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ON FACTOR STATES

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1. Introduction. Let **A** be a (complex) C^*-algebra, f a state on **A**. Let $\{\pi_f, H_f, x_f\}$ denote the *GNS* triple corresponding to f. f is a *factor state* if $\pi_f(\mathbf{A})''$ is a factor ($'$ denotes taking the commutant). Traditionally, the set of factor states has played an important role in the integration and disintegration theory of states and representation of C^*-algebras ([**6**], Chapter 8; [**16**], Chapter 3). Recently, in work of Choi-Effros and Connes characterizing separable nuclear C^*-algebras ([**3**], [**4**], [**5**]); see also [**7**]) and work of Anderson and Bunce on the Stone-Weierstrass problem ([**1**]), the set of factor states has also been important. The detailed study of factor states was started by Kadison in [**11**] (see Theorem *A*, p. 306). In view of the above recent work, it therefore seems worthwhile to continue this study, and this paper is thus a contribution in that direction.

In §2 of the present paper, we give a characterization of factor states analogous to the Segal characterization of pure states, and use this to obtain an extension theorem for factor states. In §3 we characterize commutative and elementary C^*-algebras by a condition on their set of factor states, in a way which exhibits these two classes of C^*-algebras as opposite extremes of a common phenomenon. In Section 4 we present a vector state characterization of irreducibility of C^*-algebras which grew out of the investigations of the two preceding sections.

We review and fix our notation for the sequel. If **A** is a C^*-algebra, f a state on **A**, π_f, H_f, and x_f will denote respectively the representation, Hilbert space, and unit cyclic vector arising in the *GNS* construction corresponding to f. $\mathbf{A}_+$ and $\mathbf{A}_+^*$ denote respectively the positive elements of **A** and the positive linear functionals on **A**. We denote unitary equivalence of algebras, representations, and operators by $\sim$. If X is a normed algebra or linear space and α is a cardinal number, $\alpha \cdot X$ denotes the α-fold multiple of X with the standard algebra or normed linear space operations inherited from X. If S is a subset of X, we set Ball $S = \{s \in S: \|s\| \leqq 1\}$ and $S_1 = \{s \in S: \|s\| = 1\}$. If H is a (complex) Hilbert space, $B(H)$ will denote the W^*-algebra of all bounded operators on H and $C(H)$ will denote the C^*-algebra of compact operators on H.

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2. Extensions of factor states. Let $\mathbf{A}$ be a C^*-algebra, and denote by $B(\mathbf{A} \times \mathbf{A})$ the set of all positive semi-definite, conjugate bilinear forms on $\mathbf{A}$. We say that $\varphi \in B(\mathbf{A} \times \mathbf{A})$ is *self-adjoint* if $\varphi(ab, c) = \varphi(b, a^*c)$, for all $a, b, c \in \mathbf{A}$. By applying the *GNS* construction to $\mathbf{A}$ relative to φ, one can easily show that there exists $\rho \in \mathbf{A}_+^*$ such that $\varphi(x, y) = \rho(y^*x) \doteq \varphi_\rho(x, y)$, $x, y \in \mathbf{A}$ if and only if φ is self-adjoint.

Let f be a state on $\mathbf{A}$. For each $a \in \text{Ball } \mathbf{A}_+$, the form defined by $\varphi_{f,a}$: $(x, y) \to f(y^*ax)$, $x, y \in \mathbf{A}$ is an element of $B(\mathbf{A} \times \mathbf{A})$. Let $B_f = \{\varphi_{f,a}: a \in \text{Ball } \mathbf{A}_+\}$ and let S_f denote the set of self-adjoint point-wise limit points of B_f. Set $S_f = \{\rho \in \mathbf{A}_+^*: \varphi_\rho \in S_f\}$. S_f can be described alternately as the set of $\rho \in \mathbf{A}_+^*$ for which there is a net $\{a_\alpha\} \subseteqq \text{Ball } \mathbf{A}_+$ such that $\rho(y^*x) = \lim_\alpha f(y^*a_\alpha x)$ for each $(x, y) \in \mathbf{A} \times \mathbf{A}$. S_f is a wk*-compact, convex subset of $[0, f] = \{\rho \in \mathbf{A}_+^*: \rho \leqq f\}$. Set $Q_f = \{0\} \cup \{\alpha\rho: 0 \neq \rho \in S_f, 0 \leqq \alpha \leqq \|\rho\|^{-1}\}$. Geometrically, Q_f is the union of all line segments starting at 0, ending on the surface of Ball $\mathbf{A}_+^*$, and passing through an element of S_f.

The following is a characterization of factor states which is analogous to the familiar one for pure states (cf. **[6]**, Definition 2.5.2 and Proposition 2.5.5):

THEOREM 2.1. *Let f be a state on* $\mathbf{A}$. *The following are equivalent*:
(a) *f is a factor state,*
(b) $S_f = \{\lambda f: 0 \leqq \lambda \leqq 1\}$, *and*
(c) *f is an extreme point of Q_f.*

PROOF. (a) $\Rightarrow$ (b). Let $\rho \in S_f$. Since $\rho \leqq f$, by **[6]**, Proposition 2.5.1, there is an element $z \in \pi_f(\mathbf{A})'$, $0 \leqq z \leqq 1$, such that $\rho(a) = (\pi_f(a)zx_f, x_f)$, $a \in \mathbf{A}$. We have $\rho(y^*x) = (z\pi_f(x)x_f, \pi_f(y)x_f)$, $x, y \in \mathbf{A}$, and since $\rho \in S_f$, it follows that there is a net $\{a_\alpha\}$ in Ball $\mathbf{A}_+$ such that $\pi_f(a_\alpha) \to z(WOT)$. We conclude that z is in the center of $\pi_f(\mathbf{A})''$, and this latter is a factor, so $z = \lambda$ for some scalar λ between 0 and 1. Thus $\rho = \lambda f$.

