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EXTENSIONS OF ℓ-HOMOMORPHISMS

ELEANOR R. ARON, ANTHONY W. HAGER[1] AND JAMES J. MADDEN

ABSTRACT. It is shown that, in abelian ℓ-groups, each morphism to a complete vector lattice extends over any majorizing embedding. This extends a result of the first author for Archimedean f-algebras with identity, and the recent Luxemburg-Schep theorem for vector lattices, and solves a problem of Conrad and McAlister. The proof presented here differs substantially from the Luxemburg-Schep proof. Ours uses the Yosida representation and Gleason's theorem on topological projectivity—this is novel, and seems relatively economical and transparent. The ℓ-group theorem is shown to imply, and with some modestly categorical machinery, to be implied by, certain similar statements in subcategories of ℓ-groups.

1. Introduction. Recall that, in a category $\mathscr{C}$, an object V is called injective if given the morphisms $\phi : G \to V$ and $\mu : G \to H$, with μ monic, there is a morphism $\varepsilon: H \to V$ with $\varepsilon \circ \mu = \phi$. We consider the category of Archimedean ℓ-groups (i.e., lattice-ordered groups), with morphisms the ℓ-homomorphisms, (i.e., group homomorphisms preserving finite meets and joins). Here, there are no injectives [**4**], but the theorem of the abstract, stated precisely below, shows that the complete vector lattices behave like injectives with respect to a restricted class of monics.

THEOREM 1.1 *Let $\phi: G \to V$ and $\mu: G \to H$ be morphisms of Abelian ℓ-groups. Then, there is a morphism $\varepsilon: H \to V$ with $\varepsilon \circ \mu = \phi$ provided that*

(a) *V is a complete vector lattice, and*
(b) *μ is a majorizing embedding.*

Here, *complete* means Dedekind complete; *embedding* is another word for monic or one-to-one; the subset S of the ℓ-group H is said to *majorize* H if given $h \in H$ there is $s \in S$ with $|h| \leqq s$; and the morphism $\mu: G \to H$ is called *majorizing* if $\mu(G)$ majorizes H.

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§4 discusses vector lattices, including the sharpness of 1.1, and §5 discusses various kinds of unitary morphisms.

The proof of 1.1 given here differs substantially from that of the vector lattice theorem in **[14]**; this is one of the main points of this paper. The proof in **[14]** is lengthy, proceeds through several special cases, and in the end it is not easy to tell what the extension ε really is; ε comes indirectly from a theorem of Kantorovich on extending positive linear transformations. Our proof can be described as follows; First, consider the diagrams

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, "\phi"'] & H \arrow[dl, "\varepsilon"] \\ V & \end{tikzcd} \qquad \xleftrightarrow{\text{Stone duality}} \qquad \begin{tikzcd} S_G & S_H \arrow[l, "\mu_\#"'] \\ S_V \arrow[u, "\phi_\#"] \arrow[ur, "\varepsilon_\#"'] & \end{tikzcd}$$

(Boolean algebras) (Boolean spaces)

Here, μ is monic if and only if $\mu_\#$ (the associated continuous function) is epic, and V is complete if and only if the Stone space S_V is extremally disconnected (see **[16]**). Now apply the following theorem.

THEOREM 1.2. (GLEASON **[10]**). *In the category of compact Hausdorff spaces, extremally disconnected is equivalent to projective.*

"Projective" is the exact dual of "injective". (Note that in compact Hausdorff spaces, epic is equivalent to onto.) Thus, given μ and ϕ, there is a morphism $\varepsilon_\#$ completing the diagram of Boolean spaces. Then, since S is a duality, there is a morphism ε completing the diagram of Boolean algebras. (This is Sikorski's Theorem, that in Boolean algebras, complete implies injective **[16]**.) Now consider

$$\begin{tikzcd} G \arrow[r, "\mu"] \arrow[d, "\phi"'] & H \arrow[dl, "\varepsilon"] \\ V & \end{tikzcd} \qquad \xrightarrow{\text{Yosida functor}} \qquad \begin{tikzcd} Y_G & Y_H \arrow[l, "\mu_\#"'] \\ Y_V \arrow[u, "\phi_\#"] \arrow[ur, "\varepsilon_\#"'] & \end{tikzcd}$$

