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INDUCED SHAPE FIBRATIONS AND FIBER SHAPE EQUIVALENCE

MAHENDRA JANI*

ABSTRACT. In this paper we prove that a map induced from a shape fibration is a shape fibration. We define a fiber shape equivalence relation between shape fibrations. Also, generalizing the homotopy relation, we define a strong equivalence relation in the set of maps between compact metric spaces. Then we prove that two strongly equivalent maps induce fiber shape equivalent shape fibrations. As a corollary we show that the fibers over two points connected by a strong shape path are of the same shape. Finally, we prove that a fiber shape equivalence induces a relative shape map which induces an appropriate isomorphism on relative shape groups.

1. Introduction. In a recent paper [**11**] S. Mardešić and T.B. Rushing have defined an important notion of 'shape fibration' by generalizing an approximate fibration of Coram and Duvall [**4**]. One expands a map $p: E \to B$ between compact metric spaces into a map $\mathbf{p}: \mathbf{E} = (E_n, r_{nm}) \to \mathbf{B} = (B_n, q_{nm})$ of inverse sequences of compact ANR's. The map p is a shape fibration if $\mathbf{p}$ has the following approximate homotopy lifting property: each n and each $\varepsilon > 0$ admit an index $m \geqq n$ and $\delta > 0$ such that for any topological space X, whenever the maps $h: X \to E_m$ and $H: X \times I \to B_m$ satisfy $d(p_m h, H_0) \leqq \delta$, then there is an homotopy $G: X \times I \to E_n$ satisfying $d(G_0, r_{nm}h) < \varepsilon$ and $d(p_n G, q_{nm}H) < \varepsilon$.

Analogously to fibrations, one may ask the following questions for shape fibrations. Is a map induced from a shape fibration a shape fibration? Is there a notion of a fiber shape map? In what sense are two shape fibrations fiber shape equivalent? Is it true that two homotopic maps induce equivalent shape fibrations?

In this paper we have studied all these questions and have found satisfactory positive answers. §2 contains basic definitions and some basic results that we need. In §3 we prove that a map induced from a shape fibration is a shape fibration. §4 contains the definition of a fiber shape equivalence.

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Let p: $E \to B$ and p': $E' \to B$ be shape fibrations expanded to maps $\mathbf{p}$: $\mathbf{E} = (E_n, r_{nm}) \to \mathbf{B} = (B_n, q_{nm})$ and $\mathbf{p}'$: $\mathbf{E}' = (E'_n, r'_{nm}) \to \mathbf{B}$ of inverse sequences of compact ANR's respectively. Roughly, a fiber shape map $\mathbf{f}$: $\mathbf{p} \to \mathbf{p}'$ is an equivalence class of a map $\mathbf{f}$: $\mathbf{E} \to \mathbf{E}'$ of ANR-sequences satisfying the following condition: for every n and for every $\varepsilon > 0$ there is $n^* \geqq n$ such that for all $\ell \geqq m \geqq n^*$ there is a homotopy H: $r'_{m\ell}f_\ell \cong f_m r_{\alpha(m)\alpha(\ell)}$ such that for every $t \in I$ the maps $q_{nm}p'_m H_t$ and $q_{n\alpha(\ell)}p_{\alpha(\ell)}$ are ε-close.

Two such maps $\mathbf{f}$, $\mathbf{g}$: $\mathbf{E} \to \mathbf{E}'$ are said to be equivalent if for every n and for every $\varepsilon > 0$ there is $\hat{n} \geqq n$ such that for all $m \geqq \hat{n}$, there exist $\ell \geqq \alpha(m)$ and an homotopy L: $f_m r_{\alpha(m)\ell} \cong g_m r_{\alpha(m)\ell}$ such that for every $t \in I$, the maps $q_{nm}p'_m L_t$ and $q_{n\ell}p_\ell$ are ε-close.

Finally, two shape fibrations p and p' are fiber shape equivalent if there are fiber shape maps $[\mathbf{f}]$: $p \to p'$ and $[\mathbf{g}]$: $p' \to p$ such that $\mathbf{gf} \sim \mathbf{1_p}$ and $\mathbf{fg} \sim \mathbf{1_{p'}}$.

For example, the shape fibration p: $W \to W/A \approx S^1$, where W is the Warsaw circle and A is its limit arc, is fiber shape equivalent to the obvious shape fibration 1_{S^1}: $S^1 \to S^1$.

In §5 we prove the main result that strongly equivalent maps induce fiber shape equivalent shape fibrations.

Two maps f and g: $C \to B$ between comapct metric spaces are strongly equivalent if there are maps $\mathbf{f}$, $\mathbf{g}$: $\mathbf{C} \to \mathbf{B}$ of inclusive inverse sequences of compact ANR's such that (i) for every n there exist $n(*)$ such that for all $m \geqq n(*)$ there is a homotopy H^m: $q_{nm}f_m \cong q_{nm}g_m$ and (ii) for $m' \geqq n'(*)$ where $n' \geqq n$ there is a homotopy $q_{nn'}H^{m'} \cong H^m(q_{mm'} \times 1_I)$ (rel $\dot{I}$).

Two homotopic maps are strongly equivalent. Hence, in particular two induced shape fibrations by homotopic maps are fiber shape equivalent.

One immediate corollary of the theorem is that if B is of a trivial shape then a shape fibration p: $E \to B$ is fiber shape equivalent to a trivial shape fibration π_B: $F_x \times B \to B$ where $F_x = p^{-1}(x)$ for $x \in B$.

The strong equivalence notion leads naturally to a notion of strong shape path connectedness which we have discussed in §6.

Two points x and y of a compact metric space B are connected by a strong shape path if for any ANR-sequence $\mathbf{B} = (B_n, q_{nm})$ with inj lim $\mathbf{B} = B$ there is a family of paths $\boldsymbol{\omega} = \{\omega$: $I \to B_n | \omega_0 = x, \omega_1 = y\}$ such that for all $m \geqq n$, $q_{nm}\omega_m \cong \omega_n$ (rel $\dot{I}$).

A space is strongly shape path connected if any two of its points can be connected by a strong shape path. Clearly, path connected spaces are strongly shape path connected. Moreover, plane compact connected metric spaces and pointed 1-movable compact connected metric spaces are strongly shape path connected. But the fact that the dyadic solenoid is not strongly shape path connected shows that, unlike Borsuk's approximate 0-connectedness [2], not all compact connected spaces are strongly shape path connected.

S. Mardešić and T.B. Rushing have asked the following question in [11]. For a shape fibration, is it true that two fibers over points lying in the same component are of the same shape?

We have partially answered this question by the following corollary of the main theorem: for a shape fibration, two fibers over points connected by a strong shape path are of the same shape.

Finally, in §7 we have proved that a fiber shape equivalence $[\mathbf{f}]: p \to p'$ induces a pointed relative shape map $\mathbf{f}$: $(E, F, e) \to (E', F', e')$ which induces an appropriate isomorphism $\mathbf{f}_*: \check{\pi}_q(E, F, e) \to \check{\pi}_q(E', F', e')$ of shape groups.

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By private correspondence the author came to know that A. Matsumoto has proved the theorem 3.1 independently in [13] and J. Krasinkewicz and P. Minc [7] have defined a notion called 'generalized paths' similar to the notion of 'strong shape path'. Also, the referee pointed out that the authors in [7] have proved Theorem 6.1 for 'generalized paths' and have described the example of Proposition 6.3.

2. Preliminaries. All spaces considered will be metric spaces. Denote by $d(x, y)$ the distance between two points x and y. For a number $\delta > 0$, two maps (continuous functions) f, g: $X \to Y$ are δ-close if for each $x \in X$, $d(f(x), g(x)) < \delta$. For such f and g we will write $d(f, g) < \delta$. The maps f and g are δ-homotopic if there is an homotopy H: $X \times I \to Y$ such that $H_0 = f$, $H_1 = g$ and for each $x \in X$, $d(H(x, t), H(x, t')) < \delta$ for all $t, t' \in I$. For the interior of a space X we write int X. Let A, Y be subspaces of the space X. Then Y is said to be a neighbourhood of A in X if $A \subset \operatorname{int} Y$.

By an ANR we mean absolute neighbourhood retract for metric spaces. It is well-known that if Y is a compact ANR, then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that any two δ-close maps from a space X to Y are ε-homotopic [5]. We use this result freely without mentioning it further. For a map f: $X \to Y$ between compact ANR's and a number $\varepsilon > 0$, $\Lambda(f, \varepsilon)$ denotes the set of all δ's such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ and $\Gamma(Y, \varepsilon)$ denotes the set of all η's such that two η-close maps from a metric space to Y are ε-homotopic. Note that if $0 < \delta' < \delta$ and $0 < \eta' < \eta$, then $\delta' \in \Lambda(f, \varepsilon)$ and $\eta' \in \Gamma(Y, \varepsilon)$ if $\delta \in \Lambda(f, \varepsilon)$ and $\eta \in \Gamma(Y, \varepsilon)$. By convention if $\delta \in \Lambda(f, \varepsilon)$ and $\eta \in \Gamma(Y, \varepsilon)$, then $\delta, \eta < \varepsilon$.

A space E is a convenient ANR if each compact metric space X in E

has the following property: for each neighbourhood U of X in E there is a compact ANR $M \subset U$ with $X \subset \text{int } M$. Every polyhedron is convenient and by the triangulation theorem [3] every Q-manifold is convenient where Q is the Hilbert cube. Also if E is a locally compact ANR, then $E \times Q$ is convenient [4].

An ANR-sequence $\mathbf{E} = (E_n, r_{nm})$ is an inverse sequence of compact ANR's. A level map $\mathbf{p} = (p_n)\colon \mathbf{E} \to \mathbf{B} = (B_n, q_{nm})$ of ANR-sequences is a map of inverse sequences, i.e., $\mathbf{p}$ is a family of maps $p_n\colon E_n \to B_n$ such that for $m \geqq n$, $p_n r_{nm} = q_{nm} p_m$. Let inj lim $\mathbf{E} = (E, r_n)$ and inj lim $\mathbf{B} = (B, q_n)$. Then the unique map $p\colon E \to B$ is said to be the limit map of the level map $\mathbf{p}$ if for each n, $q_n p = p_n r_n$.

Generalizing Coram and Duvall's approximate fibration [4] S. Mardešić and T.B. Rushing [11] have defined shape fibration. A map $p\colon E \to B$ between compact metric spaces is called a shape fibration if it is a limit map of a level map $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ of ANR-sequences which has the following approximate homotopy lifting property(AHLP): each n and each $\varepsilon > 0$ admit an $m \geqq n$ and a $\delta > 0$ such that for given maps $h\colon X \to E_m$ and $H\colon X \times I \to B_m$ with

(1) $$d(p_m h, H_0) \leqq \delta$$

there is a homotopy $G\colon X \times I \to E_n$ such that

(2) $$d(G_0, r_{nm} h) < \varepsilon$$

and

(3) $$d(p_n G, q_{nm} H) < \varepsilon.$$

Every such m is called a lifting index for $(n. \varepsilon)$ and δ is called a lifting mesh for (n, ε). We refer to (m, δ) as a lifting pair for (n, ε). It has been shown in [11] that if a map $p\colon E \to B$ between compact metric spaces is a limit map of level maps $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ and $\mathbf{p}'\colon \mathbf{E}' \to \mathbf{B}'$ of ANR-sequences and if $\mathbf{p}$ has AHLP, then so does $\mathbf{p}'$. Also in [11] the authors have defined a homotopy lifting property (HLP) for a level map as follows: a level map $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ of ANR-sequences has the HLP if each n admits an $m \geqq n$ such that for given maps $h\colon X \to E_m$ and $H\colon X \times I \to B_m$ with

(4) $$H_0 = p_m h,$$

there is a homotopy $G\colon X \times I \to E_n$ such that

(5) $$G_0 = r_{nm} h$$

and

(6) $$p_n G = q_{nm} H.$$

For a level map $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ of ANR-sequences, we will state two results from [9] which relate the stronger lifting properties with the weaker properties.

