-dj}}{[p;p]_{k_2-i}[p;p]_{k_1}[p;p]_{i-k_1}}$$
$$\cdot\,{}_3\phi_2\left[\begin{matrix}p^{-n},\ p^{a+b+d+g+n-1},\ p^{b+g+N-j};\ p;\ p\\ p^{b+g},\ p^{a+b+d+g+N}\end{matrix}\right]$$
$$=\frac{[p^{b+d+g-d};p]_{N-i}[p^{a+c};p]_i[p^{a+d};p]_n[p^{b+d+g-c};p]_n}{[p;p]_{N-i}[p;p]_i[p^{a+c};p]_n[p^{b+g};p]_n}$$
$$p^{n(c-d)-ia+N(c-d-g)}$$
$$\cdot\,{}_4\phi_3\left[\begin{matrix}p^{-n},\ p^{a+b+d+g+n-1},\ p^d,\ p^{d+g-c};\ p;\ p\\ p^{d+g},\ p^{a+d},\ p^{b+d+g-c}\end{matrix}\right]$$
$$\cdot\,{}_3\phi_2\left[\begin{matrix}p^{-n},\ p^{a+b+d+n-1},\ p^{b+d+g-c+N-i};\ p;\ p\\ p^{a+b+d+g+N},\ p^{b+d+g-c}\end{matrix}\right].$$

Transforming the ${}_3\phi_2$'s on both the sides of (4.8) by the formula (3.7), we get

(4.9)
$$\sum_{j=0}^{N} K_N(i,j;a,b,c,d,g;p)\,Q_n^{(2)}(j)=\lambda_n Q_n^{(1)}(i),$$

where $K_N(i,j,a,b,c,d,g;p)$ is given by (4.1), $Q_n^{(1)}(i)=Q_n(i;a+c-1,b+d+g-c-1,N;p)$, $Q_n^{(2)}(j)=Q_n(j;a+d-1,b+g-1,N;p)$ and

$$\lambda_n={}_4\phi_3\left[\begin{matrix}p^{-n},\ p^{a+b+d+g+n-1},\ p^d,\ p^{d+g-c};\ p;\ p\\ p^{d+g},\ p^{a+d},\ p^{+d+g-c}\end{matrix}\right],$$

which completes the proof of (4.1).

Note that for $c=d$, the q-Hahn polynomials on the two sides of (4.9) have the same arguments and hence $Q_n(i)$ is an eigenvector of the matrix $K_N(i,j)$ corresponding to the eigenvalue λ_n.

Let us now introduce the orthonormal systems

(4.10)
$$R_n^{(k)}(i)=\{h^{(k)}(n;p)\,g^{(k)}(i;p)\}^{1/2}Q_n^{(k)}(i).$$

In terms of $R_n^{(k)}(i)$, (4.9) may be rewritten in the form

(4.11)
$$\sum_{j=0}^{N} G_N(i,j;a,b,c,d,g;p)\,R_n^{(2)}(j)=\mu_n R_n^{(1)}(i)$$

where

$$G_N(i,j;a,b,c,d,g;p) = \left\{\frac{[p;p]_j[p;p]_{N-j}[p^{a+c};p]_i[p^{b+d+g-c};p]_{N-i}}{[p;p]_i[p;p]_{N-i}[p^{a+d};p]_j[p^{b+g};p]_{N-j}} \cdot \frac{p^{N(c-d)+j(a+d)}}{p^{i(a+c)}}\right\}^{1/2} K_N(i,j;a,b,c,d,g;p)$$

and

$$\mu_n = \left\{\frac{[p^{a+d};p]_n\,[p^{b+d+g-c};p]_n}{[p^{b+g};p]_n[p^{a+c};c]_n}\, p^{(c-d)n}\right\}^{1/2} \lambda_n. \tag{4.12}$$

If $c = d$, $G_N(i,j)$ is symmetric. It can be verified by interchanging $c \leftrightarrow d$ and $g \leftrightarrow d + g - c$ that

$$\sum_{j=0}^{N} G_N(j,i;a,b,c,d,g;p)\, R_n^{(1)}(j) = \mu_n\, R_n^{(2)}(i). \tag{4.13}$$