(Archimedean ℓ-groups) (topological spaces)

The Yosida representation describes each G as an ℓ-group $\hat{G}$ of $[-\infty, +\infty]$-valued functions on a space Y_G. Interpreted properly, this provides a functor, so that, given μ and ϕ, there are $\mu_\#$ and $\phi_\#$; if V is complete, Y_V is extremally disconnected, so an application of Gleason's Theorem 1.2 provides $\varepsilon_\#$. But, Y is not a duality; in order to produce ε from $\varepsilon_\#$, another use of that fact that V is both complete and a vector lattice is required as well as some hypothesis on μ (e.g., that μ be majorizing).

This proves 1.1 for Archimedean ℓ-groups, and a short separate argument reduces the general case to this.

The details follow in §2 and §3.

2. Yosida representation. We summarize the features which will be used. We shall assume some familiarity, but the basic information can be found in [2] and [15]. The material on functoriality comes from [12] and [13].

Let G be an ℓ-group and let $e \in G^+$. $Y_G(e)$ denotes the set of ideals (solid subgroups) which are maximal with respect to not containing e, endowed with the hull-kernel topology; $Y_G(e)$ is a compact Hausdorff space. For each $g \in G$, we define $\hat{g}\colon Y_G(e) \to [-\infty, +\infty]$ in a well-known way, so that $\hat{e}$ is constantly 1, each $\hat{g}$ is continuous, and $\{\operatorname{coz} \hat{g} \mid g \in G\}$ is an open basis in $Y_G(e)$. (Here, $\operatorname{coz} \hat{g} = \{M \mid \hat{g}(M) \neq 0\} = \{M \mid g \notin M\}$.)

For disjoint $E \subseteqq G^+$ (i.e., $e \neq e' \Rightarrow e \wedge e' = 0$), $Y_G(E)$ will denote $\sum_{e \in E} Y_G(e)$, the topological sum. For each $g \in G$, we define $\hat{g}\colon Y_G(E) \to [-\infty, +\infty]$ piecewise on each $Y_G(e)$ as in the previous paragraph. $E^\perp$ denotes $\{g \in G \mid |g| \wedge e = 0 \text{ for each } e \in E\}$.

For Y a topological space, $D(Y)$ is the set of all continuous $f\colon Y \to [-\infty, +\infty]$ with $f^{-1}(-\infty, +\infty)$ dense. With natural definitions of operations, $D(Y)$ is a lattice but usually not a group (nor much else), though it has "subgroups" (and various other "substructures"); in case Y is extremally disconnected, $D(Y)$ is an ℓ-group (vector lattice, ℓ-ring, and ℓ-algebra).

For the rest of this section, G and H denote Archimedean ℓ-groups. Morphism (as in 1.1) means ℓ-homomorphism. The version of the Yosida representation which we shall use is as follows.

THEOREM 2.1. *Let $E \subseteqq G^+$ be disjoint. Then $\hat{G}$ is an "ℓ-subgroup" of $D(Y_G(E))$, and $G \mapsto \hat{G}$ is a morphism which preserves all existing joins and meets, and whose kernel is $E^\perp$.*

2.1 is standard, if a little more detailed than usual. Note that $E^\perp = (0)$ if and only if E is maximal disjoint, and then $G \mapsto \hat{G}$ is an isomorphism. Note too, that an "ℓ-subgroup" of any $D(Y)$ must be Archimedean, and therefore any $G/E^\perp = \hat{G}$ is Archimedean. In fact, for any ℓ-subgroup $S \subseteqq G$ and maximal disjoint $E \subseteqq S^+$, we have $S^\perp = E^\perp$, whence $G/S^\perp$ is Archimedean.