(I) If $\mathbf{p}$ has the AHLP, then it has the stronger lifting property obtained where (2) is replaced by (5).

(II) $\mathbf{p}$ has the AHLP if it has the weaker lifting property obtained where (1) is replaced by (4).

The following proposition follows immediately from (II).

PROPOSITION 2.1. *Let* $\mathbf{p}: \mathbf{E} \to \mathbf{B}$ *be a level map of* ANR-*sequences. If* $\mathbf{p}$ *has the* HLP, *then the limit map* $p: E \to B$ *is a shape fibration.*

The reader is advised to refer to [10] for maps of ANR-sequences and the equivalence relation between them, to [15] for the shape groups and to [8] for the category pro-$\mathscr{G}$ and to [6] for the detailed proofs of the results of this paper.

3. Induced shape fibration. In this section we will prove that a map induced from a shape fibration by a map is a shape fibration. First we need the following proposition.

PROPOSITION 3.1. *For maps* $p: E \to B$ *and* $f: C \to B$ *between compact metric spaces, there are level maps* $\mathbf{p}: \mathbf{E} \to \mathbf{B}$ *and* $\mathbf{f}: \mathbf{C} \to \mathbf{B}$ *of* ANR-*sequences with limit maps* p *and* f *respectively.*

PROOF: Embed E, B, C in the Hilbert cube Q. Since Q is an AR and E, C are compact, maps p and f can be extended to $\tilde{p}: Q \to Q$ and $\tilde{f}: Q \to Q$ respectively. Choose for $\mathbf{B}$ a decreasing sequence of compact ANR-neighbourhoods B_n of B with $\bigcap_n B_n = B$.

By induction we can choose a decreasing sequence of compact ANR-neighbourhoods E_n of E and C_n of C with $\bigcap_n E_n = E$, $\bigcap_n C_n = C$, $p(E_n) \subseteqq B_n$ and $f(C_n) \subseteqq B_n$. For $m \geqq n$, let $r_{nm}: E_m \hookrightarrow E_n$ and $q_{nm}: C_m \hookrightarrow C_n$ be inclusions and for each n, let $p_n = \tilde{p}|E_n$ and $f_n = \tilde{f}|C_n$. Hence $\mathbf{p} = (p_n): \mathbf{E} \to \mathbf{B}$ and $\mathbf{f} = (f_n): \mathbf{C} \to \mathbf{B}$ are level maps of ANR-sequences with limit maps p and f respectively.

For maps $p: E \to B$ and $f: C \to B$ between compact metric spaces, a triple $(Z; p', f')$ is a pull-back of $(B; p, f)$ in the category of compact metric spaces and maps where $Z = \{(e, c) \in E \times C \mid p(e) = f(c)\}$ and $p': Z \to C$ and $f': Z \to E$ are the projections. Note that Z is also compact. We say that $p' = p_*(f)$ is a map induced from p by f.

We now have the main theorem of this section.

THEOREM 3.1. *Let* $p: E \to B$ *be a shape fibration and* $f: C \to B$ *be a map between compact metric spaces. Then the map* $p_*(f)$ *induced from* p *by* f *is also a shape fibration.*

PROOF. Let $(Z; p', f')$ be the pull-back of $(B; p, f)$. We want to show that

p' is a shape fibration. Let $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ and $\mathbf{f}\colon \mathbf{C} \to \mathbf{B}$ be level maps of ANR-sequences with limit maps p and f respectively. Without loss of generality we can assume that for each n, $E_n \times C_n$ is a compact convenient ANR. If this is not the case, then consider an ANR-sequence $\mathbf{Q} = (Q_n, \alpha_{nm})$ where for each n, $Q_n = Q$, the Hilbert cube and inj lim $\mathbf{Q}$ is a point. Then inj lim $\mathbf{E} \times \mathbf{Q} = E$. Also, for each n, let $\pi_n\colon E_n \times Q_n \to E_n$ be the projection map. Then $\pi\mathbf{p} = (p_n\pi_n)\colon \mathbf{E} \times \mathbf{Q} \to \mathbf{B}$ has AHLP [11] since, for each n, $E_n \times Q_n$ is a Q-manifold and, therefore, is a convenient ANR. Note that inj lim $\mathbf{E} \times \mathbf{Q} \times \mathbf{C} = E \times C$.

Consider an ANR-sequence of compact convenient ANR's $\mathbf{E} \times \mathbf{C} = (E_n \times C_n, r'_{nm})$ where $r'_{nm} = r_{nm} \times q'_{nm}\colon E_m \times C_m \to E_n \times C_n$. Then inj lim $\mathbf{E} \times \mathbf{C} =$ inj lim $\mathbf{E} \times$ inj lim $\mathbf{C} = E \times C$. Let for each n, $(Z_n; p'_n|Z_n, f'_n|Z_n)$ be the pull-back of $(B_n; p_n, f_n)$ where $p'_n\colon E_n \times C_n \to C_n$ and $f_n\colon E_n \times C_n \to E_n$ are projections. Thus $\mathbf{Z} = (Z_n;\ r'_{nm}|Z_m)$ is an inverse sequence of compact spaces with inj lim $\mathbf{Z} = Z$ and the limit maps of $p'|\mathbf{Z}$ and $f'|\mathbf{Z}$ are p' and f' respectively.

Now by induction, for each $n = 1, 2, 3, \ldots$ we will define a closed ANR-neighbourhood E'_n of Z_n in $E_n \times C_n$, numbers $\varepsilon_n > 0$ and $\delta_n > 0$ and an integer m such that (m, δ_n) is a lifting pair for (n, ε_n) with respect to $\mathbf{p}$. Also we require

(1) $r'_{nm}(E'_m) \subset E'_n$ where $r'_{nm} = r_{nm} \times q'_{nm}|E'_m$,

(2) inj lim $E' = Z$ where $\mathbf{E}' = (E'_n, r'_{nm})$,

(3) $d(f_m p'_m|E'_m, p_m f'_m|E'_m) < \delta_n$ and

(4) for each n, $d(p_n(e_n), f_n(c_n)) < \varepsilon_n$ implies $(e_n, c_n) \in E'_n$ where $(e_n, c_n) \in E_n \times C_n$.

For each n, let d_n be the composition $d_n\colon E_n \times C_n \to B_n \times B_n \to \mathbf{R}$ where d is the distance function and $\mathbf{R}$ is the set of reals. Note that $Z_n = d_n^{-1}(0)$. Choose a sequence of ε's, $0 < \ldots\ldots < \varepsilon_n < \varepsilon'_n < \varepsilon_{n-1} < \ldots$, and compact ANR-neighbourhoods E'_n of Z_n in $E_n \times C_n$ such that

$$Z_n \subset d_n^{-1}[0, \varepsilon_n) \subset \operatorname{int} E'_n \subset E'_n \subset d_n^{-1}[0, \varepsilon'_n).$$

Also we want $d_n^{-1}(x, y) < \varepsilon'_n$ to imply $d^m(q_{nm}(x), q_{nm}(y)) < \varepsilon_m$. Clearly $\mathbf{E}' = (E'_n, r'_{nm})$ is the required ANR-sequence.

To show that p' is a shape fibration, by Proposition 2.1, it is enough to show that $\mathbf{p}'$ has HLP.

By I of §2, we can assume that $\mathbf{p}$ has the stronger lifting property where (2) is replaced by (5). Let $(m, \delta_n = \varepsilon'_m)$ be the lifting pair of (n, ε_n). Let $h\colon X \to E'_m$ and $H\colon X \times I \to B_m$ be the maps such that

$$H_0 = p'_m h. \tag{5}$$

See Figure 1. By (3) and (5),

$$d(p_m f'_m h, f_m H_0) < \delta_n \tag{6}$$

Since (m, δ_n) is a lifting pair for (n, ε_n), there is a map $G': X \times I \to E_n$ such that

$$G'_0 = r_{nm} f'_m h \tag{7}$$

and

$$d(p_n G', q_{nm} f_m H) < \varepsilon_n. \tag{8}$$

Define $G: X \times I \to E'_n$ by $G(x, t) = (G'(x, t), q'_{nm}H(x, t))$ for $(x, t) \in X \times I$. Note that by (8) and (4) and $q_{nm} f_m = f_n q'_{nm}$, for every $(x, t) \in X \times I$, $G(x, t) \in E'_n$. Hence $G(X \times I) \subset E'_n$. Also by (8) and (6) $G_0 = (G'_0, q'_{nm}H_0) = (r_{nm} f'_m h, q'_{nm} p'_m h) = r'_{nm} h$ and $p'_n G = p'_n(G', q'_{nm}H) = q'_{nm}H$.

Thus $\mathbf{P}'$ has HLP, which shows that p' is a shape fibration. We will refer to $p' = p_*(f)$ as a shape fibration induced from p by f.

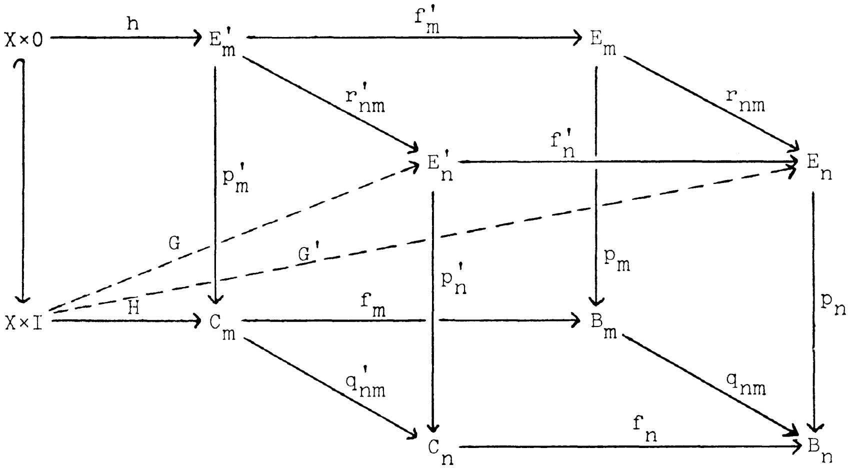

Figure 1

4. Fiber shape Equivalence. Analogous to fiber homotopy equivalence, we will define in this section the concept of a fiber shape equivalence. First we define a fiber morphism.

DEFINITION 4.1. A fiber morphism $F = (\mathbf{f}, \mathbf{h}): (\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ between two level maps of ANR-sequences is defined to be a pair of maps $\mathbf{f} = (\alpha, f_n): \mathbf{E} = (E_n, r_{nm}) \to \mathbf{E}' = (E'_n, r'_{nm})$ and $\mathbf{h} = (\beta, h_n): \mathbf{B} = (B_n, q_{nm}) \to \mathbf{B}' = (B'_n, q'_{nm})$ of ANR-sequences such that for every n and for every $\varepsilon > 0$ there is an index $n^* = F(n, \varepsilon)$ satisfying the following conditions:

(A) for all $m \geqq n^*$, $d(q'_{nm} p'_m f_m r_{\alpha(m)\ell}, q'_{nm} h_m q_{\beta(m)\ell} p_\ell) < \varepsilon$ where $\ell \geqq \mathrm{Max}(\alpha(m), \beta(m))$; and

(B) if $m' \geqq m$, there are homotopies $K: f_m r_{\alpha(m)\alpha(m')} \cong r'_{nm'} f_{m'}$ and $H: h_m q_{\beta(m)\beta(m')} \cong q'_{mm'} h_{m'}$ such that for every $t \in I$

(i) $d(q'_{nm} h_m q_{\beta(m)\beta(m')}, q'_{nm} H_t) < \varepsilon$ and

(ii) $d(q'_{nm} p'_m K_t r_{\alpha(m')\ell'}, q'_{nm} H_t q_{\beta(m')\ell'} p_{\ell'}) < \varepsilon$ where $\ell' \geqq \mathrm{Max}(\alpha(m'), \beta(m'))$.