(*b*) $\Rightarrow$ (*c*). In this case, $S_f = Q_f$, and f therefore is clearly an extreme point of Q_f.

(*c*) $\Rightarrow$ (*a*). Suppose $\pi_f(\mathbf{A})''$ contains a nontrivial central projection P. Then for all $a \in \mathbf{A}$,

$$f(a) = (\pi_f(a)x_f, x_f) = (\pi_f(a)Px_f, x_f) + (\pi_f(a)(I - P)x_f, x_f). \tag{2.1}$$

If we define ρ_1: $a \to (\pi_f(a)Px_f, x_f)$, ρ_2: $a \to (\pi_f(a)(I - P)x_f, x_f)$, $a \in \mathbf{A}$, then by the Kaplansky density theorem $\rho_1, \rho_2 \in S_f$. Thus by (2.1) and the nontriviality of P, we can write

$$f = \|Px_f\|^2(\rho_1/\|Px_f\|^2) + \|(I - P)x_f\|^2(\rho_2/\|(I - P)x_f\|^2),$$

a nontrivial convex combination of elements of Q_f, which is not possible. Thus P is trivial, and f is a factor state.

One of the most important applications of the Segal characterization of pure states deals with pure extensions of pure states. In fact, if $\mathbf{A}$ is a C^*-algebra and S is a self-adjoint linear *subspace* of $\mathbf{A}$, if f is an extreme point of the set of positive linear functionals on S of norm not exceeding 1, and if $E_f = \{\rho \in \text{Ball } \mathbf{A}_+^*: \rho \text{ extends } f\}$, then the Segal characterization quickly implies that each extreme point of E_f is a pure state of $\mathbf{A}$ ([**6**], Lemma 2.10.1). In particular, if B is a C^*- subalgebra of $\mathbf{A}$ and f is a pure state of B, then f extends to a pure state of $\mathbf{A}$. We will show next that if B is hereditary and f is a factor state of B, then Theorem 2.1 can be used to obtain the analogous extension theorem. Later we will indicate an alternate proof of the same fact.

COROLLARY 2.2. *Let* $\mathbf{A}$ *be a* C^**-algebra,* B *an hereditary* C^**-subalgebra of* $\mathbf{A}$. *Let* f *be a factor state of* B, *and set* $E_f = \{\rho \in \text{Ball } \mathbf{A}_+^*: \rho \textit{ extends } f\}$. *Then each extreme point of* E_f *is a factor state of* $\mathbf{A}$. *In particular,* f *extends to a factor state of* $\mathbf{A}$.

PROOF. Since E_f is nonempty, wk*-compact, and convex, it has extreme points. Let g be such an extreme point. Define

$$T = \{\rho \in \text{Ball } \mathbf{A}_+^*: \exists \lambda \in [0, 1] \text{ such that } \rho|_B = \lambda f\}.$$

CLAIM 1. g is an extreme point of T. Let $g = \alpha\rho_1 + (1 - \alpha)\rho_2$ be a convex combination of elements of T. Restricting to B, we get $f = (\alpha\lambda_1 + (1 - \alpha)\lambda_2)f$, λ_1, $\lambda_2 \in [0, 1]$, and so $1 = \alpha\lambda_1 + (1 - \alpha)\lambda_2$, whence $\lambda_1 = \lambda_2 = 1$. Thus $\rho_1, \rho_2 \in E_f$. Since g is an extreme point for E_f, we conclude that $\rho_1 = \rho_2 = g$.

Now, let S_g and Q_g be defined relative to g as in Theorem 2.1.

CLAIM 2. $S_g \subseteqq T$. Let $\rho \in S_g$. There exists a net $\{a_\alpha: \alpha \in \mathfrak{A}\}$ in Ball $\mathbf{A}_+$ such that $\rho(y^*x) = \lim_\alpha g(y^*a_\alpha x)$ for each $(x, y) \in \mathbf{A} \times \mathbf{A}$. Let $\{u_i: i \in I\}$ be an approximate identity for B. Since B is hereditary, the set $\{u_i a_\alpha u_i: (\alpha, i) \in \mathfrak{A} \times I\}$ is a net in Ball B_+ relative to the product ordering on $\mathfrak{A} \times I$. Now let $b_1, b_2 \in B$. Then

$$\begin{aligned}\rho(b_2^*b_1) &= \lim_{\alpha} g(b_2^*a_\alpha b_1) = \lim_{\alpha, i} g((b_2^*u_i)a_\alpha(u_ib_1)) \\ &= \lim_{\alpha, i} f(b_2^*(u_ia_\alpha u_i)b_1).\end{aligned}$$

Thus $\rho|_B \in S_f$, and so by Theorem 2.1 (*b*), $\rho|_B = \lambda f$ for some $\lambda \in [0, 1]$.