Lastly, multiplying (4.11) by $R_n^{(2)}(x)$ and summing with respect to n from 0 to N and using (2.8), we get

$$\sum_{n=0}^{N} \mu_n\, R_n^{(1)}(i) R_n^{(2)}(j) = G_N(i,j;a,b,c,d,g;p) \tag{4.14}$$

or

$$\begin{aligned}\sum_{n=0}^{N} & \frac{[p^{-N};p]_n[p^{a+b+d+g-1};p]_n[p^{a+d};p]_n(1-p^{a+b+d+g+2n-1})}{[p;p]_n[p^{a+b+d+d+N};p]_n[p^{b+g};p]_n} \\ & \cdot \frac{p^{-n/2(n+1)-n(a+d-1-N)}}{(1-p^{a+b+d+g-1})}\, {}_4\phi_3\left[\begin{matrix}p^{-n}, p^{a+b+d+g+n-1}, p^d, p^{d+g-c}; p; p \\ p^{d+g}, p^{a+d}, p^{b+d+g-c}\end{matrix}\right] \\ & \cdot Q_n^{(1)}(i)\, Q_n^{(2)}(j) \\ & = \frac{[p^{a+b+d+g};p]_N[p;p]_j[p;p]_{N-j}\, p^{-N(a+d)+j(a+d)}}{[p;p]_N[p^{a+d};p]_j[p^{b+g};p]_{N-j}} \\ & \cdot K_N(i,j;a,b,c,d,g;p).\end{aligned} \tag{4.15}$$

SPECIAL CASES. (i) (4.15) for $a = 0$ yields the following bilinear formula for q-Hahn polynomials:

$$\begin{aligned}\sum_{n=0}^{N} & \frac{[p^{-N};p]_n[p^{b+d+g-1};p]_n[p^d;p]_n[p^b;p]_n[p^c;p]_n(1-p^{b+d+g+2n-1})}{[p;p]_n[p^{b+d+g+N};p]_n[p^{b+g};p]_n[p^{b+d+g-c};p]_n[p^{d+g};p]_n(1-p^{b+d+g-1})} \\ & \cdot p^{n/2(n+1)+n(g-c+1+N)} Q_n(i; c-1, b+d+g-c-1, N; p) \\ & \cdot Q_n(j; d-1, b+g-1, N; p) \\ & = \frac{[p^{b+d+g};p]_N[p;p]_{N-j}[p;p]_{N-i}\, p^{N(g-c)}}{[p;p]_N[p^{b+g};p]_{N-j}[p^{b+d+g-c};p]_{N-i}} \\ & \cdot \sum_{k_2=\max(i,j)}^{N} \frac{[p^g;p]_{k_2-j}[p^b;p]_{N-k_2}[p;p]_{k_2}[p^{d+g-c};p]_{k_2-i}\, p^{k_2(c-g)}}{[p;p]_{k_2-j}[p;p]_{N-k_2}[p^{d+g};p]_{k_2}[p;p]_{k_2-i}}\end{aligned} \tag{4.16}$$

Furthermore, (4.16) for $c = d$ gives

$$\sum_{n=0}^{N} \frac{[p^{-N};p]_n[p^{b+d+g-1};p]_n[p^d;p]_n[p^b;p]_n[p^d;p]_n(1-p^{b+d+g+2n-1})}{[p;p]_n[p^{b+d+g+N};p]_n[p^{b+g};p]_n[p^{b+g};p]_n[p^{d+g};p]_n(1-p^{b+d+g-1})}$$

$$\cdot\, p^{n/2(n+1)+n(g-c+1+N)}Q_n(i;\, d-1,\, b+g-1,\, N;p)$$

(4.17)
$$\cdot\, Q_n(j;\, d-1,\, b+g-1,\, N;\, p)$$

$$= \frac{[p^{b+d+g};p]_N[p;p]_{N-j}[p;p]_{N-i}\,p^{N(g-d)}}{[p;p]_N[p^{b+g};p]_{N-j}[p^{b+g};p]_{N-i}}$$

$$\cdot \sum_{k_2=\max(i,j)}^{N} \frac{[p^g;p]_{k_2-j}[p^b;p]_{N-k_2}[p;p]_{k_2}[p^g;p]_{k_2-i}\,p^{k_2(d-g)}}{[p;p]_{k_2-j}[p;p]_{N-k_2}[p^{d+g};p]_{k_2}[p;p]_{k_2-i}}$$