REMARK 4.1. Cleary, $\mathbf{1_p} = (\mathbf{1_E}, \mathbf{1_B})$: $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}, \mathbf{p}, \mathbf{B})$ is a fiber morphism.

REMARK 4.2. Let E, E', B be compact ANR's, $p: E \to B$, $p': E' \to B$ be maps and $f: E \to E'$ be a map over B (i.e., $p'f = p$). Then the trivial morphism $\mathbf{f} = (f): E \to E'$ is a fiber morphism.

Now we will define an equivalence relation in the set of fiber morphisms between level maps of ANR-sequences.

DEFINITION 4.2. Two fiber morphisms $F = (\mathbf{f}, \mathbf{h})$, $G = (\mathbf{g}, \mathbf{k})$: $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ where $\mathbf{f} = (\alpha, f_n)$, $\mathbf{h} = (\beta, h_n)$, $\mathbf{g} = (\gamma, g_n)$, $\mathbf{k} = (\delta, k_n)$ are said to be equivalent (in symbols, $F \sim G$) if for every n and for every $\varepsilon > 0$ there is an index $n' = (F, G)(n, \varepsilon)$ with the following property: for every $m \geqq n'$ there is an index ℓ, $\ell \geqq \mathrm{Max}(\alpha(m), \beta(m), \gamma(m), \delta(m))$ and there are homotopies $L: f_m r_{\alpha(m)\ell} \cong g_m r_{\gamma(m)\ell}$ and $M: h_m q_{\beta(m)\ell} \cong k_m q_{\delta(m)\ell}$ such that for every $t \in I$, $d(q'_{nm} h_m q_{\beta(m)\ell}, q'_{nm} M_t) < \varepsilon$ and $d(q'_{nm} p'_m L_t, q'_{nm} M_t p_\ell) < \varepsilon$.

REMARK 4.3. Denote the set of all such ℓ by $(F, G)(m; n, \varepsilon)$. Clearly, if $\ell' \geqq \ell$, then $\ell' \in (F, G)(m; n, \varepsilon)$ since $p_\ell r_{\ell\ell'} = q_{\ell\ell'} p_{\ell'}$.

REMARK 4.4. If $F = (\mathbf{f}, \mathbf{h}) \sim G = (\mathbf{g}, \mathbf{k})$, then $\mathbf{f} \cong \mathbf{g}$ and $\mathbf{h} \cong \mathbf{k}$.

PROPOSITION 4.1. *The relation* $\sim$ *of Definition* 4.2 *is an equivalence relation.*

PROOF. Only transitivity requires a proof. Let $F = (\mathbf{f}, \mathbf{h})$, $F' = (\mathbf{f}', \mathbf{h}')$, $F'' = (\mathbf{f}'', \mathbf{h}'')$: $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ be morphisms of level maps such that $F \sim F'$ and $F' \sim F''$. For given n and $\varepsilon > 0$, let $\ell' = (F, F')(m; n, \varepsilon/2)$ and $\ell'' \in (F', F'')(m; n, \varepsilon/2)$. There are homotopies $L': f_m r_{\alpha(m)\ell'} \cong f'_m r_{\alpha'(m)\ell'}$, $M': h_m q_{\beta'(m)\ell'} \cong h'_m q_{\beta'(m)\ell''}$ and $L'': f'_m r_{\alpha'(m)\ell''} \cong f''_m r_{\alpha''(m)\ell''}$, M'': $h'_m q_{\beta'(m)\ell''} \cong h''_m q_{\beta''(m)\ell''}$ satisfying the required conditions. For $\ell \geqq \mathrm{Max}(\ell', \ell'')$, define the required homotopies $L: f_m r_{\alpha(m)\ell} \cong f''_m r_{\alpha''(m)\ell}$ and $M: h_m q_{\beta(m)\ell} \cong h''_m q_{\beta''(m)\ell}$ by composing homotopies L' with L'' and M' with M'' respectively.

REMARK 4.5. Let $F = (\mathbf{f}, \mathbf{h})$: $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ be a fiber morphism of level maps where $\mathbf{f} = (\alpha, f_n)$ and $\mathbf{h} = (\beta, h_n)$. Define an index function $\hat{\alpha} = \mathrm{Max}(\alpha, \beta): N \to N$ and maps $\hat{\mathbf{f}} = (\hat{\alpha}, \hat{f}_n)$, $\hat{\mathbf{h}} = (\hat{\alpha}, \hat{h}_n)$ (with the same index function) by $\hat{f}_n = f_n r_{\alpha(n)\hat{\alpha}(n)}$ and $\hat{h}_n = h_n q_{\beta(n)\alpha(n)}$ for every n. Then clearly, $\hat{F} = (\hat{\mathbf{f}}, \hat{\mathbf{h}})$: $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ is a fiber morphism of level maps and is equivalent to F.

By this remark, from now on we will always consider fiber morphisms with one index function.

Selecting appropriate indices, composing homotopies that are given by hypothesis and using well-known properties of ANR's, one can prove the following propositions. The proofs are complicated in details but the arguments are straightforward and so we omit them.

PROPOSITION 4.2. *The composition of two fiber morphisms of level maps is a fiber morphism of level maps. In other words, if* $F = (\mathbf{f}, \mathbf{h}):(\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ *and* $G = (\mathbf{g}, \mathbf{h})\colon (\mathbf{E}', \mathbf{p}', \mathbf{B}') \to (\mathbf{E}'', \mathbf{p}'', \mathbf{B}'')$ *are fiber morphisms of level maps of* ANR-*sequences, then* $GF = (\mathbf{gf}, \mathbf{kh})\colon (\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}'', \mathbf{p}'', \mathbf{B}'')$ *is a fiber morphism of level map of* ANR-*sequences.*

PROPOSITION 4.3. *Let* $F, F'\colon (\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ *and* $G, G'\colon (\mathbf{E}', \mathbf{p}', \mathbf{B}') \to (\mathbf{E}'', \mathbf{p}'', \mathbf{B}'')$ *be fiber morphisms of level maps. Then* $F \sim F'$ *implies* $GF \sim GF'$ *and* $G \sim G'$ *implies* $GF \sim G'F$.

We will define an equivalence relation in the set of level maps.

DEFINITION 4.3. Two level maps $(\mathbf{E}, \mathbf{p}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{p}', \mathbf{B}')$ are said to be equivalent (in symbols, $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \cong (\mathbf{E}', \mathbf{p}', \mathbf{B}')$) if there are fiber morphisms $F\colon (\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ and $G\colon (\mathbf{E}', \mathbf{p}', \mathbf{B}') \to (\mathbf{E}, \mathbf{p}, \mathbf{B})$ such that $GF \sim 1_{(\mathbf{E},\mathbf{p},\mathbf{B})}$ and $FG \sim 1_{(\mathbf{E}',\mathbf{p}',\mathbf{B})}$.

By Propositions 4.2 and 4.3 the following result is clear.

PROPOSITION 4.4. *The relation* $\cong$ *is an equivalence relation in the set of level maps of* ANR-*sequences.*

Now we are in a position to define the concept of fiber shape equivalence.

DEFINITION 4.4. Let E, E', B be compact metric spaces. Two shape fibrations $p\colon E \to B$ and $p'\colon E' \to B$ are said to be fiber shape equivalent if there are level maps $(\mathbf{E}, \mathbf{p}, \mathbf{B})$ and $(\mathbf{E}', \mathbf{p}', \mathbf{B}')$ with limit maps p and p' respectively such that $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \cong (\mathbf{E}', \mathbf{p}', \mathbf{B}')$.

The following theorem justifies Definition 4.4.

THEOREM 4.1. *Let* $p\colon E \to B$ *be a map between compact metric spaces. If* $(\mathbf{E}, \mathbf{p}, \mathbf{B})$ *and* $(\mathbf{E}', \mathbf{p}', \mathbf{B}')$ *are level maps of* ANR-*sequences with limit map* p, then $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \cong (\mathbf{E}', \mathbf{p}', \mathbf{B}')$.

First we will state an useful lemma from [11].

LEMMA M. (MARDEŠIĆ). *Let* $\mathbf{X} = (X_n, r_{nm})$ *be an* ANR-*sequence with* $\operatorname{inj\,lim} \mathbf{X} = (X, r_n)$ *and let* Y *be a compact* ANR. *Then the following assertions hold*:

(i) *for every* $\varepsilon > 0$ *and for every map* $f\colon X \to Y$ *there is an index* n^* *such* *that for each $n \geqq n^*$ there is a map $f_n: X_n \to Y$ with $d(f_n r_n, f) < \varepsilon$; and*

(ii) *if $\varepsilon > 0$ and f_n, $g_n: X_n \to Y$ are maps such that $d(f_n r_n, g_n r_n) < \varepsilon$, there exists $\hat{n} \geqq n$ such that $d(f_n r_{nm}, g_n r_{nm}) < \varepsilon$ for every $m \geqq \hat{n}$.*

PROOF OF THEOREM 4.1. For every $m = 1, 2, 3, \ldots$ we will select positive numbers $(\varepsilon_m, \delta_m, \tilde{\varepsilon}_m, \tilde{\delta}_m)$ where $\varepsilon_1 = 1$, $\varepsilon_m \to 0$ as $m \to \infty$, $\delta_m \leqq \varepsilon_m$ for $m = 1, 2, \ldots$, and by induction on m, we will construct maps $\mathbf{f} = (\alpha, f_n)$: $\mathbf{E} \to \mathbf{E}'$ and $\mathbf{h} = (\alpha, h_n): \mathbf{B} \to \mathbf{B}'$ of ANR-sequences such that the following conditions are satisfied:

(a) for every n and for every $\varepsilon > 0$ there is an index n^* such that for all $m \geqq n^*$, $\varepsilon_m \in \Lambda(q_{nm}, \varepsilon)$; and

(b) (i) $d(r'_m, f_m r_{\alpha(m)}) < \tilde{\delta}_m/2$, $d(q'_m, h_m q_{\alpha(m)}) < \delta_m/2$

(ii) $d(p'_m f_m, h_m p_{\alpha(m)}) < \delta_m$

(iii) for every $m' \geqq m$ there are $\tilde{\varepsilon}_m$-and $\varepsilon_m/3$-homotopies K: $f_m r_{\alpha(m)\alpha(m')} \cong r'_{mm'} f_{m'}$ and $H: h_m q_{\alpha(m)\alpha(m')} \cong q'_{mm'} h_{m'}$ respectively such that for every $t \in I$, $d(p'_m K_t, H_t p_{\alpha(m')}) < \varepsilon_m$.

Since $\varepsilon_m \in \Lambda(q_{nm}, \varepsilon)$, by (ii) and (iii), for every $m \geqq n^*$ $d(q'_{nm} p'_m f_m, q'_{nm} h_m p_{\alpha(m)}) < \varepsilon$ and for every $t \in I$, $d(q'_{nm} p'_m K_t, q'_{nm} H_t p_{\alpha(m)}) < \varepsilon$. Hence $F = (\mathbf{f}, \mathbf{h}): (\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ is a fiber morphism of level maps.