To finish the proof, it suffices by Theorem 2.1 (*c*) to show that g is an extreme point of Q_g. Hence, suppose

$$g = \alpha\alpha_1\rho_1 + (1 - \alpha)\alpha_2\rho_2, \tag{2.2}$$

where $\rho_i \in S_g$, $0 \leqq \alpha_i \leqq \|\rho_i\|^{-1}$, $i = 1, 2$, $0 < \alpha < 1$. (We assume here that ρ_1 and ρ_2 are both nonzero. If either one is zero, use the argument to follow, appropriately modified.) (2.2) implies $1 = \alpha\alpha_1\|\rho_1\| + (1-\alpha)\alpha_2\|\rho_2\|$, whence $\alpha_i = \|\rho_i\|^{-1}$, $i = 1, 2$. We have $\rho_i|_B = \lambda_i f$, $\lambda_i \in [0, 1]$, $i = 1, 2$ by Claim 2. Thus $\alpha_i\rho_i|_B = \alpha_i\lambda_i f = \lambda_i\|\rho_i\|^{-1}f$, and so $\alpha_i\rho_i \in T$ since $\lambda_i \leqq \|\rho_i\|$, $i = 1, 2$. By Claim 1, $\alpha_1\rho_1 = \alpha_2\rho_2 = g$, and we are done.

In view of the previous results, the extreme points of E_f appear to be good candidates for factorial extensions of factor states. The next proposition gives a necessary and sufficient condition for an extreme point of E_f to be a factor state. It is formulated in terms of a condition depending only on the subalgebra, and gives an "algebraic" verification of and counterpart to the "geometric" argument used in the proof of Corollary 2.2.

PROPOSITION 2.3. *Let* $\mathbf{A}$ *be a* C^**-algebra,* B *a* C^**-subalgebra of* $\mathbf{A}$*,* f *a factor state of* B*. An extreme point* g *of* E_f *is a factor state of* $\mathbf{A}$ *if and only if* QPQ *is in* $(\pi_g(B)Q)''$ *for each central projection* P *of* $\pi_g(\mathbf{A})''$*, where* Q *is the projection of* H_g *onto the subspace* $[\pi_g(B)x_g]$.

PROOF. With no loss of generality, we suppose that $\mathbf{A}$ contains an identity e and $e \in B$. The "only if" part is clear. Suppose P is a central projection in $\pi_g(A)''$. Let $K = [\pi_g(B)x_g]$. If we consider H_g as the external direct sum $K \oplus K^\perp$, we can from the invariance of K for $\pi_g(B)$ write $\pi_g(b)$ for $b \in B$ as a 2×2 operator matrix of the form

$$\begin{bmatrix} \pi_g(b)Q & 0 \\ 0 & \pi_g(b)(I-Q) \end{bmatrix}.$$

We have

$$P = \begin{bmatrix} U & V \\ V^* & W \end{bmatrix}, \tag{2.3}$$

where

$$0 \leqq U, W \leqq I,\ U^2 + VV^* = U,\ UV + VW = V,\ W^2 + V^*V = W. \tag{2.4}$$

Now P commutes with $\pi_g(B)$, and so $U = QPQ \in (\pi_g(B)Q)'$ (commutant taken with respect to $B(K)$). We conclude by hypothesis that U is in the center of $(\pi_g(B)Q)''$. Since $(\pi_g|_B)|_K \sim \pi_f$ and f is a factor state of B, it follows that $U = \lambda I_K$ for some nonnegative scalar λ.

We evaluate λ as follows: from (2.3), $Px_g = Ux_g \oplus V^*x_g = \lambda x_g \oplus V^*x_g$, and from (2.4), $VV^* = \lambda(1-\lambda)I_K$. Thus $\|V^*x_g\|^2 = (VV^*x_g, x_g) = \lambda(1-\lambda)$, whence $\|Px_g\|^2 = \|\lambda x_g\|^2 + \|V^*x_g\|^2 = \lambda^2 + \lambda(1-\lambda) = \lambda$. Since $x_g \in K$, we conclude that

$$QPx_g = \|Px_g\|^2 x_g. \tag{2.5}$$

Suppose P is nontrivial. By (2.5), $x_g - \|Px_g\|^{-2}Px_g \perp K$, and so for each $b \in B$,

$$\begin{aligned} 0 &= (x_g - \|Px_g\|^{-2}Px_g, \pi_g(b^*)x_g) \\ &= (\pi_g(b)x_g, x_g) - (\pi_g(b)y_g, y_g), \text{ where } y_g = Px_g/\|Px_g\|, \\ &= f(b) - (\pi_g(b)y_g, y_g), \end{aligned}$$

Thus if we define g_1: $a \to (\pi_g(a)y_g, y_g)$, $a \in \mathbf{A}$, then $g_1 \in E_f$. A similar calculation shows that if we define g_2: $a \to (\pi_g(a)z_g, z_g)$, where $z_g = (I - P)x_g/\|(I - P)x_g\|$, then $g_2 \in E_f$. Arguing as in the proof of Theorem 2.1, it follows that $g = \|Px_g\|^2 g_1 + \|(I - P)x_g\|^2 g_2$, a nontrivial convex combination of elements of E_f. This being impossible, we conclude that P is trivial and g is a factor state.

REMARK. The author does not know of a C^*-algebra $\mathbf{A}$, a C^*-subalgebra B of $\mathbf{A}$, and a factor state of B which does not extend to a factor state of $\mathbf{A}$.