THEOREM 2.2. *Let $\varphi\colon G \to H$ be a morphism, let $E \subseteqq G^+$ be disjoint, (so that $\varphi(E) \subseteqq H^+$ is disjoint), and let $\hat{G} \subseteqq D(Y_G(E))$ and $\hat{H} \subseteqq D(Y_H(\varphi(E)))$ be the morphic images of 2.1. Then,*

(a) *a morphism $\hat{\varphi}\colon \hat{G} \to \hat{H}$ is defined by $\hat{\varphi}(\hat{g}) \equiv \varphi(g)\hat{}$,*

(b) *for each $e \in E$, a continuous map $\varphi_\#^e\colon Y_H(\varphi(e)) \to Y_G(e)$ is defined by $\varphi_\#^e(M) \equiv \varphi^{-1}(M)$, and $\varphi_\# \equiv \sum_{e \in E} \varphi_\#^e$ (the topological sum) defines a continuous map $\varphi_\#\colon Y_H(\varphi(E)) \to Y_G(E)$,*

(c) *$\hat{\varphi}(\hat{g}) = \hat{g} \cdot \varphi_\#$ for each $g \in G$, and $\varphi_\#$ is the unique map with these properties, and*

(d) *if φ is one-to-one (resp., onto), then $\varphi_\#$ is onto (resp., one-to-one).*

2.2 is an immediate consequence of §2 of **[13]** (and comes from §2 of **[12]**), except for the "onto implies $1 - 1$" part of (d) which we don't need.

It is easy to see that the process of 2.1 and 2.2 maps $1\colon G \to G$ to $1\colon Y_G(E) \to Y_G(E)$, and maps the composition

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

to the composition

$$Y_A(E) \xleftarrow{\alpha*} Y_B(\alpha(E)) \xleftarrow{\beta*} Y_C(\beta(\alpha(E))).$$

Thus we have a functor to topology defined on the category with objects (G, E) and morphisms $\varphi : (G, E) \to (G', E')$ the ℓ-group morphisms $\varphi\colon G \to G'$ with $\varphi(E) = E'$. This may be awkward-looking, but the category is a natural analogue of rings with identity, which fact is exploited here in §5 (and elsewhere).

The following is closely related to 13.4.2 of **[2]**.

PROPOSITION 2.3. *Let $E \subseteq G^+$ be disjoint.*

(a) *If G has the projection property (in particular, if G is a complete ℓ-group), then $Y_G(E)$ is extremally disconnected.*

(b) *If G is a complete divisible ℓ-group (whence a complete vector lattice), then $\hat{G}$ is a solid subgroup of the ℓ-group $D(Y_G(E))$.*

PROOF. (a) A topological sum is extremally disconnected if and only if each summand is, and so it suffices that each $Y_G(e)$ be extremally disconnected. A typical open set is of the form $\operatorname{coz} \hat{I} = \bigcup\{\operatorname{coz} \hat{g} \mid g \in I\} = \{M \mid I$ is not contained in $M\}$, for some ideal I in G. The hull-kernel topology has the property that $\operatorname{cl}(\operatorname{coz} \hat{I}) = \{M \mid I^\perp \subseteq M\}$. With the projection property, $I^\perp \subseteq M$ if and only if $I^{\perp\perp}$ is not contained in M, and thus $\operatorname{cl}(\operatorname{coz} \hat{I}) = \operatorname{coz}(I^{\perp\perp})\hat{}$ which is open.

(b) Suppose $f \in D(Y_G(E))^+$ with $f \leqq \hat{g}_0$, and let $s = \bigvee\{\hat{g} \in \hat{G} \mid \hat{g} \leqq f\}$. Then $s \in \hat{G}$ and $s \leqq f$. If $s \neq f$, there is $\operatorname{coz} \hat{g}_1 \neq \emptyset$ with closure compact and contained in $\operatorname{coz}(f - s)$. We may suppose $0 \leqq \hat{g}_1 \leqq 1$ (by replacing $\hat{g}_1$ by $|g_1| \wedge e$ for appropriate $e \in E$). Then, for large enough n, $0 < ((1/n) \cdot g_1)\hat{} < (f - s)$, whence $s < s + ((1/n) \cdot g_1)\hat{} < f$. This contradicts the definition of s, so $f = s \in \hat{G}$, as required.