Let $m = 1$. Select $\varepsilon_1 = 1$, $\delta_1 \in \Gamma(B'_1, \varepsilon_1/3)$, $\tilde{\varepsilon}_1 \in \Lambda(p'_1, \delta_1/3)$ and $\tilde{\delta}_1 \in \Gamma(E'_1, \tilde{\varepsilon}_1/2)$. By lemma M(i) there is an index n_1 and there are maps $\tilde{f}_1: E_{n_1} \to E'_1$ and $\tilde{h}_1: B_{n_1} \to B'_1$ such that

$$d(r'_1, \tilde{f}_1 r_{n_1}) < \tilde{\delta}_1/2 \text{ and } d(q'_1, \tilde{h}_1 q_{n_1}) < \delta_1/2. \tag{1}$$

By the choice of $\tilde{\delta}_1$ and by $p'_1 r'_1 = q'_1 p$, $q'_{n_1} p = p_{n_1} r_{n_1}$

$$d(p'_1 \tilde{f}_1 r_{n_1}, \tilde{h}_1 p_{n_1} r_{n_1}) < \delta_1. \tag{2}$$

By the lemma M(ii), there is an index $\alpha(1) \geqq n_1$ say, such that

$$d(p'_1 \tilde{f}_1 r_{n_1\alpha(1)}, \tilde{h}_1 q_{n_1\alpha(1)} P_{\alpha(1)}) < \delta_1. \tag{3}$$

Write $\tilde{f}_1 r_{n_1\alpha(1)} = f_1$ and $\tilde{h}_1 q_{n_1\alpha(1)} = h_1$. Now by (1) and (3)

$$d(r'_1, f_1 r_{\alpha(1)}) < \tilde{\delta}_1/2, \ d(q'_1, h_1 q_{\alpha(1)}) < \delta_1/2 \tag{4}$$

and

$$d(p'_1 f_1, h_1 p_{\alpha(1)}) < \delta_1. \tag{5}$$

Let $m = 2$. Select ε_2, δ_2, $\tilde{\varepsilon}_2$ and $\tilde{\delta}_2$ as follows: $\varepsilon_2 \in \Lambda(q'_{12}, \delta_1/2^2)$, $\delta_2 \in \Gamma(B'_2, \varepsilon_2/3)$, $\tilde{\varepsilon}_2 \in \Lambda(p'_2, \delta_2/3) \cap \Lambda(r'_{12}, \tilde{\delta}_1/2^2)$ and $\tilde{\delta}_2 \in \Gamma(E'_2, \tilde{\varepsilon}_2/2)$. As in the case $m = 1$, by the lemma M(i), there is an index $n_2 \geqq \alpha(1)$ and there are maps $\tilde{f}_2: E_{n_2} \to E'_2$ and $\tilde{h}_2: B_{n_2} \to B'_2$ such that

$$d(r'_2, \tilde{f}_2 r_{n_2}) < \tilde{\delta}_2/2, d(q'_2, \tilde{h}_2 q_{n_2}) < \delta_2/2 \tag{6}$$

and

(7) $$d(p_2'\tilde{f}_2,\ \tilde{h}_2 p_{n_2}) < \delta_2.$$

By the choices of $\tilde{\delta}_2$ and δ_2 and by the equalities $r_{12}'r_2' = r_1'$, $q_{12}'q_2' = q_1'$, (6) implies

(8) $$d(r_1',\ r_{12}'\tilde{f}_2 r_{n_2}) < \tilde{\delta}_1^2/2^2 \text{ and } d(q_1',\ q_{12}'\tilde{h}_2 q_{n_2}) < \delta_1/2^2.$$

By (4) and (8),

(9) $$\begin{aligned} d(f_1 r_{\alpha(1)},\ r_{12}'\tilde{f}_2 r_{n_2}) &< \delta_1/2 + \tilde{\delta}_1/2^2 \\ d(h_1 q_{\alpha(1)},\ q_{12}'\tilde{h}_2 q_{n_2}) &< \delta_1/2 + \delta_1/2^2. \end{aligned}$$

Since $r_{\alpha(1)} = r_{\alpha(1)n_2} r_{n_2}$ and $q_{\alpha(1)} = q_{\alpha(1)n_2} q_{n_2}$,

(10) $$\begin{aligned} d(f_1 r_{\alpha(1)n_2} r_{n_2},\ r_{12}'\tilde{f}_2 r_{n_2}) &< \tilde{\delta}_1 \\ d(h_1 q_{\alpha(1)n_2} q_{n_2},\ q_{12}'\tilde{h}_2 q_{n_2}) &< \delta_1. \end{aligned}$$

By the lemma M(ii), there is an index n_2^* such that for all indices $\alpha(2) \geqq n_2^*$ we have

(11) $$d(f_1 r_{\alpha(1)\alpha(2)},\ r_{12}'\tilde{f}_2 r_{n_2\alpha(2)}) < \tilde{\delta}_1$$

and

(12) $$d(h_1 q_{\alpha(1)\alpha(2)},\ q_{12}'\tilde{h}_2 q_{n_2\alpha(2)}) < \delta_1.$$

Write $\tilde{f}_2 r_{n_2\alpha(2)} = f_2$ and $\tilde{h}_2 q_{n_2\alpha(2)} = h_2$. Now, by the choice of $\tilde{\delta}_1$ and δ_1, there are $\tilde{\varepsilon}_1/2$- and $\varepsilon_1/3$-homotopies $K\colon f_1 r_{\alpha(1)\alpha(2)} \cong r_{12}' f_2$ and $H\colon h_1 q_{\alpha(1)\alpha(2)} \cong q_{12}' h_2$. Hence

(13) $$d(f_1 r_{\alpha(1)\alpha(2)},\ K_t) < \tilde{\varepsilon}_1$$

and

(14) $$d(h_1 q_{\alpha(1)\alpha(2)},\ H_t) < \varepsilon_1/3$$

for every $t \in I$. By the choice of $\tilde{\varepsilon}_1$,

(15) $$d(p_1' f_1 r_{\alpha(1)\alpha(2)}, p_1' K_t) < \delta_1/3 < \varepsilon_1/3$$

for every $t \in I$. Therefore by (14), for every $t \in I$,

(16) $$d(h_1 q_{\alpha(1)\alpha(2)} p_{\alpha(2)},\ H_t p_{\alpha(2)}) < \varepsilon_1/3.$$

By (5), we have

(17) $$d(p_1' f_1 r_{\alpha(1)\alpha(2)},\ h_1 p_{\alpha(1)} r_{\alpha(1)\alpha(2)}) < \delta_1 < \varepsilon_1/3.$$

Since $p_{\alpha(1)} r_{\alpha(1)\alpha(2)} = q_{\alpha(1)\alpha(2)} p_{\alpha(2)}$, by (15), (16), and (17)

(18) $$d(p_1' K_t,\ H_t p_{\alpha(2)}) < \varepsilon_1$$

for every $t \in I$.

Let $m \geqq 3$. Select $\varepsilon_m \in \bigcap_{n=1}^{m-1} \Lambda(q_{nm}',\ \delta_n/2^{m-n+1})$, $\delta_m \in \Gamma(B_m',\ \varepsilon_m/3)$, $\tilde{\varepsilon}_m \in$

$\Lambda(p'_m, \delta_m/3) \cap \bigcap_{n=1}^{m-1} \Lambda(r'_{nm}, \tilde{\delta}_n/2^{m-n+1})$ and $\tilde{\delta}_m \in \Gamma(E'_m, \tilde{\varepsilon}_m/2)$. The rest of the construction follows as above.

CONSTRUCTION OF G. For every $n = 1, 2, 3, \ldots$, similar to the numbers $(\varepsilon_n, \delta_n, \tilde{\varepsilon}_n, \tilde{\delta}_n)$, we can select numbers $(\lambda_n, \mu_n, \tilde{\lambda}_n, \tilde{\mu}_n)$ satisfying the following additional condition:

$$\text{(c)} \qquad \begin{aligned} &\lambda_{\alpha(m)} \in \Lambda(h_m, \delta_m/2),\ \mu_{\alpha(m)} \in \Gamma(B_{\alpha(m)}, \lambda_{\alpha(m)}/3), \\ &\tilde{\lambda}_{\alpha(m)} \in \Lambda(f_m, \tilde{\delta}_m/2) \ \text{ and } \ \tilde{\mu}_{\alpha(m)} \in \Gamma(E_{\alpha(m)}, \tilde{\lambda}_{\alpha(m)}/2). \end{aligned}$$

Similar to the maps $\mathbf{f}$, $\mathbf{h}$, we can construct maps $\mathbf{q}\colon \mathbf{E}' \to \mathbf{E}$ and $\mathbf{k}\colon \mathbf{B}' \to \mathbf{B}$ of ANR-sequences such that $G = (\mathbf{q}, \mathbf{k})\colon (\mathbf{E}', \mathbf{p}', \mathbf{B}') \to (\mathbf{E}, \mathbf{p}, \mathbf{B})$ is a fiber morphism of level maps. See Figure 2. To show that $FG \sim 1_{\mathbf{p}'}$, for given n and $\varepsilon > 0$, let n' be such that for all $m \geqq n'$, $\varepsilon_m \in \Lambda(q'_{nm}, \varepsilon)$. By construction of F and G we have

$$\text{(13)} \qquad d(r'_m, f_m r_{\alpha(m)}) < \tilde{\delta}_m/2,\ d(q'_m, h_m q_{\alpha(m)}) < \delta_m/2$$

and

$$\text{(14)} \qquad \begin{aligned} &d(r_{\alpha(m)}, g_{\alpha(m)} r'_{\beta\alpha(m)}) < \tilde{\mu}_{\alpha(m)}/2, \\ &d(q_{\alpha(m)}, k_{\alpha(m)} q'_{\beta\alpha(m)}) < \mu_{\alpha(m)}/2. \end{aligned}$$

By the choices of $\tilde{\mu}_{\alpha(m)}$ and $\mu_{\alpha(m)}$

$$\text{(15)} \qquad \begin{aligned} &d(q'_{m\beta\alpha(m)} q'_{\beta\alpha(m)}, h_m k_{\alpha(m)} q'_{\beta\alpha(m)}) < \delta_m, \\ &d(r'_{m\beta\alpha(m)} r'_{\beta\alpha(m)}, f_m g_{\alpha(m)} r'_{\beta\alpha(m)}) < \tilde{\delta}_m. \end{aligned}$$

By lemma M(ii), there is an index m^* such that for all $\ell \geqq m^*$

$$\text{(16)} \qquad \begin{aligned} &d(r'_{m\ell}, f_m g_{\alpha(m)} r'_{\beta\alpha(m)\ell}) < \tilde{\delta}_m, \\ &d(q'_{m\ell}, h_m k_{\alpha(m)} q'_{\beta\alpha(m)\ell}) < \delta_m. \end{aligned}$$

By the choices of δ_m and $\tilde{\delta}_m$ there are $\tilde{\varepsilon}_m/2$- and $\varepsilon_m/3$-homotopies $L\colon r'_{m\ell} \cong f_m g_{\alpha(m)} r'_{\beta\alpha(m)\ell}$ and $M\colon q'_{m\ell} \cong h_m k_{\alpha(m)} q'_{\beta\alpha(m)\ell}$ such that

$$\text{(17)} \qquad d(r'_{m\ell}, L_t) < \tilde{\varepsilon}_m/2 \text{ and } d(q'_{m\ell}, M_t) < \varepsilon_m/3$$

for every $t \in I$. But, by the choice of $\tilde{\varepsilon}_m$, $d(q'_{nm} p'_m L_t, q'_{nm} M_t p'_\ell) < \varepsilon$ for every $t \in I$. Since this is true for all $m \geqq n'$, $FG \sim 1_{\mathbf{p}'}$.

Now, if GF is not equivalent to $1_{\mathbf{p}}$, then similar to the fiber morphism G we can construct a fiber morphism $F' = (\mathbf{f}', \mathbf{h}')\colon (\mathbf{E}, \mathbf{p}, \mathbf{B}) \to (\mathbf{E}', \mathbf{p}', \mathbf{B}')$ satisfying the conditions (a), (b), (c) and such that $GF' \sim 1_{\mathbf{p}}$. From these conditions it is clear that $F \sim F'$.

Note that $(FGF')\cdot G \sim FG \sim 1_{\mathbf{p}'}$ and $G\cdot(FGF') \sim GF' \sim 1_{\mathbf{p}}$ which shows that $(\mathbf{E}, \mathbf{p}, \mathbf{B}) \cong (\mathbf{E}', \mathbf{p}', \mathbf{B}')$.