(ii) In (4.15) setting $a = g - c$ and letting $b \to 0$, we get the following interesting bilinear formula:

$$\sum_{n=0}^{N} \frac{[p^{-N};p]_n[p^{d-c-1};p]_n[p^{g-c};p]_n(1-p^{d-c-1+2n})p^{-n/2(n+1)-n(g-c-1-N)}}{[p;p]_n[p^{d-c+N};p]_n[p^{d+g};p]_n(1-p^{d-c-1})}$$

$$\cdot\, Q_n(i;g-1,d+g-c-1,\, N;\, p)Q_n(j;\, d+g-c-1,\, g-1,\, N;p)$$

(4.18)
$$= \frac{[p^{d-c};p]_N[p^d;p]_j[p^c;p]_i\,p^{(i+j-N)(g-c)}}{[p^{d+g};p]_N[p^{d+g-c};p]_j[p^g;p]_i}$$

$$\cdot\, {}_4\phi_3\left[\begin{matrix}p^{g-c},\, p^{1-d-g-N},\, p^{-i},\, p^{-j};\, p;\, p\\ p^{1-d-j},\, p^{1-c-j},\, p^{-N}\end{matrix}\right].$$

5. Before obtaining the q-analogue of a connection relation for Hahn polynomials due to Ismail [7] we need the following transform formula

(5.1) $$L_N\{Q_j(n;\, a,\, b,\, N;\, q);\, \phi_N;\, x\} = p_j(x;\, a,\, b;\, q)$$

with L_N is defined as

(5.2) $$L_N\{f;\, \phi_N;\, x\} = \sum_{n=0}^{N} \frac{(1-q)^n x^n(-)^n\, q^{n(n-1)/2}}{[q;\, q]_n}\, \phi_N^{(n)}(x)\, f(n),$$

$$\phi_N^{(n)}(x) = D_q^n\{\phi_N(x)\},$$

where $D_qf(x) = (f(x) - f(xq))/x(1-q)$ and $D_q^n f(x) = D_q(D_q^{n-1}f(x))$. In (5.2) setting $f(n) = Q_j(n;a,b,N;q)$,

$$\phi_N(x) = \sum_{k=0}^{N} \frac{[q^{-N};\, q]_k}{[q;\, q]_k}\, (xq)^k$$

so that

$$\phi_N^{(n)}(x) = \sum_{k=n}^{N} \frac{(-)^n[q^{-N};\, q]_k[q^{-k};\, q]_n}{[q;\, q]_k(1-q)^n}\, x^{k-n}\, q^{-n/2(n+1)+kn}$$

and rearranging the series, we get (5.1) on some simplification.

Now, using (5.1) in the orthogonality relation (2.3) for q-Jacobi polynomials, we get

$$\sum_{n=0}^{N}\sum_{m=0}^{N} \frac{[q^{-N};q]_n[q^{-N};q]_m[q^{1+a};q]_{m+n}[q^{1+b};q]_{N-n}(-)^{n+m}}{[q;q]_m[q;q]_n[q^{2+a+b};q]_{m+N}\, q^{n(1+a+m)}}$$

$$(5.3)\qquad \cdot\, q^{m/2(m+1)-n/2(n-1)+N(1+a+m+n)}\, {}_2\phi_1\left[\begin{matrix} q^{-N+m},\ q^{1+a+m+n};\ q;\ q^{1+N-n} \\ q^{2+a+b+m+N}\end{matrix}\right]$$

$$\cdot\, Q_j(n;a,b,N;q)\, Q_t(m;a,b,N;q)$$

$$= \frac{[q;q]_j[q^{1+b};q]_j(1-q^{1+a+b})\, q^{j(1+a)}}{[q^{1+a};q]_j[q^{1+a+b};q]_j(1-q^{1+a+b+2j})}$$

Comparing (5.3) and (2.7) and using the uniqueness of the orthogonality relation (2.7), we get a connection relation

$$(5.4)\qquad \sum_{m=0}^{N} \theta(n,m)\, Q_j(m;a,b,N;q) = \mu_j\, Q_j(n;a,b,N;q)$$

where

$$\mu_j = \frac{[q^{-N};q]_j(-)^j\, q^{-j/2(j+1)+j(N-1)}}{[q^{2+a+b+N};q]_n},$$

$$\theta(n,m) = \frac{[q^{-N};q]_m[q^{1+a+n};q]_m[q^{1+b};q]_{N-n}(-)^m\, q^{m/2(m+1)+m(N-n)}}{[q;q]_m[q^{2+a+b+N};q]_m[q^{1+a};q]_{N-n}}$$

$$\cdot\, {}_2\phi_1\left[\begin{matrix} q^{-N+m},\ q^{1+a+m+n};\ q;\ q^{1+N-n} \\ q^{2+a+b+m+N}\end{matrix}\right].$$