3. Degeneration of states. Let $\mathbf{A}$ be a C^*-algebra, and denote respectively by P, F, and S and set of pure states, factor states, and states of $\mathbf{A}$. We have $P \subseteqq F \subseteqq S$. In this section, we will characterize when equality holds at both ends of this chain of inclusions. We begin by studying factor states of C^*-algebras of compact operators. The next few results are more general than what will be needed in the sequel, but they perhaps have some independent interest.

The next few lemmas make use of Mackey's concept of disjointness of representations [12]. Two representations π_1 and π_2 of a C^*-algebra $\mathbf{A}$ are *disjoint* if they have no non-zero unitarily equivalent subrepresentations. We denote this by $\pi_1 \between \pi_2$. We say that π_1 and π_2 are *quasi-equivalent*, denoted $\pi_1 \sim_q \pi_2$, if there exists an isomorphism ρ of $\pi_1(\mathbf{A})''$ onto $\pi_2(\mathbf{A})''$ such that $\rho(\pi_1(a)) = \pi_2(a)$, for all $a \in \mathbf{A}$. By a *factor representation* of a C^*-algebra, we of course mean a representation π for which $\pi(\mathbf{A})''$ is a factor.

LEMMA 3.1. *Let π_1 and π_2 be representations of a C^*-algebra $\mathbf{A}$. Then there exist central projections P_i in $\pi_i(\mathbf{A})''$, $i = 1, 2$ such that $\pi_1|_{P_1} \sim_q \pi_2|_{P_2}$ and $\pi_1|_{1-P_1} \between \pi_2|_{1-P_2}$.*

PROOF. Given any representation π of $\mathbf{A}$ on H, π^{**} is a $\sigma(\mathbf{A}^{**}, \mathbf{A}^*)$-ultraweakly continuous extension of π to a representation of $\mathbf{A}^{**}$, with $\pi^{**}(\mathbf{A}^{**}) \subseteqq \pi(\mathbf{A})''$. Thus there exists a central projection z in $\mathbf{A}^{**}$ such that $\ker \pi^{**} = z\mathbf{A}^{**}$. We set $s(\pi) = 1 - z$.

Now, let P be the supremum of all central projections c in $\mathbf{A}^{**}$ such that $c \leqq s(\pi_i)$, $i = 1, 2$. Let $F_i = s(\pi_i) - P$, $i = 1, 2$. Since P, $s(\pi_i)$, and F_i, $i = 1, 2$ are all projections in an abelian W^*-algebra (the center of $\mathbf{A}^{**}$), they can be considered as characteristic functions of measurable sets ([16], Proposition 1.18.1), and we hence deduce that $F_1 \perp F_2$. Let $P_i = \pi_i^{**}(P)$, $i = 1, 2$. Then P_i is a central projection in $\pi_i(\mathbf{A})''$, $i = 1, 2$, and we have $s(\pi_i|_{P_i}) = P$, $s(\pi_i|_{1-P_i}) = F_i$, $i = 1, 2$. Since $F_1 \perp F_2$, it follows that $\pi_1|_{1-P_1} \between \pi_2|_{1-P_2}$ by **[18]**, Theorem 3.8.11. and since $s(\pi_1|_{P_1}) = s(\pi_2|_{P_2})$, we have $\pi_1|_{P_1} \sim_q (\mathrm{id}|_{\mathbf{A}})|_P \sim_q \pi_2|_{P_2}$ by **[18]**, Theorem 3.8.2.

The following corollary is well known (**[6]**, Corollary 5.3.6).

COROLLARY 3.2. *Two factor representations of a C^*-algebra are either quasi-equivalent or disjoint.*

LEMMA 3.3. *Let $\pi = \oplus\{\pi_\alpha: \alpha \in \mathfrak{A}\}$ be a direct sum of nondegenerate representations of a C^*-algebra $\mathbf{A}$. The following are equivalent:*

(i) *π is a factor representation,*

(ii) *π_α is a factor representation for each $\alpha \in \mathfrak{A}$, and for each $(\alpha_1, \alpha_2) \in \mathfrak{A} \times \mathfrak{A}$, if ρ_i is a nonzero subrepresentation of π_{α_i}, $i = 1, 2$, then ρ_1 and ρ_2 are not disjoint, and*

(iii) *π_α is a factor representation for each $\alpha \in \mathfrak{A}$, and for each $(\alpha_1, \alpha_2) \in \mathfrak{A} \times \mathfrak{A}$, π_{α_1} and π_{α_2} are quasi-equivalent.*

PROOF. Let H_α denote the representation space of π_α, P_α the projection of $H = \oplus_\alpha H_\alpha$ onto H_α.

(i) $\Rightarrow$ (ii). Let E be a projection in $\pi(\mathbf{A})'$. We claim that $\pi|_E$ is a factor representation. By the double commutant theorem, $\pi(\mathbf{A})E$ is dense in $\pi(\mathbf{A})''E$ with respect to the weak operator topology, and so $(\pi(\mathbf{A})E)'' = \pi(\mathbf{A})''E$ (the former commutant taken in $B(E(H))$). Now the center of $E\pi(\mathbf{A})'E$ = (center of $\pi(\mathbf{A})')E$ (**[17]**, Proposition 7), and so $E\pi(\mathbf{A})'E$ is a factor, since $\pi(\mathbf{A})'$ is a factor. Thus $(E\pi(\mathbf{A})'E)' = \pi(\mathbf{A})''E$ (**[17]**, Corollary 5) $= (\pi(\mathbf{A})E)''$ is a factor. This verifies the claim. Thus, setting $E = P_\alpha$ for $\alpha \in \mathfrak{A}$, we deduce that π_α is a factor representation for each $\alpha \in \mathfrak{A}$.