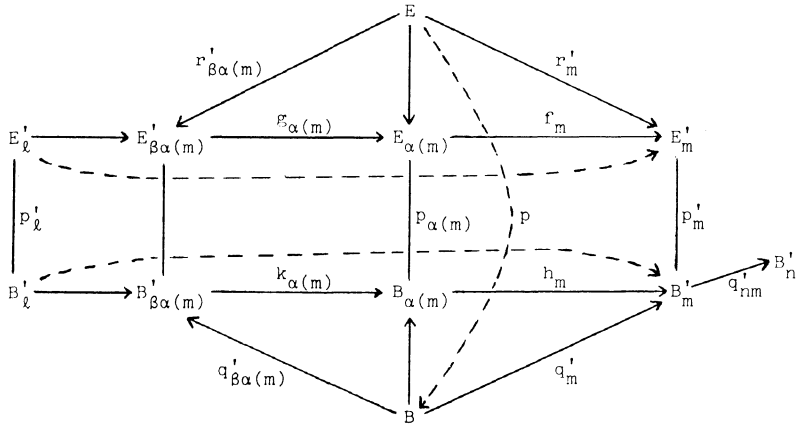

Figure 2

EXAMPLE. In the plane $\mathbf{R}^2$, let c be the closure of the diagram of the function $y = \sin(\pi/x)$ for $0 < x \leqq 1$ and let ℓ be a simple arc with endpoints $(0, -1)$ and $(1, 0)$ such that its interior lies in $\mathbf{R}^2 - c$. Denote the resulting space by W. Let $A = \{(0, y) \,|\, |y| \leqq 1\}$. It is known that the quotient map $p\colon W \to W/A \approx S^1$ is a shape fibration. The identity map $p' = 1_{S^1}\colon W/A \to W/A$ is also a shape fibration. It is very easy to see that p and p' are fiber shape equivalent.

Consider annuli $E_1 \supset E_2 \supset \cdots$ which are neighbourhoods of W in $\mathbf{R}^2$ shrinking to W. For every n, there is a retract $p_n\colon E_n \to W/A$ with $p_n|E_{n+1} = p_{n+1}$. Then $\mathbf{p} = (p_n)\colon \mathbf{E} \to W/A$ is a level map. Let $\mathbf{p}' = (p' = 1_{S^1})\colon W/A \to W/A$ be a trivial level map.

Let $F = (\mathbf{f})\colon \mathbf{p}' \to \mathbf{p}$ be a fiber morphism of level maps such that for each n, $f_n\colon S^1 \to E_n$ is a simple closed curve, $f_n \simeq f_{n+1}$ in E_n and for every $\varepsilon > 0$ there is an index $n^* \geqq n$ such that for $\ell \geqq m \geqq n^*$, $d(p_n H_t, 1_{S^1}) < \varepsilon$ for every $t \in I$ where $H\colon f_\ell \simeq f_m$ in E_n.

Let $G = (\mathbf{g})\colon \mathbf{p} \to \mathbf{p}'$ be a fiber morphism of level maps such that for each n, $g_n\colon E_n \to S^1$ is a retract of $f_n(S^1)$. Clearly, $FG \sim 1_{\mathbf{p}}$ and $GF \sim 1_{\mathbf{p}'}$.

5. Strongly equivalent maps induce equivalent shape fibrations. In this section we will define a notion called 'strong equivalence' and prove the main result that two strongly equivalent maps induce fiber shape equivalent shape fibrations.

Note that in this section we will only consider ANR-sequences with bonding maps as inclusions.

DEFINITION 5.1. Two level maps $\mathbf{f}, \mathbf{g}\colon \mathbf{C} = (C_n, q'_{nm}) \to \mathbf{B} = (B_n, q_{nm})$ of ANR-sequences are said to be strongly equivalent (in symbols, $\mathbf{f} \cong \mathbf{g}$)

if for every n there is $n^* \geqq n$ such that for all $m \geqq n^*$ there is an homotopy $H^{mn} = H\colon C_m \times I \to B_n$ with $H_0 = f_n q'_{nm} = q_{nm} f_m H_1 = g_n q'_{nm} = q_{nm} g_m$ and if for $n' \geqq n$, $m' \geqq m$, $H'^{mn'} = H'\colon f_{n'} q'_{n'm'} \cong g_{n'} q'_{n'm'}$, then there is a homotopy $G\colon C_{m'} \times I \times I \to B_n$ such that $G_0 = q_{nn'} H'$, $G_1 = H(q'_{mm'} \times 1_I)$ and for every $x \in C_{m'}$, $t \in I$, $G(x, 0, t) = q_{nm'} f_{m'}(x)$ and $G(x, 1, t) = q_{nm'} g_{m'}(x)$.

By choosing appropriate indices, composing homotopies one can prove the following proposition.

PROPOSITION 5.2. *The relation $\cong$ is an equivalence relation in the set of level maps of* ANR-*sequences.*

DEFINITION 5.2. Two maps $f, g\colon C \to B$ between compact metric spaces are said to be strongly equivalent (in symbols, $f \sim g$) if there are level maps $\mathbf{f}, \mathbf{g}\colon \mathbf{C} \to \mathbf{B}$ of ANR-sequences with limit maps f and g respectively and $\mathbf{f} \cong \mathbf{g}$.

THEOREM 5.1. *The relation $\sim$ is an equivalence relation.*

First we will prove two useful lemmas.

LEMMA 5.1. *Let $\mathbf{g}, \mathbf{g}'\colon \mathbf{C} \to \mathbf{B}$ be level maps of* ANR-*sequences with limit map g. Then $\mathbf{g} \cong \mathbf{g}'$.*

PROOF. Consider $C \times I \subset Q \times I \approx Q$. Let $H\colon C \times I \to B$ be the constant homotopy $H(c, t) = g(c)$ for all $c \in C$, $t \in I$. By induction, for each $n = 1, 2, 3, \ldots$ we can find an index $\hat{n} \geqq n$ and an homotopy $H^n\colon C_{\hat{n}} \times I \to B_n$ which is an extension of H such that $H^n\colon g_n r_{n\hat{n}} \cong g'_n r_{n\hat{n}}$ and $H^n | C_{n+1} \times I = H^{n+1}$. By reindexing we get $\mathbf{g} \cong \mathbf{g}'$.

LEMMA 5.2 *Let f, g, $h\colon C \to B$ be maps between compact metric spaces, $\mathbf{f}, \mathbf{g}\colon \mathbf{C} = (C_n, r_{nm}) \to \mathbf{B} = (B_n, q_{nm})$ and $\mathbf{g}', \mathbf{h}\colon \mathbf{C}' = (C'_n, r'_{nm}) \to B' = (B'_n, q'_{nm})$ be level maps of* ANR-*sequences such that* inj lim $\mathbf{f} = f$, inj lim $g =$ inj lim $\mathbf{g}' = g$, inj lim $\mathbf{h} = h$, $\mathbf{f} \cong \mathbf{g}$ *and* $\mathbf{g}' \cong \mathbf{h}$. *Then there are* ANR-*sequences $\tilde{\mathbf{C}}$, $\tilde{\mathbf{B}}$ and level maps $\tilde{\mathbf{f}}$, $\tilde{\mathbf{g}}$, $\tilde{\mathbf{g}}'$, $\tilde{\mathbf{h}}\colon C \to B$ such that* inj lim $\tilde{\mathbf{f}} = f$, inj lim $\tilde{\mathbf{g}} =$ inj lim $\tilde{\mathbf{g}}' = g$, inj lim $\tilde{\mathbf{h}} = h$, $\tilde{\mathbf{f}} \cong \tilde{\mathbf{g}}$ *and* $\tilde{\mathbf{g}}' \cong \tilde{\mathbf{h}}$.

PROOF. Let $B_1 \bigcup B'_1$ be disjoint union. For each $b \in B$ identify $b \in B_1$ to $b \in B'_1$. Let the quotient space be denoted by $B_1 \bigcup_B B'_1$. Note that for each n, $B_{n+1} \bigcup_B B'_{n+1} \subset B_n \bigcup_B B'_n$. Embed $B_1 \bigcup_B B'_1$ in Q. Choose a decreasing sequence $\tilde{\mathbf{B}}$ of compact ANR-neighbourhoods $\tilde{B}_n$ such that $\bigcap \tilde{B}_n = B$. For every n there is $n' \geqq n$ such that $B_{n'} \bigcup_B B'_{n'} \subset \operatorname{int} \tilde{B}_n$.

Now, consider an ANR-sequence $\mathbf{C} \times \mathbf{C}' = \{C_n \times C'_n, r_{nm} \times r'_{nm}\}$. Without loss of generality we may assume that for each n, $C_n \times C'_n$ is a compact convenient ANR. Let $\Delta(C) = \{(c, c) \in C \times C\} \subset C_n \times C'_n$ be the diagonal. Since $\Delta(c)$ is a closed subspace of $C_n \times C'_n$, for each n, we can select a decreasing sequence $\tilde{\mathbf{C}}$ of compact ANR-neighbourhoods $\tilde{C}_n$ of $\Delta(C)$ in $C_n \times C'_n$ such that for each n, $\tilde{C}_n \subset \operatorname{int} C_n \times C'_n$ and $\bigcap \tilde{C}_n = \Delta C \approx C$.

For each n, let $\pi_n: C_n \times C'_n \to C_n$ and $\pi'_n: C_n \times C'_n \to C'_n$ be the projections. We will denote $\pi_n \mid \tilde{C}_n$ by π_n and $\pi'_n \mid \tilde{C}_n$ by π'_n. Clearly, by reindexing if necessary, $\mathbf{f}\pi$, $\mathbf{g}\pi$, $\mathbf{g}'\pi'$, $\mathbf{h}\pi'$,: $\mathbf{C} \to \mathbf{B}$ are level maps of ANR-sequences with inj lim $\mathbf{f}\pi = f$, inj lim $\mathbf{g}\pi$ = inj lim $\mathbf{g}'\pi' = g$, inj lim $\mathbf{h}\pi' = h$, $\mathbf{f}\pi \cong \mathbf{g}\pi$ and $\mathbf{g}'\pi' \cong \mathbf{h}\pi'$.

Proof of Theorem 5.1. Only transitivity requires a proof. Let f, g, h: $C \to B$ be maps between compact metric spaces such that $f \sim g$ and $g \sim h$. Then by Lemma 5.2 there are level maps of ANR-sequences $\mathbf{f}$, $\mathbf{g}$, $\tilde{\mathbf{g}}$, $\mathbf{h}$: $\mathbf{C} \to \mathbf{B}$ such that inj lim $\mathbf{f} = f$, inj lim $\mathbf{g}$ = inj lim $\tilde{\mathbf{g}} = g$, inj lim $\mathbf{h} = h$ and $\mathbf{f} \cong \mathbf{g}$, $\tilde{\mathbf{g}} \cong \mathbf{h}$. By Lemma 5.1, $\mathbf{g} \cong \tilde{\mathbf{g}}$ and by Proposition 5.2, then $\mathbf{f} \cong \mathbf{h}$ which proves the theorem.

The following proposition shows that this equivalence relation is a generalization of a homotopy relation.

Proposition 5.3. *Two homotopic maps* $\mathbf{f}$, $\mathbf{g}$: $C \to B$ *between compact metric spaces are strongly equivalent.*

Proof. Let $H: C \times I \to B$ be the homotopy with $H_0 = f$ and $H_1 = g$. Embed $C \times I$ in $Q \times I \approx Q$ and B in Q. Select a decreasing sequence $\mathbf{B}$ of compact ANR-neighbourhoods B_n of B in Q with $\bigcap B_n = B$. Let $\tilde{H}$: $Q \to Q$ be an extension of H. For each n, we can select a compact ANR-neighbourhood $C_n \times I$ of $C \times I$ in $\tilde{H}^{-1}(B_n)$ such that $C_{n+1} \times I \subset C_n \times I$ and $\bigcap(C_n \times I) = C \times I$. Write $H_n = \tilde{H}|C_n \times I$. Note that for every n, $H_n|C \times I = H$. Write $f_n = H_n(0)$ and $g_n = H_n(1)$. Thus $\mathbf{f} = (f_n) \cong \mathbf{g} = (g_n)$, showing that f and g are strongly equivalent.