3. Proof of Theorem 1.1. We first reduce to the Archimedean case. Suppose given (in Abelian ℓ-groups) the morphism $\phi\colon G \to V$, with V a complete vector lattice (hence itself Archimedean (**[2]** or **[15]**), and the majorizing embedding $\mu\colon G \to H$. Consider Diagram 1.

In Diagram 1 (ker ϕ) is the generated ideal in H, ν exists by a homomorphism theorem, and I is an ideal chosen (by Zorn's lemma) maximal with repsect to $I \cap \nu[H/(\text{ket } \phi)] = (0)$. To lift ϕ over μ, it suffices to

$$
\begin{tikzcd}
G \arrow[r, "\mu"] \arrow[dr] \arrow[dd, "\psi"'] & H \arrow[dr] & \\
& G/\ker\psi \arrow[r, "\nu"] \arrow[dl, hook', "i"'] & H/(\ker\psi) \arrow[d, "\pi"] \\
V & & {[H/(\ker\psi)]/I} \arrow[ll, dashed, "\varepsilon"']
\end{tikzcd}
$$

Diagram 1

produce ε with $\varepsilon \circ (\pi \circ \nu) = i$, and this amounts to proving 1.1 in Archimedean ℓ-groups, because $G/\ker \psi$ is Archimedean as a subgroup of V. Since μ is a majorizing embedding, so is ν, and then so is $\pi \circ \nu$. By maximality of I, $\pi \circ \nu$ has large image (i.e., each nonzero ideal in the codomain meets the image in a nonzero ideal) and thus $[H/(\ker \psi)]/I$ is Archimedean, by the following lemma (which is essentially 2.3 of [**15**]).

LEMMA 3.1. *If $A \to B$ is a large majorizing embedding, and A is Archimedean, then B is Archimedean.*

PROOF. Let I_A and I_B be the ideals of infinitely small elements in A and B. Since A majorizes B, we have $I_A = I_B \cap A$. Since A is Archimedean, $I_A = (0)$. Since A is large in B, we have $I_B = (0)$, i.e., B is Archimedean.

We now prove 1.1 for Archimedean ℓ-groups. Suppose given the morphism $\psi: G \to V$, with V a complete vector lattice, and the majorizing embedding $\mu: G \to H$. We prepare to apply 2.2. Consider the diagram

$$
\begin{tikzcd}
G \arrow[r, "\mu"] \arrow[d, "\psi"'] & H \arrow[r, "q"] & H/\mu(G)^{\perp} \arrow[dl, dashed] \\
V & \psi(G)^{\perp\perp} \arrow[l, hook'] &
\end{tikzcd}
$$

It suffices to produce a dotted arrow making the diagram commute. Notice that $\psi(G)^{\perp\perp}$ is a complete vector lattice, since it is an ideal in V; $\psi(G)^{\perp\perp} \supseteqq \psi(G)$ so that $\psi: G \to \psi(G)^{\perp\perp}$ is a morphism. Regarding $q \circ \mu: G \to H/\mu(G)^{\perp}$, notice that $H/\mu(G)^{\perp}$ is Archimedean, as pointed out after 2.1, $\mu(G)^{\perp} \cap \mu(G) = 0$, whence q is one-to-one on $\mu(G)$, whence $q \circ \mu$ is an embedding, and $q \circ \mu$ is majorizing because μ is majorizing and q is onto. Finally note that whenever $E \subseteqq G^{+}$ is maximal disjoint, then $\psi(E)$ and $q(\mu(E))$ are maximal disjoint in $\psi(G)^{\perp\perp}$ and $H/\mu(G)^{\perp}$, respectively.

Thus, it suffices to prove 1.1 for $\phi\colon G \to V$ and $\mu\colon G \to H$ assuming a maximal disjoint $E \subseteqq G^+$ for which $\phi(E)$ and $\mu(E)$ are maximal disioint.