Let ρ_i and π_{α_i}, $i = 1, 2$ be as in (ii). We have $\rho_i = \pi_{\alpha_i}|_{E_i}$, E_i a projection in $\pi_{\alpha_i}(\mathbf{A})'$, $i = 1, 2$. Thus $E_1 \oplus E_2 \in \pi(\mathbf{A})'$, and so by the preceding claim, $\rho_1 \oplus \rho_2 = \pi|_{E_1 \oplus E_2}$ is a factor representation. By **[6]**, Corollary 5.2.5. ρ_1 and ρ_2 are not disjoint.

(ii) $\Rightarrow$ (iii). This is an immediate consequence of Corollary 3.2.

(iii) $\Rightarrow$ (i). Let P be a nonzero central projection in $\pi(\mathbf{A})''$. Then $E_\alpha = PP_\alpha$ is a central projection in $\pi_\alpha(\mathbf{A})''$ for all α, and so $E_\alpha = 0$ or P_α, for all α, since $\pi_\alpha(\mathbf{A})''$ is a factor.

Suppose $E_{\alpha_0} = 0$ for some $\alpha_0 \in \mathfrak{A}$. Fix $\alpha \neq \alpha_0$ in $\mathfrak{A}$, and let $\rho = \rho_{\alpha,\alpha_0}$ be the isomorphism of $\pi_{\alpha_0}(\mathbf{A})''$ onto $\pi_\alpha(\mathbf{A})''$ such that $\rho(\pi_{\alpha_0}(a)) = \pi_\alpha(a)$, for all $a \in \mathbf{A}$.

By the double commutant theorem, choose a net $\{a_\gamma\}$ in $\mathbf{A}$ such that $\{\pi(a_\gamma)\}$ approaches P ultraweakly on H. Then $\{\pi_i(a_\gamma)\}$ approaches E_i ultraweakly on H_i, $i = \alpha, \alpha_0$. Since ρ is ultraweakly continuous (**[16]**, Corollary 4.1.23), it hence follows that

$$0 = \rho(E_{\alpha_0}) = \lim_\gamma \rho(\pi_{\alpha_0}(a_\gamma)) = \lim_\gamma \pi_\alpha(a_\gamma) = E_\alpha.$$

Since $\alpha \neq \alpha_0$ is arbitrary, we conclude that $P = \oplus_\alpha E_\alpha = 0$, contrary to assumption. Hence $E_\alpha = P_\alpha$, for all α, and therefore $P = \oplus_\alpha E_\alpha = \oplus_\alpha P_\alpha$ = identity on H.

COROLLARY 3.4. *Let $\pi = \oplus_\alpha \pi_\alpha$ be a direct sum of irreducible representations of a C^*-algebra. π is a factor representation if and only if all representations π_α are unitarily equivalent.*

PROOF. Any two irreducible representations of a C^*-algebra are either unitarily equivalent or disjoint. The corollary now follows from Lemma 3.3.

PROPOSITION 3.5. *Let B be a C^*-algebra of compact operators on a Hilbert space H. Let f be a state on B. The following are equivalent:*

(a) *f is a factor state, and*

(b) *there is a nonzero minimal projection e in B, a nonzero vector ξ in the range of e, an orthonormal sequence $\{x_n\}$ of vectors in $[B\xi]$, and a sequence $\{\lambda_n\}$ of nonnegative real numbers with $\sum_n \lambda_n = 1$ such that $f(b) = \sum_n \lambda_n (bx_n, x_n)$, $b \in B$.*

PROOF. (a) $\Rightarrow$ (b). By [2], Theorem 1.4.4, there is a family $\{\pi_\alpha : \alpha \in \mathfrak{A}\}$ of irreducible representations of B such that $\pi_f = \oplus_\alpha \pi_\alpha$. By Corollary 3.4, all π_α's are unitarily equivalent. Thus, π_f is unitarily equivalent to card $\mathfrak{A} \cdot \pi_{\alpha_0}$, α_0 a fixed element in $\mathfrak{A}$. By [2], Theorem 1.4.4, there exists a nonzero minimal projection e in B and a nonzero vector ξ in range of e such that $\pi_{\alpha_0} \sim (\mathrm{id}|_B)|_{[B\xi]}$. Thus $\pi_f \sim$ card $\mathfrak{A} \cdot (\mathrm{id}|_B)|_{[B\xi]}$. Let U be the unitary transformation implementing this equivalence, $x = Ux_f \in$ card $\mathfrak{A} \cdot [B\xi]$. Then for all $b \in B$, with $x = \oplus_n y_n$, $\{y_n\} \subseteqq [B\xi]$,

$$\begin{aligned} f(b) &= (\pi_f(b)x_f, x_f) \\ &= (\text{card } \mathfrak{A} \cdot (\mathrm{id}|_B)|_{[B\xi]}(b)x, x) \\ &= ((\bigoplus_{\alpha \in \mathfrak{A}} b)(x), x) \\ &= \sum_n (by_n, y_n). \end{aligned}$$

Since $\{y_n\} \subseteqq [B\xi]$, we conclude that $f(b) = f(PbP)$, for all $b \in B$, where P is the projection of H onto $[B\xi]$. By [2], Proposition 1.4.3, $BP = C([B\xi])$, and the existence of an orthonormal sequence $\{x_n\}$ and nonnegative numbers $\{\lambda_n\}$ with the desired properties is thus guaranteed by [6], Corollary 4.1.3.