Remark 5.1. Two strongly equivalent maps may not be homotopic. Let X be the $\sin(1/x)$ curve in $\mathbf{R}^2$ with domain $(0, 1]$ and let A be the closure of X. Let $C = \{*\}$ be a space with one point. Define f and g from C to X such that $f(*) = a \in A - X$ and $g(*) = b \in X$. Choose a decreasing sequence $\mathbf{A} = (A_n)$ of compact ANR-neighbourhoods of A in $\mathbf{R}^2$ such that for each n, A_n is contractible in itself and $\bigcap A_n = A$. Hence for every n, there is a path from a to b in A_n which shows that f and g are strongly equivalent. But there is no path from a to b in A. So f and g are not homotopic.

Now, we will prove the main theorem of this paper.

Theorem 5.2. *Let E, B, C be compact metric spaces, p: $E \to B$ be a shape fibration and f, g: $C \to B$ be strongly equivalent maps. Then the shape fibrations $p' = p_*(f)$ and $p'' = p_*(g)$ induced from p by f and g respectively are fiber shape equivalent.*

PROOF. Let $\mathbf{p}: \mathbf{E} \to \mathbf{B}$, $\mathbf{f}, \mathbf{g}: \mathbf{C} \to \mathbf{B}$ be level maps of ANR-$e$equences with limit maps p, f, g respectively and $\mathbf{f} \cong \mathbf{g}$. For each n, let $(Z'_n; f'_n, p'_n)$ and $(Z''_n; g''_n, p''_n)$ be the pull-backs of $(B_n; f_n, p_n)$ and (B_n, g_n, p_n) respectively. Then $\mathbf{Z}' = (Z'_n, r'_{n,n+1})$ and $\mathbf{Z}'' = (Z''_n, r''_{n,n+1})$ are inverse sequences of compact metric spaces with inj lim $\mathbf{Z}' = (Z', f', p')$ and inj lim $\mathbf{Z}'' = (Z'', g'', p'')$, the pullbacks of $(B; f, p)$ and $(B; g, p)$ respectively.

Now by induction, for each $n = 1, 2, 3, \ldots$ we can select positive numbers ε_n and η_n ($\eta_n < \varepsilon_n$), compact ANR-neighbourhoods E'_n and E''_n of Z'_n and Z''_n respectively such that

(1) $r'_{n,n+1}(E'_{n+1}) \subset E'_n$, $r''_{n,n+1}(E''_{n+1}) \subset E''_n$,

(2) inj lim $\mathbf{E}' = Z'$, inj lim $\mathbf{E}'' = Z''$,

(3) $d(p_n f'_n|E'_n, f_n p'_n|E'_n) < \varepsilon_n$, $d(p_n g''_n|E''_n, g_n p''_n|E''_n) < \varepsilon_n$ where $f'_n, g''_n: E_n \times C_n \to E_n$ and $p'_n, p''_n: E_n \times C_n \to C_n$ are the projections

(4) for $e', e'' \in E_n$ and $x, y \in C_n$, $d(p_n(e'), f_n(x)) < \eta_n$ implies $(e', x) \in E'_n$ and $d(p_n(e''), g_n(y)) < \eta_n$ implies $(e'', y) \in E''_n$,

(5) as $n \to \infty$, $\varepsilon_n \to 0$ (so does η_n) and $\varepsilon_{n+1} \in \Lambda(q_{n,n+1}, \eta_n)$.

Selections of the neighbourhoods are similar to those that are shown in the proof of Theorem 3.1.

Now we will define fiber morphisms $\mathbf{h}': \mathbf{E}' \to \mathbf{E}''$ and $\mathbf{h}'': \mathbf{E}'' \to \mathbf{E}'$ such that $(\mathbf{h}'', \mathbf{1_C}) \cdot (\mathbf{h}', \mathbf{1_C}) \sim \mathbf{1}_{\mathbf{p}'}$ and $(\mathbf{h}', \mathbf{1_C}) \cdot (\mathbf{h}'', \mathbf{1_C}) \sim \mathbf{1}_{\mathbf{p}''}$.

For the fiber morphism $\mathbf{h}'': \mathbf{E}'' \to \mathbf{E}'$, since $\mathbf{p}$ has AHLP and $\varepsilon_n \to 0$ as $n \to \infty$ every pair (n, ε) has a lifting pair (m, ε_m). Let (i, ε_i) be a lifting pair for (n, η_n) and (j, ε_j) be a lifting pair for (i, η_i). Since $\mathbf{f} \cong \mathbf{g}$, for j there is an index m and a homotopy $H': C_m \times I \to B_j$ such that $H'_0 = f_j q'_{jm}$ and $H'_1 = g_j q'_{jm}$. Define a homotopy $H^m: E''_m \times I \to B_j$ by $H^m = H' \cdot (p''_m \times 1_I)$. Let $h^m: E''_m \to E_j$ be the composition $h^m = r_{jm} g''_m$. Consider Figure 3.

$$\begin{tikzcd}
E_i \arrow[d, "p_i"] & E_j \arrow[l, "r_{ij}"'] \arrow[d, "p_j"] & E''_m \times 1 \arrow[l, "h^m"'] \arrow[d, hook] \\
B_i & B_j \arrow[l, "q_{ij}"] & E''_m \times I \arrow[l, "H^m"] \arrow[ull, dashed, "G^m"']
\end{tikzcd}$$

Figure 3

By (3) and the choice of ε_m, $d(p_jh^m, H_1^m) < \varepsilon_j$. Since (j, ε_j) is a lifting pair for (i, η_i), there is a homotopy $G^m: E''_m \times I \to E_i$ such that

(6) $$G_1^m = r_{ij}h^m$$

and

(7) $$d(p_iG^m, q_{ij}H^m) < \eta_i.$$

By the choice of η_i and (4), there is a map $\tilde{h}''_n = (G_0^m, q'_{im}p''_m): E''_m \to E'_i$. Write $h''_n = r'_{ni}\tilde{h}''_n$.

Similarly for $n' \geqq n$, there is a map $h''_{n'} = r'_{n'i'}\tilde{h}''_{n'}: E''_{m'} \to E'_{n'}$. Now we have to show that for every $\varepsilon > 0$ there is a homotopy $G: r'_{nn'}h''_{n'} \cong h''_n r''_{mm'}$ such that for every $t \in I$, $d(p'_nG_t, q'_{nm'}p''_{m'}) < \varepsilon$.

Since $\mathbf{f} \cong \mathbf{g}$, there is a homotopy $\hat{H}: q_{jj'}H'' \cong H' \cdot (q'_{mm'} \times 1_I)$ where $H'': f_jq'_{j'm'} \cong g_jq'_{j'm'}$, such that $\hat{H}(x, 0, t) = f_jq'_{jm'}(x)$ and $\hat{H}(x, 1, t) = g_jq'_{jm'}(x)$ for every $x \in C'_m$ and $t \in I$. Write $H = q_{ij}\hat{H}(p''_m \times 1_I \times 1_I)$. Let $g: E''_{m'} \times J \to E_i$ be defined by

$$g(x, s, t) = \begin{cases} G^m(r''_{mm'} \times 1_I)(x, s) & t = 1 \\ r_{im'}g''_{m'}(x) & s = 1 \\ r_{ii'}G^{m'}(x, s) & t = 0 \end{cases}$$

where $J = I \times 1 \cup 1 \times I \cup I \times 0$. Note that for $t = 1$,

$$\begin{aligned} H|E''_{m'} \times I \times 1 &= q_{ij}\hat{H}(p''_{m'} \times 1_I \times 1_I)|E''_{m'} \times I \times 1 \\ &= q_{ij}H'(q'_{mm'} \times 1_I)(p''_{m'} \times 1_I) \\ &= q_{ij}H^m(r''_{mm'} \times 1_I), \end{aligned}$$

for $s = 1$,

$$\begin{aligned} H|E''_{m'} \times 1 \times I &= q_{ij}\hat{H}|p''_{m'}(E''_{m'}) \times 1 \times I \\ &= q_{im'}g_{m'}p''_{m'} \end{aligned}$$

and for $t = 0$, $H|E''_{m'} \times I \times 0 = q_{ij'}H''(p''_{m'} \times 1_I) = q_{ij}H^{m'}$.

Hence, by the choice of $\varepsilon_{m'}$ and $\eta_{i'}$,

(8) $$d(p_ig, H|E''_{m'} \times J) < \eta_i.$$

Since (i, ε_i) is a lifting pair for (n, η_n), there is a homotopy $\hat{G}: E''_{m'} \times I \times I \to E_n$ such that $\hat{G}|E''_{m'} \times J = r_{ni}g$ and $d(p_n\hat{G}, q_{ni}H) < \eta_n$. Write $G = (\hat{G}, q'_{nm'}P''_{m'} \times 0 \times 1_I)$. We have to show that for every $(x, t) \in E''_{m'} \times I$, $d(p_n\hat{G}(x, 0, t), f_nq'_{nm'}p''_{m'}(x, 0, t)) < \eta_n$ which by the choice of η_n implies that $(\hat{G}(x, 0, t), q'_{nm'}p''_{m'}(x, 0, t)) \in E'_n$.

For $(x, t) \in E''_{m'} \times I$, we have $d(p_n\hat{G}(x, 0, t), q_{ni}H(x, 0, t) < \eta_n$. By the definition of H, $d(p_n\hat{G}(x, 0, t), q_{ni}q_{ij}\hat{H}(p''_{m'}(x), 0, t) < \eta_n$. This implies that $d(p_n\hat{G}(x, 0, t), q_{nm'}f_{m'}p''_{m'}(x, 0, t)) < \eta_n$. Hence, $d(p_n\hat{G}(x, 0, t),$

$f_n q'_{nm'} p''_{m'}(x, 0, t)) < \eta_n$. This shows that $(\hat{G}(x, 0, t), q'_{nm'} p''_{m'}(x, 0, t)) \in E'_n$ for every $(x, t) \in E''_{m'} \times I$. So, G is well-defined.

Now,

$$\begin{aligned} G_0 &= (\hat{G}(x, 0, 0), q'_{nm'} p''_{m'}(x, 0, 0)) \\ &= (r_{ni} g(x, 0, 0,), q'_{nm'} p''_{m'}(x)) \\ &= r'_{nn'} r'_{n'i'}(G^{m'}(x, 0), q'_{i'm'} p''_{m'}(x)) = r'_{nn'} h''_{n'} \end{aligned}$$

and

$$\begin{aligned} G_1 &= (\hat{G}(x, 0, 1), q'_{nm'} p''_{m'}(x, 0, 1)) \\ &= (r_{ni} g(x, 0, 1), q'_{nm'} p''_{m'}(x)) \\ &= r'_{ni}(G^m(r''_{mm'}(x), 0), q'_{im} p''_m r''_{mm'}(x)) \\ &= r'_{ni}(G_0^m, q'_{im} p''_m) r''_{mm'}(x) = h''_n r''_{mm'}(x) \end{aligned}$$

for $x \in E''_{m'}$. Thus, G is the required homotopy: $r'_{nn'} h''_{n'} \cong h''_n r''_{mm'}$ and $(\mathbf{h}'', 1_{\mathbf{C}}): (\mathbf{E}'', \mathbf{p}'', \mathbf{C}) \to (\mathbf{E}', \mathbf{p}', \mathbf{C})$ is a fiber morphism of level maps.