Multiplying both the sides of (5.4) by $h(j;a,b;q)$ and $Q_j(y;a,b,N;q)$, adding for $j = 0, 1, \ldots N$ and using the dual orthogonality relation (2.8), we get the bilinear formula

$$(5.5)\qquad \sum_{j=0}^{N} \mu_j h(j;a,b,q)\, Q_j(n;a,b,N;q)\, Q_j(y,a,b,N;q) = \frac{\theta(n,y)}{g(y;a,b;q)}.$$

On the other hand multiplying (5.4) by $\mu_j h(j;a,b;q)$ and $Q_j(y;a,b,N;q)$, adding for j from 0 to N and using (5.5), we get a second bilinear formula for q-Hahn polynomials

$$(5.6)\qquad \sum_{j=0}^{N} \mu_j^2 h(j;a,b;q) Q_j(n;a,b,N;q)\, Q_j(y;a,b,N;q) = \sum_{m=0}^{N} \theta(n,m)\frac{\theta(m,y)}{g(y;a,b;q)}.$$

Similarly one can use (5.4) and (5.6) to generate more complicated bilinear formulas for q-Hahn polynomials.

6. Feldheim [4] obtained an explicit formula (a ${}_3F_2$ hypergeometric function) for the connection coefficients between two arbitrary Jacobi polynomials. Later on, Gasper [5] proved a discrete analogue of Feldheim's result by obtaining an explicit formula for the connection coefficients between two arbitrary Hahn polynomials and discussed some of its special cases. Recently, Andrews and Askey [2] obtained the q-analogue of Feldheim's formula. In this section we obtain an explicit formula for the connection coefficients between two q-Racah polynomials and discuss some of its interesting special cases. In fact we prove

$$\sum_{s=0}^{n} \frac{[q^{-n};q]_s[q^{1+\alpha+\beta+n};q]_s[q^{-x};q]_s[q^{1+c+d+x};q]_s\, q^s}{[q;q]_s[q^{1+\alpha};q]_s[q^{1+\beta+d};q]_s[q^{1+c};q]_s} A_s = \sum_{m=0}^{n} a_{m,n}, P_m(\mu(x);a,b,c,d;q) \tag{6.1}$$

where

$$A_s = \frac{[q^{(a_t)};q]_s}{[q^{(b_t)};q]_s},$$

$$a_{m,n} = \frac{(-)^m q^{m(m+1)/2}[q^{-n};q]_m[q^{1+\alpha+\beta+n};q]_m[q^{1+b+d};q]_m[q^{1+a};q]_m}{[q;q]_m[q^{1+\alpha};q]_m[q^{1+\beta+d};q]_m[q^{1+a+b+m};q]_m}$$
$$\cdot \sum_{r=0}^{n-m} \frac{[q^{-n+m};q]_r[q^{1+\alpha+\beta n+m};q]_r[q^{1+a+m};q]_r[q^{1+b+d+m};q]_r}{[q;q]_r[q^{1+\alpha+m};q]_r[q^{1+\beta+d+m};q]_r[q^{2+a+b+2m};q]_r} q^r A_{r+m}$$

and either $1+a$, $1+b+d$ or $1+c$ is of the form $-N$ (N a non-negative integer).

Proof of (6.1). Multiplying (6.1) by $\omega(x;q)P_t(\mu(x);a,b,c,d;q)$ with $\omega(x;q)$ defined by (2.14), summing with respect to x from 0 to N (assuming that either $1+a$, $1+b+d$ or $1+c$ is of the form $-N$) and using the orthogonality relation (2.12), we get