(b) $\Rightarrow$ (a). If f is of the indicated form, it is easy to see that π_f is unitarily equivalent to $\sigma = \oplus_n (\mathrm{id}|_B)|_{[Bx_n]} = \aleph_0 \cdot (\mathrm{id}|_B)|_{[B\xi]}$ restricted to the subspace $E = [(bx_1, bx_2, \ldots) : b \in B]$. Since $B|_{[B\xi]} = C([B\xi])$ and $\{x_n\}$ is an orthonormal set in $[B\xi]$, it follows that $E = \aleph_0 \cdot [B\xi]$, and so $\pi_f \sim \sigma$. By [2], Proposition 1.4.3, $(\mathrm{id}|_B)|_{[B\xi]}$ is irreducible on B, whence by Corollary 3.4, σ is a factor representation, and so therefore is π_f.

The following result refines Corollary 4.1.4 of **[6]**.

COROLLARY 3.6. *Let B be a C^*-subalgebra of compact operators on H. Let f be a state on B. The following are equivalent*:

(a) *f is pure, and*

(b) *there is a nonzero minimal projection e of B, a nonzero vector ξ in the range of e, and a unit vector $x \in [B\xi]$ such that $f(b) = (bx, x)$, for all $b \in B$.*

COROLLARY 3.7. *Every state on $C(H)$ is a factor state.*

PROOF. This is immediate from Proposition 3.5 and **[6]**, Corollary 4.1.

A C^*-algebra $\mathbf{A}$ is *elementary* if it is isomorphic to $C(H)$ for some Hilbert space H. Recalling our notations at the beginning of this section, it follows by Corollary 3.7 that $F = S$ if $\mathbf{A}$ is elementary. If $\mathbf{A}$ is commutative, then it is easy to see that $F = P$. The main result of this section asserts that both of these implications have a converse.

Before we prove this, some terminology needs explaining. The statement "$\mathbf{A}$ has only one irreducible representation" should be interpreted to mean $\mathbf{A}$ has only one unitary equivalence class of irreducible representations. Also, by an ideal of a C^*-algebra we always mean a uniformly closed, two-sided ideal.

THEOREM 3.8. *Let $\mathbf{A}$ be a C^*-algebra.*

(a) *Every factor state of $\mathbf{A}$ is pure if and only if $\mathbf{A}$ is commutative.*

(b) *Every state of $\mathbf{A}$ is factorial if and only if $\mathbf{A}$ is elementary.*

PROOF. (a) Suppose $\mathbf{A}$ is not commutative. Then $\mathbf{A}$ is not 1-homogeneous, so there exists an irreducible representation ρ of $\mathbf{A}$ such that the dimension of the representation space H_ρ of ρ exceeds 1. Let $\xi_1, \xi_2 \in H_\rho$ be linearly independent vectors with $\|\xi_1\|^2 + \|\xi_2\|^2 = 1$. Set $\sigma_i(a) = (\rho(a)\xi_i, \xi_i)$, $a \in \mathbf{A}$, $i = 1, 2$. Let $f = \sigma_1 + \sigma_2$. f is a state on $\mathbf{A}$. We have for each $a \in \mathbf{A}$,

$$\begin{aligned}(\pi_f(a)\xi_f, \xi_f) &= \sigma_1(a) + \sigma_2(a) \\ &= (\rho(a)\xi_1, \xi_1) + (\rho(a)\xi_2, \xi_2) \\ &= (\rho \oplus \rho(a)(\xi_1, \xi_2), (\xi_1, \xi_2)).\end{aligned}$$

By Kadison's transitivity theorem (**[10]**, Theorem 1), (ξ_1, ξ_2) is cyclic for $\rho \oplus \rho$, and so $\pi_f \sim \rho \oplus \rho$. Therefore by Corollary 3.4, f is factorial. Thus to establish (a), it suffices to show that f is not pure. Suppose f is pure. Since $f = \|\xi_1\|^2(\sigma_1/\|\xi_1\|^2) + \|\xi_2\|^2(\sigma_2/\|\xi_2\|^2)$, we have

$$f = \sigma_1/\|\xi_1\|^2 = \sigma_2/\|\xi_2\|^2. \tag{3.2}$$

By the Kadison transitivity theorem, we can find an $a \in \mathbf{A}$ such that $\rho(a)\xi_1 = \xi_1$, $\rho(a)\xi_2 = 0$, whence $\sigma_1(a) = \|\xi_1\|^2 \neq 0$ and $\sigma_2(a) = 0$, contradicting (3.2). Thus f is not pure.

(b) Suppose every state of $\mathbf{A}$ is factorial. We first prove a simple lemma.

LEMMA 3.9. *Let* $\mathbf{A}$ *be a* C^**-algebra with only one irreducible representation. Then* $\mathbf{A}$ *is simple.*

PROOF. Let I be a nonzero ideal in $\mathbf{A}$, and suppose $I \neq \mathbf{A}$, so that $\mathbf{A}/I \neq (0)$. Let ρ_1 and ρ_2 be nonzero irreducible representations of I and $\mathbf{A}/I$, respectively. Let π_1 be the unique extension of ρ_1 to an irreducible representation of $\mathbf{A}$. If $\pi: \mathbf{A} \to \mathbf{A}/I$ is the quotient map, then $\pi_2 = \pi \circ \rho_2$ is an irreducible representation of $\mathbf{A}$. Thus $\pi_1 \sim \pi_2$. But $I \subseteqq \ker \pi_2$, $I \nsubseteq \ker \pi_1$, and so $\ker \pi_1 \neq \ker \pi_2$, a contradiction.