For the fiber morphism $\mathbf{h}'$: $\mathbf{E}' \to \mathbf{E}''$ let (i_m, ε_{i_m}) be a lifting pair for (m, η_m) and (j_m, ε_{j_m}) be a lifting pair for (i_m, η_{i_m}). Define a homotopy $K: E'_\ell \times I \to B_{j_m}$ by $K = K'(p'_\ell \times 1_I)$ where $K': C_\ell \times I \to B_{j_m}$ is a homotopy $K': f_{j_m} q'_{j_m \ell} \cong g_{j_m} q'_{j_m \ell}$. Let $k: E'_\ell \to E_{j_m}$ be the composition $k = r_{j_m \ell} f'_\ell$. Note that $d(p_{j_m} k, K_0) < \varepsilon_{j_m}$. Since (j_m, ε_{j_m}) is a lifting pair for (i_m, η_{i_m}), there is a map $\hat{K}$: $E'_\ell \times I \to E_{i_m}$ such that $\hat{K}_0 = r_{i_m j_m} k$ and $d(p_{i_m} \hat{K}, q_{i_m j_m} K) < \eta_{i_m}$. Again, by the choice of the η_{i_m}, there is a map $\tilde{h}'_m = (\hat{K}_1, q'_{i_m \ell} p'_\ell)$: $E'_\ell \to E''_{i_m}$. Write $h'_m = r''_{m i_m} \tilde{h}'_m$: $E'_\ell \to E''_m$. By similar arguments asbefore one can show that $(\mathbf{h}', 1_{\mathbf{C}}): (\mathbf{E}', \mathbf{p}', \mathbf{C}) \to (\mathbf{E}'', \mathbf{p}'', \mathbf{C})$ is a fiber morphism of level maps.

To show that $\mathbf{h}''\mathbf{h}' \sim \mathbf{1}_{\mathbf{p}}$, consider Figure 4.

$$\begin{tikzcd}
E'_n \arrow[d, "p'_n"'] \arrow[dr, phantom, "r'_{n\ell}"] & E''_m \arrow[l, "h''_n"'] \arrow[d, "p''_m"] & E'_\ell \arrow[l, "h'_m"'] \arrow[d, "p'_\ell"] \\
C_n & C_m \arrow[l] & C_\ell \arrow[l]
\end{tikzcd}$$

Figure 4

Observe that $p'_n r'_{n\ell} = q'_{n\ell} p'_\ell$ and

$$\begin{aligned} p'_n h''_n h'_m &= p'_n r'_{ni} \tilde{h}''_n r''_{mi_m} \tilde{h}'_m \\ &= p'_n r'_{ni}(G_0^m, q'_{i_m} p''_m) r''_{mi_m}(J_1, q'_{i_m\ell} p'_\ell) \\ &= q'_{ni} q'_{i_m} p''_m r''_{mi_m}(J_1, q'_{i_m\ell} p'_\ell) \\ &= q'_{ni_m} q'_{i_m\ell} p'_\ell = q'_{n\ell} p'_\ell. \end{aligned}$$

Therefore, $p'_n h''_n h'_m = p'_n r'_{n\ell}$.

We have to show that for every $\varepsilon > 0$, there is a homotopy $L\colon h''_n h'_m \cong r'_{n\ell}$ such that $d(p'_n L_t, q'_{n\ell} p'_\ell) < \varepsilon$ for every $t \in I$.

Since $f \sim g$, there is a homotopy $\hat{\Phi}\colon C_\ell \times I \times I \to B_j$ such that $\hat{\Phi}$: $q_{jj_m} K' \cong H'(q'_{m\ell} \times 1_I)$. Write $\Phi = q_{ij}\hat{\Phi}(p'_\ell \times 1_I \times 1_I)\colon E'_\ell \times I \times I \to B_i$. Let $\phi\colon E'_\ell \times J \to E_i$ be a map defined by

$$\phi(x, s, t) = \begin{cases} r_{ii_m}\hat{K}(x, s) & t = 0 \\ r_{ii_m}\hat{K}_1(x) & s = 1 \\ G^m(h'_m \times 1_I)(x, s) & t = 1 \end{cases}$$

where $J = I \times 1 \cup 1 \times I \cup I \times 0 \approx I \times 0$. Observe that for $t = 0$, $d(p_i\phi, \Phi | E'_\ell \times I \times 0) = d(q_{ii_m} p_{i_m} J, q_{ij_m} K)$, for $s = 1$, $d(p_i\phi, \Phi | E'_\ell \times 1 \times I) = d(q_{ii_m} p_{i_m} J_1, q_{ii_m} q_{i_m j_m} K_1)$ and for $t = 1$, $d(p_i\phi, \Phi | E'_\ell \times I \times 1) = d(p_i G^m(h'_m \times 1_I), q_{ij} H'(p'_m \times 1_I)(h'_m \times 1_I))$. Hence $d(p_i\phi, \Phi | E'_\ell \times J) < \eta_i$ where $\eta_i \leqq \varepsilon_i$.

Since (i, ε_i) is a lifting pair for (n, η_n), there is a map $\Psi\colon E'_\ell \times I \times I \to E_n$ such that $\Psi | E'_\ell \times J = r_{ni}\phi$ and $d(p_n\Psi, q_{ni}\Phi) < \eta_n$. Define a homotopy $L\colon E'_\ell \times I \to E'_n$ by $L(x, t) = (\Psi(x, 0, t), q'_{n\ell} p'_\ell(x))$ for $x \in E'_\ell$ and $t \in I$.

Since $q_{ni}\Phi(x, 0, t) = f_n q'_{n\ell} p'_\ell(x)$ and $d(p_n \Psi(x, 0, t), q_{ni}\Phi(x, 0, t)) < \eta_n$ for $x \in E'_\ell$ and $t \in I$, $L(E'_\ell \times I) \subset E'_n$. Hence L is well-defined. Also, $L(x, 0) = r'_{n\ell}(x)$ and $L(x, 1) = h''_n h'_m(x)$ for $x \in E'_\ell$ and $p'_n L_t = q'_{n\ell} p'_\ell$ for every $t \in I$. Hence L is the required homotopy.

By similar arguments, one can show that $\mathbf{h}'\mathbf{h}'' \sim \mathbf{1}_\mathbf{p}$ which proves Theorem 5.2.

6. Strong shape path connectedness. In this section we will define a concept of strong shape path connectedness and will find the class of such spaces. Then as a corollary of Theorem 5.2 we will show that with respect to a shape fibration, the fibers over two points which are connected by a strong shape path are of the same shape. This partially answers the question raised by Mardešić and Rushing in [11].

Let B be a compact metric space and $\mathbf{B} = (B_n, q_{nm})$ be an ANR-sequence with inj lim $\mathbf{B} = (B, q_n)$. Let $b, c \in B$ be any two points and $\mathbf{b} = (b_n = q_n(b))$. $\mathbf{c} = (c_n = q_n(c))$ be sequences of points with limits b and c respectively. First, we will define a path-family from $\mathbf{b}$ to $\mathbf{c}$ in $\mathbf{B}$.

DEFINITION 6.1. A path-family from $\mathbf{b} = (b_n)$ to $\mathbf{c} = (c_n)$ in an ANR-sequence $\mathbf{B} = (B_n, q_{nm})$ is a seuqnce of paths $\boldsymbol{\omega} = \{\omega_n: I \to B_n | \omega_n(0) = b_n, \omega_n(1) = c_n\}$ such that for each $m \geqq n$, $\omega_n \cong q_{nm}\omega_m (\text{rel } \dot{I})$ where $\dot{I} = \{0, 1\}$.

A routine check shows the following proposition.

PROPOSITION 6.1. *Let* $\mathbf{B} = (B_n, q_{nm})$ *and* $\mathbf{B}' = (B'_n, q'_{nm})$ *be* ANR-*sequences such that* inj lim $\mathbf{B}$ = inj lim $\mathbf{B}' = B$. *For points* $b, c \in B$, *let* $\mathbf{b} = (b_n = q_n(b))$, $\mathbf{c} = (c_n = q_n(c))$, $\mathbf{b}' = (b'_n = q_n(b))$ *and* $\mathbf{c}' = (c'_n = q_n(c))$. *If there is a path-family* $\boldsymbol{\omega}$ *from* $\mathbf{b}$ *to* $\mathbf{c}$ *in* $\mathbf{B}$, *then there is a path-family* $\boldsymbol{\omega}'$ *from* $\mathbf{b}'$ *to* $\mathbf{c}'$ *in* $\mathbf{B}'$.

Now we can define the principal concept of this section.

DEFINITION 6.2. A compact metric space B is said to be strongly shape path connected if there is an ANR-sequence $\mathbf{B} = (B_n, q_{nm})$ with inj lim $\mathbf{B} = (B, q_n)$ and for every pair of points $b, c \in B$ there is a path-family $\boldsymbol{\omega} = (\omega_n)$ from $\mathbf{b} = (b_n = q_n(b))$ to $\mathbf{c} = (c_n = q_n(c))$.

REMARK 6.1. Let $C = \{*\}$ be a space with one point and B be a strongly shape path connected space. Then any two maps $f, g: C \to B$ will be strongly equivalent (Definition 5.2).

Let $f(*) = x$ and $g(*) = y$. If $p: E \to B$ is a shape fibration, then the shape fibrations induced from p by f and g are $p_*(f): F_x \to C$ and $p_*(g): F_y \to C$ respectively where $F_x = p^{-1}(x)$ and $F_y = p^{-1}(y)$. Analogous to the path component, we can define strong shape path component. Then by Remark 6.1, we have the following corollary of Theorem 5.2.

COROLLARY 5.2.1. *Let* $p: E \to B$ *be a shape fibration and* $x, y \in B$ *be any two points in the same strong shape path component. Then the fibers* $F_x = p^{-1}(x)$ *and* $F_y = p^{-1}(y)$ *are of the same shape.*

REMARK 6.2. S. Mardešić has showed in [9] that the result of Corollary 5.2.1 can also be obtained by using the notion of 'generalized paths' and the results of [7].

The rest of the section is devoted to finding the class of strongly shape path connected spaces.

(i) By Definition 6.2 it is clear that path connected spaces are strongly shape path connected.

(ii) The spaces homotopy dominated by strongly shape path connected spaces are strongly shape path connected.

Let a compact metric space Y homotopy dominate a compact metric space X and Y be strongly shape path connected. We want to show that X is then strongly shape path connected.

Let $f: X \to Y$ and $g: Y \to X$ be maps such that $gf \cong 1_X$. We can find

ANR-sequences $\mathbf{X} = (X_n, r_{nm})$ and $\mathbf{Y} = (Y_n, q_{nm})$ with bonding maps as inclusions such that $\bigcap X_n = X$, $\bigcap Y_n = Y$, and also we can find level maps $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g}: \mathbf{Y} \to \mathbf{X}$ with limit maps f and g respectively. Let x_1, x_2 be any two points of X. Write $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Since gf is homotopic to 1_X, there are paths λ and μ from x_1 to $gf(x_1)$ and from x_2 to $gf(x_2)$ in X respectively. Since Y is strongly shape path connected, there is a path-family $\boldsymbol{\omega}$ from $\mathbf{y}_1$ to $\mathbf{y}_2$ in $\mathbf{Y}$. Then $\boldsymbol{\omega}' = (\boldsymbol{\lambda})\mathbf{g}(\boldsymbol{\omega})(\boldsymbol{\mu})$ is a path-family from $\mathbf{x}_1$ to $\mathbf{x}_2$ in $\mathbf{X}$ where $(\boldsymbol{\lambda}) = (\lambda_n = \lambda)$ and $(\boldsymbol{\mu}) = (\mu_n = \mu)$.

To get other classes of strongly shape path connected spaces we will state a property $(*)$ which, in general, is known as the Mittag-Leffier property.