By 2.1, $G \to \hat{G} \subseteqq D(Y_G(E))$, $H \to \hat{H} \subseteqq D(Y_H(\mu(E)))$ and $V \to \hat{V} \subseteqq D(Y_V(\phi(E)))$ are isomorphisms. So, to produce the desired $\varepsilon\colon H \to V$, it suffices to produce a suitable $\hat{\varepsilon}\colon \hat{H} \to \hat{V}$.

Consider the diagrams

$$\begin{tikzcd} \hat{G} \arrow[r, "\hat{\mu}"] \arrow[d, "\hat{\phi}"'] & \hat{H} \arrow[dl, dashed, "\hat{\varepsilon}"] \\ \hat{V} & \end{tikzcd} \qquad\qquad \begin{tikzcd} Y_G(E) & Y_H(\mu(E)) \arrow[l, "\mu_\#"'] \\ Y_V(\phi(E)) \arrow[u, "\phi_\#"] \arrow[ur, dashed, "\varepsilon_\#"'] & \end{tikzcd}$$

(A) $\qquad\qquad$ (B)

That is, if $\hat{\varepsilon}$ exists in (A), then, by 2.2, $\hat{\varepsilon}$ is produced by composition with a certain continuous map $\varepsilon_\#$, with (B) commuting. We shall produce $\varepsilon_\#$, then $\hat{\varepsilon}$.

There is $\varepsilon_\#$ with $\mu_\# \circ \varepsilon_\# = \phi_\#$ because the diagram (B) is the sum of the diagrams

$$(B_e) \qquad \begin{tikzcd} Y_G(e) & Y(e) \arrow[l, "\mu^e_\#"'] \\ Y_V(e) \arrow[u, "\phi^e_\#"] \arrow[ur, "\varepsilon^e_\#"'] & \end{tikzcd} \qquad (e \in E).$$

Gleason's Projectivity Theorem 1.2 produces $\varepsilon^e_\#$ because $Y_V(e)$ is extremally disconnected (by 2.4 (a)), and $\mu^e_\#$ is onto (by 2.2). Then let $\varepsilon_\# \equiv \sum_{e\in E} \varepsilon^e_\#$, the topological sum. (There is a generalization of Gleason's theorem in [8] which applies to (B) directly to produce $\varepsilon_\#$.)

Define $\hat{\varepsilon}(\hat{h}) \equiv \hat{h} \circ \varepsilon_\#$ for $h \in H$. Note that for $g \varepsilon G$, we have $\hat{\varepsilon}(\hat{\mu}(\hat{g})) = (\hat{\mu}(\hat{g})) \circ \varepsilon_\# = (\hat{g} \circ \mu_\#) \circ \varepsilon_\# = \hat{g} \circ (\mu_\# \circ \varepsilon_\#) = \hat{g} \circ \phi_\# = \hat{\phi}(\hat{g})$.

Regarding the other required features of $\hat{\varepsilon}$, we digress a bit.

REMARK 3.2. With $R^* = [-\infty, +\infty]$ and Y any topological space, the set of continuous functions $C(Y, R^*)$ is a lattice containing $D(Y)$. Any continuous $\tau_\#\colon Z \to Y$ induces a lattice homomorphism $\tau\colon C(Y, R^*) \to C(Z, R^*)$ defined by $\tau(f) = f \circ \tau$. We need not have $\tau(D(Y)) \subseteqq D(Z)$. However, if A is an "ℓ-subgroup" of $D(Y)$ and B is an "ℓ-subgroup" of $D(Z)$, and if $\tau(A^+) \subseteqq B$, then the restriction $\tau\colon A \to B$ is a morphism. This comes directly from the definition in §2 of the partial operations in $D(Y)$. Clearly, the same is true for other kinds of "algebraic substructures" of $D(Y)$.

