A NOTE ON
THE SOLUTION SETS OF DIFFERENTIAL INCLUSIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. Aronszajn [1] proved that the set S of solutions of the initial value problem

$$(1) \qquad\qquad x' = f(t, x), \ x(0) = x_0,$$

where $x \in \mathbf{R}^n$, $t \in I = [0, T]$ and f is bounded and continuous on $I \times \mathbf{R}^n$, is an R_δ-set in the space $C(I)$ of continuous functions from I into $\mathbf{R}^n$. (Recall that an R_δ-set is defined to be the intersection of a decreasing sequence of compact absolute retracts.) Aronszajn's result implies, among other things, that S is topologically equivalent to a point, a trival fact when solutions of (1) are unique.

The purpose of this note is to extend Aronszajn's theorem to the set of solutions of the differential inclusion

$$(2) \qquad\qquad x' \in F(t, x), \ x(0) = 0,$$

where F is a set-valued function whose values are nonempty, compact, convex subsets of $\mathbf{R}^n$. It is assumed that F is bounded (all values lie in a fixed ball in $\mathbf{R}^n$) and that F is upper semicontinuous ($F^{-1}(K) = \{(t, x) |$ $F(t, x) \cap K \neq \varnothing\}$ is closed for each closed subset K of $\mathbf{R}^n$).

In this note it is shown that by combining the techniques recently developed by Lasry and Robert [5] with a characterization of R_δ-sets due to Hyman [4], the desired result follows rather easily. Both Lasry and Robert and the authors [3] have considered this problem of extending Aronszajn's theorem. In [3] the case $n = 1$ was treated while in [5] Lasry and Robert proved that the solution set of (2) is acyclic (Corollary 2.14). The main point of this paper is that using Lasry and Robert's techniques together with Hyman's characterization the conclusion that S is acyclic in [5] can be strengthened to the statement that S is an R_δ-set.

The approach in both [3] and [5] was to approximate (2) by a control problem

$$(3) \qquad\qquad x' = f_n(t, x, u), \; x(0) = 0,$$

where (3) has, as u ranges over all admissible controllers, a nice solution set $S_n \supset S$, and then take a limit as $n \to \infty$. Lasry and Robert showed that S_n is acyclic and hence $S = \bigcap_n S_n$ is also acyclic. In this note it is shown that S_n is contractible and hence, using Hyman's characterization of R_δ-sets, $S = \bigcap_n S_n$ is an R_δ-set. The construction of (3) in this paper is that of Lasry and Robert.

There are several possible extensions of the problem (2). For example, Lasry and Robert consider differential inclusions with retardations [5, Theorem 2.12] and show that the appropriate solution set is acyclic. The extension of the Theorem given in §3 to this case follows again using Hyman's result. Secondly, the boundedness condition on F can be relaxed somewhat. For instance, boundedness of F can be replaced by the following condition:

(∗) Assume there exist $\alpha, \beta \in L^1(I)$ such that for each $x \in \mathbf{R}^n$, for each $v \in F(t, x)$, $|v| \leqq \alpha(t) + \beta(t) |x|$ a.e. $(t \in I)$.

A standard application of Grönwall's inequality can be used to reduce a problem satisfying (∗) to a bounded situation. Finally, only the autonomous case of (2) will be treated since the time dependent case can be reduced to an autonomous situation by introducing the auxiliary equation $t' = 1$.

2. Background material. In this section the technique of Lasry and Robert is summarized, Hyman's result is stated, and a sketch of a proof of a result due to Yorke [6] is given. Yorke's result on the contractibility of solution sets of differential inclusions shows the applicability of Hyman's theorem to the situation considered here.

Consider the differential inclusion

$$(2a) \qquad\qquad x' \in F(x), \; x(0) = 0,$$

where F an upper semicontinuous multifunction defined on $\mathbf{R}^n$ having as its values nonempty, compact, convex subsets of $\mathbf{R}^n$. Also assume that F is bounded, i.e., there is an $r > 0$ such that, for each $x \in \mathbf{R}^n$, $F(x) \subset B(0, r)$, where $B(0, r)$ denotes the closed ball of radius r about the origin.

Since $F(x) \subset B(0, r)$ for each x, any solution of (2a) must exist on $[0, T]$ and have its values in $B(0, rT)$. To simplify notation, let $B = B(0, rT)$, and let $\mathscr{K}$ denote the set of nonempty, compact, convex subsets of $B(0, r)$.

PROPOSITION 1. (Lasry-Robert, Proposition 2.19 and Remarks 2.21, 2.22). *Let $F: B \to \mathscr{K}$ be upper semicontinuous. Then there exists a sequence (F_n) of upper semicontinuous multifunctions, $F_n: B \to \mathscr{K}$, which satisfy*

for each $x \in B$,

   (i) $F(x) \subset F_n(x)$ *for* $n = 1, 2, \ldots$,

   (ii) $F_{n+1}(x) \subset F_n(x)$ *for* $n = 1, 2, \ldots$,

   (iii) $\bigcap_n F_n(x) = F(x)$, *and*

   (iv) $F_n(x) = \sum_{i=1}^{p} \phi_i(x)C_i$ *where* $C_1, \ldots, C_p \in \mathscr{K}$ *and* $\phi_1, \ldots, \phi_p$ *form a Lipschitz continuous partition of unity of B.*

REMARK. The C'_is, ϕ_i's, and p may depend on n, but this dependency has been suppressed in the notation.

The control problem mentioned in the introduction takes the form

$$(3a) \qquad\qquad x' = \sum_{i=1}^{p} \phi_i(x)u_i, \; x(0) = 0,$$

when $u_i \in C_i$ for $i = 1, 2, \ldots, p$. The admissible class of controllers is the set of measurable functions

$$\mathscr{M}_c = \{u = (u_1, \ldots, u_p) | u \colon I \to C_1 \times \cdots \times C_p\}.$$

$\mathscr{M}_c$ is convex and is compact and metrizable in the weak topology $\sigma(L^1(I, \mathbf{R}^{np}), L^\infty(I, \mathbf{R}^{np}))$.