$$a_{t,n} = \pi(t;q)\sum_{x=0}^{N}\sum_{s=0}^{x} \frac{[q^{-n};q]_s[q^{1+\alpha+\beta+n};q]_s[q^{-x};q]_s[q^{1+c+d+x};q]_s\, q^s A_s}{[q;q]_s[q^{1+\alpha};q]_s[q^{1+\beta+d};q]_s[q^{1+c};q]_s}$$
$$\omega(x;q)P_t(\mu(x);a,b,c,d;q)$$
$$= \pi(t;q)\frac{[q^{1+a-d};q]_t[q^{1+a+b-c};q]_t\, q^{t(d-a+c)}}{[q^{1+b+d};q]_t[q^{1+c};q]_t}$$
$$\cdot \sum_{s=0}^{n}\sum_{k=0}^{t} \frac{[q^{-n};q]_s[q^{-t};q]_k}{[q;q]_s[q;q]_k}$$
$$\cdot \frac{[q^{1+\alpha+\beta+n};q]_s[q^{1+a+b+t};q]_k[q^{2+c+d};q]_{2s}[q^{1+a};q]_{s+k}[q^{a-c-d-s};q]_k}{[q^{1+\alpha};q]_s[q^{1+\beta+d};q]_s[q^{1+a};q]_k[q^{1+a-d};q]_k\,[q^{1+a+b-c};q]_k} \tag{6.2}$$
$$\cdot \frac{[q^{1+b+d};q]_s\, A_s(-)^s\, q^{k-s(s+1)/2-s(a+b)}}{[q^{1+c-b};q]_s[q^{1+c+d-a};q]_s[q^{1+d};q]_s}$$
$$\cdot {}_6\phi_5\left[\begin{matrix} q^{1+c+d+2s},\ q^{(3+c+d+2s)/2},\ -q^{(3+c+d+2s)/2},\ q^{1+a+s+k}, \\ q^{(1+c+d+2s)/2},\ -q^{(1+c+d+2s)/2},\ q^{1+c+d-a+s-k}, \end{matrix}\right.$$
$$\left.\begin{matrix} q^{1+c+s},\ q^{1+b+d+s};\ q;\ q^{-s-k-a-b-1} \\ q^{1+d+s},\ q^{1+c-b+s} \end{matrix}\right].$$

Summing ${}_6\phi_5$ by the summation theorem [**12**; 3.3.1.4], we get

$$a_{t,n} = \frac{[q^{1+a}; q]_t\,[q^{1+a+b}; q]_t(1 - q^{1+a+b+2t})\, q^{-t(1+a)}}{[q; q]_t[q^{1+b}; q]_t(1 - q^{1+a+b})}$$

$$\cdot \sum_{s=0}^{n} \frac{[q^{-n}; q]_s\,[q^{1+\alpha+\beta+n}; q]_s[q^{1+a}; q]_s[q^{1+b+d}; q]_s\, q^s A_s}{[q; q]_s[q^{1+\alpha}; q]_s[q^{1+\beta+d}; q]_s[q^{2+a+b}; q]_s}$$

$$\cdot\, {}_3\phi_2\left[\begin{matrix} q^{-t},\, q^{1+a+b+t},\, q^{1+a+s};\, q;\, q \\ q^{1+a},\, q^{2+a+b+s} \end{matrix}\right],$$

on summing ${}_3\phi_2$ by the q-analogue of Saalschiitz summation theorem [**13**; 3.3.2.2], we get the required result on some simplification.

SPECIAL CASES. (i) For $A_s = 1$, (6.1) yields

(6.3)
$$P_n(\mu(x); \alpha, \beta, -N - 1, d; q) = \sum_{m=0}^{n} a_{m,n}\, P_m(\mu(x); a, b, -N - 1, d; q)$$

where

$$a_{m,n} = \frac{(-)^m q^{m(m+1)/2}[q^{-n}; q]_m[q^{1+\alpha+\beta+n}; q]_m[q^{1+b+d}; q]_m[q^{1+a}; q]_m}{[q; q]_m[q^{1+\alpha}; q]_m[q^{1+\beta+d}; q]_m[q^{1+a+b+m}; q]_m}$$

$$\cdot\, {}_4\phi_3\left[\begin{matrix} q^{-n+m},\, q^{1+\alpha+\beta+n+m},\, q^{1+a+m},\, q^{1+b+d+m};\, q;\, q \\ q^{1+\alpha+m},\, q^{1+\beta+d+m},\, q^{2+a+b+3m} \end{matrix}\right].$$

(6.3) is an extension of a result of Andrews and Askey [**2**; 3.10] to which it reduces on replacing x by replacing x by $N - x$ and letting $d, N \to \infty$.