Returning to $\hat{\varepsilon}$, we see from 3.2 that to have the required morphism $\hat{\varepsilon}\colon \hat{H} \to \hat{V}$, it is enough that $\hat{\varepsilon}(\hat{h}) \in \hat{V}$ for each $h \in H^+$: Given h, choose $g \in G$ with $0 \leqq h \leqq \mu(g)$. Then $0 \leqq \hat{h} \leqq \hat{\mu}(\hat{g})$, whence $0 \leqq \hat{\varepsilon}(\hat{h}) \leqq \hat{\varepsilon}(\hat{\mu}(\hat{g})) = \hat{\phi}(\hat{g}) \in \hat{V}$ (because $\hat{\varepsilon}$ is a lattice homomorphism, and $\hat{\varepsilon} \circ \hat{\mu} = \hat{\phi}$).

In particular, this shows that $\hat{\varepsilon}(\hat{h})$ is real-valued whenever $\hat{\phi}(\hat{g})$ is real-valued, so that $\hat{\varepsilon}(\hat{h}) \in D(Y_V(\phi(E)))$. Since V is a complete vector lattice, $\hat{V}$ is a solid subgroup (by (2.3(b)) thus $\hat{\varepsilon}(\hat{h}) \in \hat{V}$, as desired.

4. Vector lattices. We discuss the special place of vector lattices in the context of the extension problem, the Luxemburg-Schep theorem, and the need for the hypotheses in 1.1. Let $\mathscr{G}$ denote the category of ℓ-groups with ℓ-homomorphisms, and $\mathscr{V}$ the category of vector lattices with vector lattice homomorphisms. Recall that each Archimedean $G \in |\mathscr{G}|$ has an essentially unique $\mathscr{G}$-completion, a dense and majorizing $\mathscr{G}$-embedding $G \hookrightarrow \bar{G}$, with $\bar{G}$ complete, and similarly for $\mathscr{V}$-completions. See [7], [2], [15].

We begin with a problem of Conrad and McAlister: Example VI of [7] shows that a complete ℓ-group containing G need not contain a $\mathscr{G}$-completion of G, and question 1. at the end of [7] asks if the following proposition is true.

PROPOSITION 4.1. *If G is an ℓ-subgroup of the complete vector lattice V, then V contains a $\mathscr{G}$-completion of G.*

PROOF. By 1.1, the hypothesized inclusion $\phi: G \to V$ extends over the embedding $G \to \bar{G}$ in the $\mathscr{G}$-completion, to $\varepsilon: \bar{G} \to V$. According to 4.2 below, ε is monic and thus $\varepsilon(\bar{G})$ is the desired completion of G within V.

LEMMA. 4.2. *Suppose ℓ-group morphisms satisfy $\varepsilon \circ \mu = \phi$. If μ has large image and ϕ is monic, then ε is monic.*

PROOF. If $\phi = \varepsilon \circ \mu$ is monic, then $\mathrm{im}(\mu) \cap \ker(\varepsilon) = 0$; since $\mathrm{im}(\mu)$ is large, $\ker(\varepsilon) = 0$.

4.3. We explain the connection between 1.1 and the Luxemburg-Schep theorem, 3.1 of [15]. This is the statement "1.1($\mathscr{V}$)": *Given $\mathscr{V}$-morphisms $\phi: G \to V$ and $\mu: G \to H$, with V complete and μ a majorizing embedding, there is a $\mathscr{V}$-morphism $\varepsilon: H \to V$ with $\varepsilon \circ \mu = \phi$.*

Our method of proving 1.1 proves 1.1($\mathscr{V}$). §2 is completely valid "in $\mathscr{V}$" and then the proof in §3 goes through without change (noting 3.2).

But further, there are interesting "formal connections" between 1.1 and 1.1($\mathscr{V}$) existing by virtue of the Conrad-Bleir vector lattice hulls. It is shown in [3] and [4] that (a) each $\mathscr{G}$-morphism between Archimedean $\mathscr{V}$-objects is already $\mathscr{V}$-morphism (i.e. $\mathscr{V}$ is full in $\mathscr{G}$), and that (b) each $G \in |\mathscr{G}|$ $\mathscr{G}$-embeds into an Archimedean $\nu G \in |\mathscr{V}|$ so that each $\mathscr{G}$-morphism $\varphi: G \to L$ with $L \in |\mathscr{V}|$ has a unique $\mathscr{V}$-extension $\nu\varphi: \nu G \to \nu L$ (we can say, arch $\mathscr{V}$ is embedding-reflective in arch $\mathscr{G}$).