By Filippov's lemma, every solution of

$$(4) \qquad\qquad x' \in \sum_{i=1}^{p} \phi_i(x)C_i, \; x(0) = 0,$$

is a solution of (3a) for some $u \in \mathscr{M}_c$, and conversely, for each $u \in \mathscr{M}_c$, (3a) has a unique solution $\psi(u)$ which is trivially a solution of (4). The map $\psi \colon \mathscr{M}_c \to C(I)$, defined in the previous sentence is continuous and onto the solution set S_n of (4). Thus $S_n = \psi(\mathscr{M}_c)$ and hence S_n is compact. Clearly S_n is metrizable. Lasry and Robert use the fact that $S_n = \psi(\mathscr{M}_c)$ to deduce that S_n is acyclic since $\mathscr{M}_c$, being convex, is trivially acyclic. If the map ψ were also one-to-one, it would follow that S_n is an absolute retract and it could be directly established that $S = \bigcap_n S_n$ is an R_δ. Unfortunately, it appears difficult to fix up the approximation (3a) in such a way that the map $\psi \colon \mathscr{M}_c \to S_n$ is a homeomorphism in the appropriate topologies. Instead it is shown that S_n is contractible, and then Hyman's characterization of $\mathbf{R}_\delta$'s stated in Proposition 2 below is used.

Recall that a topological space X is contractible if there is a continuous function $h \colon [0, 1] \times X \to X$ and a point $x_0 \in X$ such that, for all $x \in X$, $h(0, x) = x_0$ and $h(1, x) = x$.

PROPOSITION 2. (Hyman). *For a compact metric space X the following are equivalent*:

   (i) *X is the intersection of a decreasing sequence of compact contractible metric spaces; and*

(ii) *X is the intersection of decreasing sequence of compact absolute retracts, i.e., X is an R_δ-set.*

Since it will be shown that $S = \bigcap_n S_n$, where S is the solution set of (2a) and S_n is the solution set of (3a), and since it is known that S is compact metric, Proposition 2 reduces the problem of showing that S is an R_δ to that of showing S_n is contractible. This latter problem seems much more amenable than showing that S_n is an absolute retract, especially for solution sets of equations. In particular, the following special case of a proposition due to Yorke [6] will establish the required contractibility.

PROPOSITION 3. (Yorke, Theorem 1.3). *Let B and $\mathscr{K}$ be as in Proposition 1 and let $G: B \to \mathscr{K}$ be upper semicontinuous. If there exists a Lipschitzian function $g: B \to \mathbf{R}^n$ such that $g(x) \in G(x)$ for all $x \in B$, then the solution set X of $x' \in G(x)$, $x(0) = 0$, is contractible.*

Since Yorke did not publish a proof of his result, a proof of Proposition 3 is outlined. Let g be the Lipschitzian selector for G. Let x_0 be the unique solution of the problem $x' = g(x)$, $x(0) = 0$, on I, and let x be any solution of $x' \in G(x)$, $x(0) = 0$, on I. For each $\tau \in [0, 1)$, let y_τ denote the unique solution of $x' = g(x)$, $y_\tau(\tau T) = x(\tau T)$, for $\tau T \leq t \leq T$. Then define a function z_τ by

$$z_\tau(t) = \begin{cases} x(t) & \text{for } 0 \leq t \leq \tau T \\ y_\tau(t) & \text{for } \tau T \leq t \leq T. \end{cases}$$

For each $\tau \in [0, 1)$, z_τ is a solution of $x' \in G(x)$, $x(0) = 0$. Finally, define $h: [0, 1] \times X \to X$ by

$$h(\tau, x) = \begin{cases} z_\tau & \text{for } \tau \in [0, 1) \\ x & \text{for } \tau = 1. \end{cases}$$

Then h is the desired contraction of X onto x_0. For clearly $h(0, x_0) = x_0$ and $h(1, x) = x$ for each $x \in X$. Also, the continuity of h with respect to (τ, x) follows from the standard continuity theorem for ordinary differential equations (cf. Coppel [2], Theorem 5, page 20).

3. Principal result. Let $F: B \to \mathscr{K}$ be as in Proposition 1.

THEOREM. *The solution set S of $x' \in F(x)$, $x(0) = 0$, on an interval $[0, T]$, is an R_δ-set in $C([0, T])$.*

PROOF. By Proposition 1, there is a sequence (F_n) of upper semicontinuous multifunctions from B to $\mathscr{K}$ which decrease to F and which are of the form $F_n = \sum_{i=1}^p \phi_i C_i$. For fixed $u_i^0 \in C_i$, $i = 1, 2, \ldots, p$, $\sum_{i=1}^p \phi_i u_i^0$ is a Lipschitzian selector for F_n and hence, by Proposition 3, the solution set S_n of $x' \in F_n(x)$, $x(0) = 0$, is contractible. It is known

that S and S_n are compact and metrizable. Also, since $F_{n+1}(x) \subset F_n(x)$, the S_n's are decreasing. Finally, to see that $S = \bigcap_n S_n$, note first that $S \subset S_n$ for each n since $F(x) \subset F_n(x)$ for each x and n. Thus $S \subset \bigcap_n S_n$. Next, suppose $x \in \bigcap_n S_n$. Then, for each n, $x'(t) \in F_n(x(t))$, and hence $x'(t) \in \bigcap_n F_n(x(t)) = F(x(t))$ by Proposition 1 (iii). The theorem now follows from Proposition 2.

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