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CONDITIONS FOR COUNTABILITY OF THE SPECTRUM OF A SEPARABLE C*-ALGEBRA

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Most of the notation will be taken from [**4**]. A will always denote a separable C^*-algebra with spectrum $\hat{A}$ [**4**; p. 9]. We shall characterize those A with countable spectrum. Other characterizations can be found in [**1**; p. 292] and [**7**]. The theorem in this paper answers a question raised by Vaughn Jones.

If f is any state of A, we may define a semi-norm $\|\cdot\|_f$ on A by $\|a\|_f = f(a^*a)^{1/2}$. The state f is faithful exactly when $\|\cdot\|_f$ is a norm. For each equivalence class $\{\phi\}$ in $\hat{A}$ we select a representative ϕ and let H_ϕ be the representation space of ϕ. Since A is separable, so is H_ϕ. Let $\{\eta_{\phi n}\}_{n=1}^{\infty}$ be an orthonormal basis in H_ϕ. There is a unique, minimal central projection x_ϕ in the second dual A'' of A such that $x_\phi A''$ is isomorphic to the weak closure $\phi(A)''$ of $\phi(A)$ on the space H_ϕ [**4**; sect. 3.8]. Let Q denote the quasi-state space of A [**4**; p. 44] and P the set of pure states of A [**4**; p. 69]. Q is compact, convex and metrizable with the weak* topology, and $P \cup \{0\}$ is the set of extreme points of Q. By Choquet's theorem [**5**; p. 19], for each $f \in Q$ there is a representing measure m on the Borel subsets of Q, supported within P, such that for any $a \in A$, $\int_Q g(a)\, dm(g) = f(a)$. For any central projection x_ϕ as above let $P_\phi = \{g \in P: g(x_\phi) = 1\}$. For any unit vector $\eta \in H_\phi$, the pure state $g_{\phi\eta}$ on A given by $g_{\phi\eta}(a) = \langle \phi(a)\eta, \eta\rangle$ has a support projection [**6**; p. 31] $p_{\phi\eta} \leqq x_\phi$ which is a 1-dimensional projection in A'' [**4**; 3.13.6]. By [**4**; 3.11.9] there is a sequence $\{a_n\} \subset A$ such that $\|a_n\| \leqq 1$ and $a_n \to (1 - p_{\phi\eta})$ strongly in A''. Since each a_n, considered as a function on Q (see [**4**; p. 69]), is continuous, hence measurable, $p_{\phi\eta}$ is also measurable. Since the series $\sum_{n=1}^{\infty} p_{\phi\eta}$ converges pointwise on Q to the function represented by x_ϕ, then x_ϕ is also a measurable function on Q. Consequently P_ϕ is a measurable set for every $\phi \varepsilon \hat{A}$.

THEOREM. *The following are equivalent.*

(1) *$\hat{A}$ is countable.*

(2) *There is a faithful state f on A such that, for any proper C^*-subalgebra B of A, the $\|\cdot\|_f$ closure of any bounded ball in B does not contain the unit ball of A.*

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(3) *There is a faithful state f on A such that, for any maximal proper hereditary (see* [**4**; p.14]) *C^*-subalgebra B of A, the unit ball of B is not dense in the unit ball of A for* $\|\cdot\|_f$.

Proof. First assume (1) holds. Then $\hat{A}$ is countable, and the pure states $\{g_{\phi\eta}\colon \phi \in \hat{A}$ and $n = 1, 2, \ldots\}$ can be arranged in a sequence $\{g_k\}_{k=1}^{\infty}$. Set $f = \sum_{k=1}^{\infty} 2^{-k} g_k$. Then f is a faithful state on A'' by [**1**; p. 292], hence by [**2**; ch. I, §4, Prop. 5], $\|\cdot\|_f$ gives the strong operator topology on bounded balls of A''. Thus if the $\|\cdot\|_f$ closure of a bounded ball of a C^*-subalgebra B of A contains the unit ball of A, then B is strongly dense in A''. In particular, B separates the points of Q and so B is not a proper subalgebra of A [**3**; 11.3.2.]. Thus (1) $\Rightarrow$ (2), and (2) $\Rightarrow$ (3) is trivial.

Assume (3) and assume that $\hat{A}$ is uncountable. If f is the faithful state promised by (3), then let m denote the representing measure for f which was described before the statement of the Theorem. Since $\hat{A}$ is uncountable and P_ϕ is measurable for each $\phi \in \hat{A}$, we may fix some $\phi \in \hat{A}$ such that $m(P_\phi) = 0$. Fix a unit vector $\eta \in H_\phi$, and let B be the maximal, hereditary C^*-subalgebra of A defined by $B = \{a \in A\colon \phi(b)\eta = \phi(b^*)\eta = 0\}$ [**4**; 3.13.6, 3.10.7, and 1.5.2]. Let $p = p_{\phi\eta}$ be the support projection of $g_{\phi\eta}$ and $\{a_n\}_{n=1}^{\infty}$ be a countable approximate unit for B [**4**; p.11]. Then $a_n \to (1 - p)$ strongly in A'' [**4**; 3.11.9]. Thus $(1 - x_\phi)(a_n a a_n - a)^*(a_n a a_n - a) \to 0$ strongly in A'' since $(1 - x_\phi) \leqq (1 - \mathrm{p})$.

Thus $g((a_n a a_n - a)^*(a_n a a_n - a)) \to 0$ for all g in $P \backslash P_\phi$, so, as a function on Q, $(a_n a a_n - a)^*(a_n a a_n - a) \to 0$ almost everywhere with respect to m. By Lebesgue's dominated convergence theorem,

$$\|a_n a a_n - a\|_f^2 = \int_Q g((a_n a a_n - a)^*(a_n a a_n - a))\, dm(g) \to 0.$$

Since B is hereditary and $0 \leqq a_n a a_n \leqq \|a\|\ a_n^2 \in B$, then $a_n a a_n \in B$ and $\|a_n a a_n\| \leqq \|a\|$. This shows that the unit ball of B is dense in the unit ball of A for $\|\cdot\|_f$, contradicting (3).

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