We show now how 1.1 implies, and is implied by 1.1($\mathscr{V}$). As at the beginning of §3, it suffices to consider only Archimedean ℓ-groups and vector lattices. Now, 1.1 implies 1.1($\mathscr{V}$), using (a): Given appropriate ϕ

and μ in arch $\mathscr{V}$, 1.1 yields a $\mathscr{G}$-morphism ε with $\varepsilon \circ \mu = \phi$; and ε is a $\mathscr{V}$-morphism. And, 1.1($\mathscr{V}$) implies 1.1 using (b): Given appropriate $\phi: G \to V$ and $\mu: G \to H$ in arch $\mathscr{G}$, we have $\mathscr{V}$-morphisms $v\phi: vG \to vV = V$ and $v\mu: vG \to vH$. One checks that the operator v preserves majorizing embeddings. So 1.1($\mathscr{V}$) provides a $\mathscr{V}$-morphism $\varepsilon': vH \to V$ with $\varepsilon' \circ (v\mu) = v\phi$. Then

$$\varepsilon \equiv G \xrightarrow{\mu} H \longrightarrow vH \xrightarrow{\varepsilon'} V$$

is as desired.

4.4. Veksler [17] has proved "4.1($\mathscr{V}$)": If in $\mathscr{V}$, $G \hookrightarrow V$ with V complete, then V contains a $\mathscr{V}$-completion of G.

Again, 4.1 implies and is implied by 4.1($\mathscr{V}$), just as in 4.3, but this additional fact is needed: for (Archimedean) $G \in |\mathscr{G}|$, the $\mathscr{V}$-completion of vG is v (the $\mathscr{G}$-completion of G) [4].

We note that Veksler's proof used the Kantorovich Theorem (as does the proof of 1.1($\mathscr{V}$) in [14]). So again we see Gleason's Theorem 1.2 replacing the Kantorovich Theorem.

5. Unitary morphisms. We have considered, and will now consider further, subcategories $\mathscr{C}$ of $\mathscr{G}$, and the statements 1.1($\mathscr{C}$): Given $\mathscr{C}$-morphisms $\phi: G \to V$ and $\mu: G \to H$, V a complete vector lattice, and μ a majorizing embedding, there is a $\mathscr{C}$-morphism $\varepsilon: H \to V$ with $\varepsilon \circ \mu = \phi$.

We distinguish rather imprecisely two types of situation where 1.1($\mathscr{C}$) holds: (1) (As arch $\mathscr{V}$ in arch $\mathscr{G}$) $\mathscr{C}$ is full in $\mathscr{C}_1$, and 1.1($\mathscr{C}_1$) holds. The proof of 1.1($\mathscr{C}$) is then the triviality "$\varepsilon \circ \mu = \phi$ in $\mathscr{C}_1$, but $\varepsilon \in \mathscr{C}$ already". (It may not be particularly trivial that $\mathscr{C}$ is full in $\mathscr{C}_1$.) (2) $\mathscr{C}$ is not full in $\mathscr{C}_1$, but $\varepsilon \circ \mu = \phi$ in $\mathscr{C}_1$ with $\mu, \phi \in \mathscr{C}$ imply $\varepsilon \in \mathscr{C}$.

5.1. ℓ-groups with weak unit. An object of $\mathscr{L}$ is a $G \in |\mathscr{G}|$ with a distinguished weak unit $\mathscr{E}_G$ (i.e., $\mathscr{E}_G^\perp = (0)$), and a morphism is a $\mathscr{G}$-morphism preserving weak unit. Then $\mathscr{L} \subseteqq \mathscr{G}$ is type (2), and 1.1($\mathscr{L}$) holds.