Let π_1 and π_2 be nonzero cyclic representations of $\mathbf{A}$. We claim that π_1 and π_2 are not disjoint. Suppose not. They by Lemma 3 of **[9]**, $(\pi_1 \oplus \pi_2)(\mathbf{A})'' = \pi_1(\mathbf{A})'' \oplus \pi_2(\mathbf{A})''$. We assert that $\pi_1 \oplus \pi_2$ is a cyclic representation of $\mathbf{A}$. Let ξ_i be a cyclic vector for π_i, $i = 1, 2$. Then

$$\begin{aligned}
[(\pi_1 \oplus \pi_2)(\mathbf{A})(\xi_1, \xi_2)] &= [(\pi_1 \oplus \pi_2)(\mathbf{A})''(\xi_1, \xi_2)] \\
&= [\pi_1(\mathbf{A})''\xi_1] \oplus [\pi_2(\mathbf{A})''\xi_2] \\
&= [\pi_1(\mathbf{A})\xi_1] \oplus [\pi_2(\mathbf{A})\xi_2] \\
&= H_1 \oplus H_2,
\end{aligned}$$

where H_i is the representation space of π_i, $i = 1, 2$, proving the assertion. Thus there is a state σ on $\mathbf{A}$ such that $\pi_1 \oplus \pi_2 \sim \pi_\sigma$. We conclude that $\pi_1 \oplus \pi_2$ is a factor representation. On the other hand $(\pi_1 \oplus \pi_2)(\mathbf{A})''$ contains the nontrivial central projection $I_{H_1} \oplus 0$. This contradiction verifies the claim. It follows from Corollary 3.2 that $\pi_1 \sim_q \pi_2$.

Let π be an arbitrary, nondegenerate representation of $\mathbf{A}$, and let ρ be a fixed irreducible representation of $\mathbf{A}$. We can write π as a direct sum of cyclic representations $\{\pi_\alpha: \alpha \in \mathfrak{A}\}$, and we deduce from the previous paragraph that $\pi_\alpha \sim_q \rho$, for all $\alpha \in \mathfrak{A}$. If φ_α is the isomorphism of $\pi_\alpha(\mathbf{A})''$ onto $\rho(\mathbf{A})'' = B(H_\rho)$ such that $\varphi_\alpha(\pi_\alpha(a)) = \rho(a)$, for all $a \in \mathbf{A}$, it follows from the automatic ultraweak continuity of φ_α that $\oplus_\alpha \varphi_\alpha$ extends to an isomorphism of $\pi(\mathbf{A})'' = ((\oplus_\alpha \pi_\alpha)(\mathbf{A}))''$ onto card $\mathfrak{A} \cdot B(H_\rho)$. We conclude that π is type I, and therefore by the arbitrariness of π, $\mathbf{A}$ is type I. By **[15]**, Theorem 2, $\mathbf{A}$ is *GCR*.

Let ρ_1 and ρ_2 be irreducible representations of $\mathbf{A}$. Then $\rho_1 \sim_q \rho_2$, and since the isomorphism implementing this quasi-equivalence is a bijection of $B(H_{\rho_1})$ onto $B(H_{\rho_2})$, it is unitarily implemented, and so $\rho_1 \sim \rho_2$. By Lemma 3.9, $\mathbf{A}$ is simple. Since every nonzero *GCR* algebra contains a nonzero *CCR* ideal (**[6]**, Proposition 4.3.4), $\mathbf{A}$ is either zero or simple and *CCR*, hence elementary.

REMARK. An old question of Naimark **[13]** (see also **[16]**, Remark 1, p. 236) asks if a C^*-algebra with only one irreducible representation must necessarily be elementary. If the C^*-algebra is separable, Rosenberg [14] has answered this question affirmatively. This is a special case of Glimm's "kernel" characterization of separable type I C^*-algebras ([8], condition ($a6$) of Theorem 1). Can the factor state characterization of elementary C^*-algebras in Theorem 3.8 be used to obtain affirmative answers to Naimark's question in nonseparable cases?

4. A vector state characterization of irreducibility. Let H be a Hilbert space, and let $\mathbf{A}$ be a C^*-subalgebra of $B(H)$ which contains the identity operator I on H. $\mathbf{A}$ is *locally irreducible* if $\mathbf{A}$ acts irreducibly on $[\mathbf{A}x]$ for each $x \in H_1$. We will show below that, modulo a degeneracy, local irreducibility implies irreducibility.

In the following, if $x \in H_1$, ω_x will denote the vector state $T \to (Tx, x)$, $T \in B(H)$.

THEOREM 4.1. *Let* $\mathbf{A}$ *be a* C^**-subalgebra of* $B(H)$ *which contains the identity operator* I *on* H. *The following are equivalent*:

(a) *every vector state is pure on* $\mathbf{A}$,

(b) $\mathbf{A}$ *is locally irreducible, and*

(c) *either* $\mathbf{A}$ *is irreducible on* H *or* $\mathbf{A} = \{\lambda I: \lambda \in \mathbf{C}\}$.