(ii) In (6.1), setting $A_s = [q^{1+\beta+d}; q]_s/[q^{-M}; q]_s$, $1 + b + d = -N$ and letting $c \to \infty$, we get the q-analogue of a result of Gasper [**5**; 3.1], viz.,

(6.4)
$$Q_n(x; \alpha, \beta, M; q) = \sum_{m=0}^{n} B_{m,n} Q_n(x; a, b, N; q)$$

where

$$B_{m,n} = \frac{(-)^m q^{m(m+1)/2}[q^{-n}; q]_m[q^{1+\alpha+\beta+n}; q]_m[q^{1+a}; q]_m[q^{-N}; q]_m}{[q; q]_m[q^{1+\alpha}; q]_m[q^{-M}; q]_m[q^{1+a+b+m}; q]_m}$$

$$\cdot\, {}_4\phi_3\left[\begin{matrix} q^{-n+m},\, q^{1+\alpha+\beta+n+m},\, q^{1+a+m},\, q^{-N+m};\, q;\, q \\ q^{1+\alpha+m},\, q^{2+a+b+2m},\, q^{-M+m} \end{matrix}\right].$$

Furthermore some of interesting special cases of (6.4) may be discussed on the lines of Gasper [**5**], we omit them for sake of brevity.

(iii) In (6.3), ${}_4\phi_3$ reduces to ${}_3\phi_2$ in two cases (i) $\alpha = a$, and (ii) $\beta = b$ which when summed by the q-analogue of Saalschiitz summation theorem [**12**; 3.3.2.2], yield the following connection relation for q-Racah polynomials

$$(6.5)\quad \begin{aligned} &P_n(\mu(x); a, \beta, c, d; q) \\ &= \sum_{m=0}^{n} \frac{(-)^m q^{-m(m+1)/2+mn+(1+b+d)n-(b+d)m}[q^{-n}; q]_m}{[q; q]_m[q^{1+\beta+d}; q]_n[q^{1+a+b+m}; q]_m} \\ &\cdot \frac{[q^{1+a+\beta+n}; q]_m[q^{1+b+d}; q]_m[q^{\beta-b}; q]_{n-m}[q^{1+a-b+m}; q]_{n-m}}{[q^{2+a+b+2m}; q]_{n-m}} \\ &\cdot P_m(\mu(x); a, b, c, d; q) \end{aligned}$$

(provided either $1 + a$ or $1 + c$ is of the form $-N$) and

$$(6.6)\quad \begin{aligned} P_n(\mu(x); \alpha, b, c, d; q) &= \sum_{m=0}^{n} \frac{(-)^m q^{-m(m+1)/2+n(1+a+m)-am}[q^{-n}; q]_m}{[q; q]_m[q^{1+\alpha}; q]_n[q^{1+a+b+m}; q]_m} \\ &\cdot \frac{[q^{1+\alpha+b+n}; q]_m[q^{1+a}; q]_m[q^{1+b+m}; q]_{n-m}[q^{\alpha-a}; q]_{n-m}}{[q^{2+a+b+2m}; q]_{n-m}} \\ &\cdot P_m(\mu(x); a, b, c, d; q) \end{aligned}$$

(provided either $1 + b + d$ or $1 + c$ is of the form $-N$). (6.5) and (6.6) for $d \to \infty$ yield the results which are q-analogues of Gasper's result for Hahn polynomials [**5**; 3.8, 3.7].

(iv) In (6.3) setting $1 + a = 1 + \alpha = -N$ and letting $\beta \to \infty$, we get

$$(6.7)\quad Q_x(n; c, d, N; q) = \sum_{m=0}^{n} H(m, n)P_m(\mu(x); -N - 1, b, c, d; q)$$

where

$$(6.8)\quad \begin{aligned} H(m,n) &= \frac{[q^{-d-N};q]_n[q^{-n};q]_m[q^{1+b+d};q]_m[q^{b-N};q]_m[q^{1+b-N};q]_{2m}(-)^m}{[q^{1+b-N}; q]_{n+m}[q; q]_m[q^{b-N}; q]_{2m}[q^{-d-N}; q]_m} \\ &\cdot q^{n(1+b+d)+m(n-b-d)-m(m+1)/2}. \end{aligned}$$