Actually, 1.1(arch $\mathscr{L}$) implies 1.1(arch $\mathscr{G}$) (whence 1.1($\mathscr{G}$)) by a method which is more-or-less "algebraically formal". The $\mathscr{G}$-diagram $\varepsilon \circ \mu = \phi$ is a subdirect product of $\mathscr{L}$-diagrams $\varepsilon^e \circ \mu^e = \phi^e$ over e ranging in a maximal disjoint $E \subseteqq (\text{domain } \mu)^+$; to produce ε it is enough to produce all ε^e, i.e., to have 1.1($\mathscr{L}$). Of course, we used this procedure in §2 and §3.

5.2. ℓ-groups with strong unit. $\mathscr{S}$ is the full subcategory of $\mathscr{L}$ with objects G for which $G = (e_G)$. $\mathscr{S} \subseteqq \mathscr{L}$ is type (1), so 1.1($\mathscr{S}$) holds. Since in $\mathscr{S}$ every morphism is majorizing, 1.1($\mathscr{S}$) says that each complete vector lattice is $\mathscr{S}$-injective. Moreover, by 3.2, within $\mathscr{S}$, a complete vector lattice is isomorphic to a $C(Y)$ for Y compact and extremally disconnected, and conversely, by the Stone-Nakano Theorem [9].

5.3. Boolean algebras. Let $\mathscr{B}$ be this category, construed as a full subcategory of $\mathscr{S}$ by the device of identifying $B \in |\mathscr{B}|$ with the ℓ-group $C(S_B)$ of continuous real-valued functions on the Stone space S_B. Then, $\mathscr{B} \subseteqq \mathscr{S}$ is type (1), whence 1.1($\mathscr{B}$) holds. (It is hardly surprising that 1.1 implies 1.1($\mathscr{B}$) since our proof of 1.1 generalizes a proof of 1.1($\mathscr{B}$). 3.6 of **[14]** notes also that 1.1($\mathscr{V}$) implies 1.1($\mathscr{B}$).) About $\mathscr{B}$, more is known, of course [*S*]: B is $\mathscr{B}$-injective if and only if B is complete and if and only if S_B is extremally disconnected.

5.4. *f*-rings with identity. An object of $\mathscr{R}$ is an *f*-ring G with ring identity 1_G which is a weak unit. Just as arch $\mathscr{V}$ is (a) full, and (b) embedding-reflective in arch $\mathscr{S}$, as described in 4.3, arch $\mathscr{R}$ is (a) full and (b) embedding-reflective in arch $\mathscr{L}$, by **[6]** and **[13]**. Then (a) says that arch $\mathscr{R} \subseteqq$ arch $\mathscr{L}$ is of type (1), whence 1.1($\mathscr{R}$) holds. Conversely, so to speak, 1.1 (arch $\mathscr{R}$) implies 1.1($\mathscr{L}$) by (b), just as in 4.3. (This requires verifying, as can be done, that the functor arch $\mathscr{L} \to$ arch $\mathscr{R}$ which embeds each G into a ring ρG has the property of preserving majorizing embeddings.) We thus have the curious observation that 1.1(arch $\mathscr{R}$) implies 1.1($\mathscr{S}$) (since by 5.1, 1.1($\mathscr{L}$) implies 1.1($\mathscr{S}$)).

[1] contains the theorem (with a proof like §3 here) that in Φ-Archimedean *f*-algebras with identity which is a weak order unit, each $G \to V$, with V complete, lifts over the Dedekind completion of G. This is a little weaker than 1.1(Φ), which itself follows from 1.1(arch $\mathscr{R}$) and 1.1($\mathscr{V}$).

Note that we have not proved 1.1($\mathscr{R}$); we don't know if this is true since there seem to be difficulties with reducing to the Archimedean case. Further, we have not proved 1.1(*f*-rings), whether or not Archimedean, and we don't know if these are true.

Added later (12/2/80). **[11]** will contain the following: the "converse" of 1.1; another proof of 1.1, which is not terribly complicated and does not use representations; a discussion of "majorizing-injectivity" qua injectivity theory.

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TRINITY COLLEGE, DUBLIN, IRELAND

WESLEYAN UNIVERSITY, MIDDLETOWN, CT 06457