PROOF. (a) $\Leftrightarrow$ (b). Let $x \in H_1$. We have $(\mathrm{id}|_A)|_{[Ax]} \sim \pi_{\omega_x}$ and so by [6], Proposition 2.5.4, $\mathbf{A}$ acts irreducibly on $[\mathbf{A}x]$ if and only if ω_x is pure on $\mathbf{A}$.

(c) $\Rightarrow$ (a). If $\mathbf{A} = \{\lambda I: \lambda \in \mathbf{C}\}$, (a) is clear. If $\mathbf{A}$ is irreducible, $\mathrm{id}|_{\mathbf{A}}$ is an irreducible representation of $\mathbf{A}$. If $x \in H_1$, then $\mathrm{id}|_{\mathbf{A}} = (\mathrm{id}|_{\mathbf{A}})|_{[\mathbf{A}x]} \sim \pi_{\omega_x}$, and so π_{ω_x} is an irreducible representation. Thus ω_x is pure on $\mathbf{A}$.

(a) $\Rightarrow$ (c). Let $z \in H_1$, and choose a maximal family $\{x_\alpha: \alpha \in \mathfrak{A}\}$ of unit vectors which contains z and for which $[\mathbf{A}x_\alpha] \perp [\mathbf{A}x_\beta]$ if $\alpha \neq \beta$. Let $E_\alpha = [\mathbf{A}x_\alpha]$. By maximality, $\oplus_\alpha E_\alpha = I$. Thus $\mathrm{id}|_{\mathbf{A}} = \oplus_\alpha (\mathrm{id}|_{\mathbf{A}})|_{E_\alpha}$. We claim that $(\mathrm{id}|_{\mathbf{A}})|_{E_\alpha} \sim (\mathrm{id}|_{\mathbf{A}})|_{E_\beta}$ for $\alpha \neq \beta$. Let $x = (x_\alpha + x_\beta)/\sqrt{2}$ Since $X_\alpha \perp E_\beta$ and E_α, $E_\beta \in \mathbf{A}'$, $\omega_x = \omega_{x_\alpha/\sqrt{2}} + \omega_{x_\beta/\sqrt{2}}$ on $\mathbf{A}$. Thus ω_x dominates $\omega_{x_\alpha/\sqrt{2}}$ and $\omega_{x_\beta/\sqrt{2}}$ on $\mathbf{A}$, and since ω_x is pure on $\mathbf{A}$, we conclude by Proposition 2.5.5 of [6] that there exists $\lambda_i \in [0, 1]$ such that $\lambda_i \omega_x = \omega_{x_i/\sqrt{2}}$ on $\mathbf{A}$, $i = \alpha, \beta$. Evaluating at $I \in \mathbf{A}$, we obtain $\lambda_\alpha = \lambda_\beta = 1/2$, whence $\omega_{x_\alpha} = \omega_{x_\beta}$ on $\mathbf{A}$, and it follows that $(\mathrm{id}|_{\mathbf{A}})|_{E_\alpha} \sim (\mathrm{id}|_{\mathbf{A}})|_{E_\beta}$. Thus if we fix an $\alpha_0 \in \mathfrak{A}$ and set $H_0 = E_{\alpha_0}$, then $\mathbf{A} \sim \text{card } \mathfrak{A} \cdot \mathbf{A}|_{H_0}$. Thus we may identify $\mathbf{A}$ with Card $\mathfrak{A} \cdot \mathbf{A}|_{H_0}$ acting on $K = \text{card } \mathfrak{A} \cdot H_0$.

Suppose $\mathbf{A} \neq \{\lambda I: \lambda \in \mathbf{C}\}$. Then $\dim H_0 \geqq 2$. Suppose Card $\mathfrak{A} \geqq 2$. Let $\{x_1, x_2\}$ be linearly independent vectors in H_0 with $\|x_1\|^2 + \|x_2\|^2 = 1$. Let y be a unit vector in K with coordinates x_1, x_2 and all others zero. Since $\mathbf{A}$ and $\mathbf{A}|_{H_0}$ are *-isomorphic under the map $\oplus_\alpha a \to a$, $a \in A|_{H_0}$ and ω_y is pure on $\mathbf{A}$, the linear functional f defined by f: $a \to (ax_1, x_1) + (ax_2, x_2)$, $a \in \mathbf{A}|_{H_0}$ is pure on $\mathbf{A}|_{H_0}$. Since f dominates ω_{x_i}, $i = 1, 2$, we conclude as before that $\omega_{x_1/\|x_1\|} = \omega_{x_2/\|x_2\|}$ on $\mathbf{A}|_{H_0}$. By the equivalence of (a) and (b), $A|_{H_0}$ is an irreducible C^*-subalgebra of $B(H_0)$. Thus by Kadison's transitivity theorem ([**10**], Theorem 1), we can find an $a \in A|_{H_0}$ such that $ax_1 = x_1$ and $ax_2 = 0$. It follows that $1 = \omega_{x_1/\|x_1\|}(a) = \omega_{x_2/\|x_2\|}(a) = 0$, a contradiction. We conclude that $\mathfrak{A}$ is a singleton, whence $x_{\alpha_0} = z$ and $[Az] = [Ax_{\alpha_0}] = H$. Since $z \in H_1$ is arbitrary, this shows that **A** acts irreducibly on H.

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