On the other hand, we get an inverse formula of (6.7) by setting in (6.3) $1 + a = 1 + \alpha = -N$ and letting $b \to \infty$, viz.,

$$(6.9)\quad \begin{aligned} P_n(\mu(x); -N - 1, \beta, c, d; q) &= \frac{(-)^n q^{n(n+1)/2+(\beta+d)n}[q^{-d-N}; q]_n}{[q^{1+\beta+d}; q]_n} \\ &\cdot \sum_{m=0}^{n} \frac{q^{-m(\beta+d)}[q^{-n}; q]_m[q^{\beta-N+n}; q]_m}{[q; q]_m[q^{-d-N}; q]_m} Q_x(n; c, d, N; q). \end{aligned}$$

(v) Lastly, substituting (6.7) in the orthogonality relation (2.7) of q-Hahn polynomials, we get

$$(6.10)\quad \begin{aligned} &\sum_{n=0}^{N} g(n; c, d; q) \sum_{m=0}^{n} H(m, n)P_m(\mu(x); -N - 1, b, c, d; q) \sum_{t=0}^{n} H(t, n) \\ &\cdot P_t(\mu(y); -N - 1, b, c, d; q) = 1/h(x; c, d; q), \end{aligned}$$

where $g(n; c, d; q)$, $h(x; c, d; q)$ and $H(m, n)$ are defined by (2.9), (2.10) and (6.8) respectively.

Now, (6.10) may be rewritten in the form

$$\sum_{m=0}^{N} \pi(m;\ -N-1,\ b,\ c,\ d)$$

$$(6.11)\quad \cdot\left\{\sum_{t=0}^{N} \sum_{n=\max(m,t)}^{N} \frac{h(x;\ c,\ d;\ q)g(n;\ c,\ d;\ q)H(m,\ n)H(t,\ n)}{\omega(x;\ -N-1,\ b,\ c,\ d;\ q)\pi(m;\ -N-1,b,\ c,\ d;\ c)}\right.$$

$$\left.\cdot\ P_t(\mu(y);\ -N-1,\ b,\ c,\ d;\ q)\right\} P_m(\mu(x);\ -N-1,\ b,\ c,\ d;\ q)$$

$$= 1/\omega(x;\ -N-1,\ b,\ c,\ d;\ q).$$

In view of the uniqueness of the dual orthogonality relation of q-Racah polynomials (2.13), (6.11) yields the connection relation

$$(6.12)\quad \lambda(x)\, P_m(\mu(x);\ -N-1,\ b,\ c,\ d;\ q) = \sum_{t=0}^{N} \sum_{n=\max(m,t)}^{N} \mu(m,\ n)H(t,\ n)$$

$$\cdot\ P_t(\mu(x);\ -N-1,\ b,\ c,\ d;\ q),$$

where

$$\lambda(x) = \frac{[q^{1+b+d};\ q]_x(-)^x q^{x(x+1)/2+x(c-b)}}{[q^{1+c-b};\ q]_x}$$

and

$$\mu(m,\ n) = \frac{[q^{1+c};\ q]_n[q^{-N};\ q]_n[q^{-b};\ q]_N\, q^{(1+c)N+(b-c)n}}{[q;\ q]_n[q^{1+b-N};\ q]_n[q^{1+c-b};\ q]_N}$$

$$\cdot\ \frac{[q^{-n};\ q]_m[q^{1+b};\ q]_m[q^{b-c-N};\ q]_m(-)^m q^{-m(m+1)/2+m(1+n+c-b)}}{[q^{-N};\ q]_m[q^{1+c};\ q]_m[q^{1+b-N+n};\ q]_m}.$$

Multiplying (6.12) by $\omega(x;\ -N-\ 1,\ b,\ c,\ d;\ q)$ and $P_z(\mu(x);\ -N-1,\ b,\ c,\ d;\ q)$, summing with respect to x from 0 to N and using the orthogonality relation (2.12) of q-Racah polynomials, we get a bilinear formula for q-Racah polynomials

$$(6.13)\quad \begin{aligned}&\sum_{x=0}^{N} \lambda(x)\, \omega(x;\ -N-1,\ b,\ c,\ d;\ q)P_m(\mu(x);\ -N-1,\ b,\ c,\ d;\ q)\\ &P_z(\mu(x);-N-1,\ b,\ c,\ d;\ q) = \sum_{n=\max(m,z)}^{N} \frac{\mu(m,\ n)\ H(z,\ n)}{\pi(z;-N-1,\ b,\ c,\ d;\ q)}\end{aligned}$$

Similarly other bilinear formulas of q-Racah polynomials may be obtained with the help of connection relations (6.5) and (6.6).

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MATHEMATICS DEPARTMENT, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA T6G2G1