DEFINITION 6.4. A pointed ANR-sequence $(\mathbf{X}, \mathbf{x})$ is said to have property $(*)$ if for every n, there is an index $n(*)$ such that for all $m \geqq n(*)$ the following condition is satisfied: for every $\omega_n \in \pi_1(X_n, x_n)$ with $\omega_n = q_{nm*}(\omega_m)$ for some $\omega_m \in \pi_1(X_m, x_m)$, there is $\omega_\ell \in \pi_1(X_\ell, x_\ell)$ for every $\ell \geqq m$ such that $q_{n\ell*}(\omega_\ell) = \omega_n$.

First we prove the following proposition.

PROPOSITION 6.2. *Let* $(\mathbf{X}, \mathbf{x})$ *and* $(\mathbf{X}', \mathbf{x}')$ *be* ANR-*sequences associated with a compact metric space* (X, x). *If* $(\mathbf{X}, \mathbf{x})$ *has the property* $(*)$, *then so does* $(\mathbf{X}', \mathbf{x}')$.

PROOF. By assumption there are pointed maps $\mathbf{f} = (\alpha, f_n): (\mathbf{X}, \mathbf{x}) \to (\mathbf{X}', \mathbf{x}')$ and $\mathbf{g} = (\beta, g_n): (\mathbf{X}', \mathbf{x}') \to (\mathbf{X}, \mathbf{x})$ such that $\mathbf{gf} \cong \mathbf{1}_{(\mathbf{X},\mathbf{x})}$ and $\mathbf{fg} \cong 1_{(\mathbf{X}',\mathbf{x}')}$. Let $\mathbf{f}_*: \pi_1(\mathbf{X}, \mathbf{x}) \to \pi_1(\mathbf{X}', \mathbf{x}')$ and $\mathbf{g}_*: \pi_1(\mathbf{X}', \mathbf{x}') \to \pi_1(\mathbf{X}, \mathbf{x})$ be the induced homomorphisms. Then $\mathbf{g}_*\mathbf{f}_* \sim \mathbf{1}_{(\mathbf{X},\mathbf{x})*}$ and $\mathbf{f}_*\mathbf{g}_* \sim 1_{(\mathbf{X}',\mathbf{x}')*}$. Note that for every $m \geqq n$ and $\ell \geqq n$, $q'_{nm*}f_{m*} = f_{n*}q_{\alpha(n)\alpha(m)*}$ and $q_{n\ell*}f_{\ell*} = g_{n*}q'_{\beta(n)\beta(\ell)*}$. For any n let $m \geqq \alpha(n)(*)$. Since $\mathbf{f}_*\mathbf{g}_* \sim \mathbf{1}_{(\mathbf{X}',\mathbf{x}')*}$, there is an index $m' \geqq \beta(m)$ such that $f_{n*}g_{\alpha(n)*}q_{\beta\alpha(n)m'*} = q_{nm'*}$. It is easy to see that for $m' = n(*)$, if $\lambda_n \in \pi_1(x'_n, x'_n)$ with $\lambda_n = q'_{nm'*}(\lambda_{m'})$, then for $\ell \geqq m'$ there is $\lambda_\ell = f_{\ell*}(\omega_{\alpha(\ell)'})$ such that $q'_{n\ell*}(\lambda_\ell) = q'_{nm'*}(\lambda_{m'})$ which proves that $(\mathbf{X}', \mathbf{x}')$ has the property $(*)$.

DEFINITION 6.5. A pointed compact metric space (X, x) has property $(*)$ if there is an ANR-sequence $(\mathbf{X}, \mathbf{x})$ associated with (X, x) which has property $(*)$.

DEFINITION 6.6. A compact metric space X is said to have property $(*)$ if (X, x) has property $(*)$ for every $x \in X$.

THEOREM 6.1. (see also [7]). *If a compact connected metric space has property* $(*)$, *then it is strongly shape path connected.*

PROOF. Embed X in the Hilbert cube Q. Since X is a compact connected metric space, we can select a decreasing sequence of compact connected

ANR's $\mathbf{X} = (X_n)$ with $\bigcap X_n = X$. Note that for each n, X_n is a compact path connected ANR.

Let $x, y \in X$ be any two points. For each n, write $q_n(x) = x_n$ and $q_n(y) = y_n$ where $q_n: X \to X_n$ is the inclusion map. By assumption $\mathbf{X}$ has the property $(*)$. So, we can select a cofinal subsequence which we will again denote by $\mathbf{X} = (X_n)$ such that $n + 1 = n(*)$.

By induction we will construct a path-family $\boldsymbol{\omega}$ from $\mathbf{x} = (x_n)$ to $\mathbf{y} = (y_n)$ in $\mathbf{X}$ as follows (for simplicity we will not distinguish a class and its representative).

First select a family of paths $\boldsymbol{\lambda} = \{\lambda_n: I \to X_n | \lambda_n(0) = x_n, \lambda_n(1) = y_n\}$ from $\mathbf{x}$ to $\mathbf{y}$ in $\mathbf{X}$. This may not be the required path-family from $\mathbf{x}$ to $\mathbf{y}$. Write $\omega_1 = q_{12}\lambda_2$. Now if $\lambda_2(q_{23}\lambda_3)^{-1}$ is not the trivial element in $\pi_1(X_2, x_2)$, then by assumption there is $\mu_3 \in \pi_1(X_3, x_3)$ such that $q_{13*}(\mu_3) = q_{12*}(\lambda_2 \cdot (q_{23}\lambda_3)^{-1})$. Write $\omega_2 = q_{23*}(\mu_3\lambda_3)$. Then $q_{12*}(\omega_2(\omega_1)^{-1})$ is the trivial element of $\pi_1(X_1, x_1)$. Again, if $\mu_3\lambda_3(q_{34}\lambda_4)^{-1}$ is the trivial element of $\pi_1(X_3, x_3)$, then write $\omega_3 = q_{34}(\lambda_4)$. If this is not the case, then there is an element $\mu_4 \in \pi_1(X_4, x_4)$ such that $q_{24*}(\mu_4) = q_{23*}(\mu_3\lambda_3(q_{34}\lambda_4)^{-1})$. Write $\omega_3 = q_{34}(\mu_4\lambda_4)$ and so on.

Since x and y are any two points of X this argument shows that X is strongly shape path connected space.

Now, a pointed compact metric space $(X, x) \subset (Q, x)$ is 1-movable [**14**] if for every neighbourhood U of X in Q, there is a neighbourhood V of X in $Q(V \subset U)$ such that for any neighbourhood W of X in $Q(W \subset V)$, any loop at x in V can be deformed within U into a loop in W leaving x fixed.

A compact metric space X is pointed 1-movable if (X, x) is 1-movable for every $x \in X$. Note that if $x, y \in X$ are in the same connected component of X, then (X, x) is 1-movable implies (X, y) is 1-movable [**1**].

Clearly, a pointed 1-movable compact metric space has property $(*)$. Hence

(iii) pointed 1-movable compact connected metric spaces are strongly shape path connected.

Borsuk has proved [**1**] that every plane pointed compact connected metric space is movable and therefore 1-movable. So, in particular,

(iii)′ every plane compact connected metric space is strongly shape path connected.

The following proposition shows that not all compact connected metric spaces are strongly shape path connected.

PROPOSITION 6.3. (see also [**7**]). *The dyadic solenoid is not strongly shape path connected.*

PROOF. The dyadic solenoid Σ_2 is an inverse limit of $\mathbf{X} = (S^1_n, q_{n,n+1})$ where for each n, $S^1_n = S^1$, the unit circle, and $q_{n,n+1}: S^1_{n+1} \to S^1_n$ is defined to be $q_{n,n+1}(z) = z^2$ for $z \in S^1_{n+1}$. It is well known that Σ_2 is not path connected. Let x, $y \in \Sigma_2$ be two points in the different path components. Assume that Σ_2 is strongly shape path connected. Then by definition there is a path-family $\boldsymbol{\omega} = \{\omega_n : I \to S^1_n | \omega_n(0) = x_n,\ \omega_n(1) = y_n\}$ from $\mathbf{x} = (x_n)$ to $\mathbf{y} = (y_n)$ in $\mathbf{X}$ where $q_n(x) = x_n$ and $q_n(y) = y_n$ for each n. Also $q_{nm}\omega_m \cong \omega_n(\mathrm{rel}\ \dot{I})$. Assume that for each n, ω_n is a 'shortest path' from x_n to y_n in X_n. Then $\omega_n = \ell_n\sigma_n\lambda_n$ where λ_n is a contractible path from x_n to y_n in S^1_n, σ_n is the generator of $\pi_1(S^1_n, x) \cong Z$ and ℓ_n is an integer. If $\ell_n = 0$, then $\omega_n = \lambda_n$. Let $\ell_n \neq 0$. Then $q_{n,n+1*}(\ell_{n+1}\sigma_{n+1}) = \ell_n\sigma_n$ and $q_{n,n+1}\lambda_{n+1} = \lambda_n$ since $q_{n,n+1}(\omega_{n+1}) \cong \omega_n(\mathrm{rel}\ \dot{I})$. Hence there is a path $\lambda : I \to \Sigma_2$ where $\lambda = \text{inj lim}\ \lambda_n$ from x to y. This contradicts our assumption that x and y are in different path components. So, Σ_2 is not strongly shape path connected.

7. Isomorphism of shape groups. In this section we will prove that a fiber shape induces an 'appropriate isomorphism' on relative shape groups which also justifies the definition of 'fiber shape map'.

THEOREM 7.1. *Let $p : E \to B$ and $p' : E' \to B$ be shape fibrations. Let $e' \in E'$, $p'(e') = b$ and $F' = (p')^{-1}(b)$. Then for any fiber shape map $\tilde{F} = (\mathbf{f}) : p' \to p$ there is $e \in F = p^{-1}(b)$ and an induced homomorphism $f_* : \check{\pi}_q(E', F', e') \to \check{\pi}_q(E, F, e)$ such that $f_* = p_*^{-1}p'_*$.*

Note that f_* is then an isomorphism.

PROOF. Let $\mathbf{p}' : \mathbf{E}' \to \mathbf{B}$ and $\mathbf{p} : \mathbf{E} \to \mathbf{B}$ be level maps of ANR-sequences and $\tilde{F} = (\mathbf{f}) : \mathbf{E}' \to \mathbf{E}$ be a fiber shape map. By [11] we can assume that $\mathbf{p}$ has HLP. For each n, let $e'_n = r'_n(e') \in E'_n$ and $b_n = q_n(b) \in B_n$. By induction, for each $n = 1, 2, 3, \ldots$ we can define a lifting index $m = m(n) > n$ and a closed neighbourhood Q_n of $b_n \approx Q$ such that

(1) $$q_{nm}(Q_m) \subseteqq \text{int}\ Q_n,\ m > n$$

and

(2) $$\text{inj lim}(Q_n, q_{nm}|Q_m) = \{b\}.$$

Furthermore we can choose closed ANR-neighbourhoods C_n of Q_n such that

(3) $$q_{nm}(C_m) \subset \text{int}\ Q_n,\ m > n,$$

and therefore

(4) $$\text{inj lim}(C_n, q_{nm}|c_m) = \{b\}.$$

Next choose closed ANR-neighbourhoods F'_n and F_n of $(p'_n)^{-1}(Q_n)$ and $p_n^{-1}(Q_n)$ respectively so small that

(5) $$F'_n \subseteqq (p'_n)^{-1}(C_n), \qquad F_n \subseteqq p_n^{-1}(C_n).$$

Notice that by (3)

(6) $q_{nm}\ p'_m(F'_m) \subset \text{int } Q_n \subset Q_n, \quad q_{nm}p_m(F_m) \subset \text{int } Q_n \subset Q_n$ for $m > n$ Select an $\varepsilon > 0$. Since $\mathbf{f}: \mathbf{E}' \to \mathbf{E}$ is a fiber shape map, for each n there is an index $n_* \geqq n$ such that for all $n' \geqq n^*$

(7) $$d(q_{nn'}p_{n'}f_{n'}, q_{n\alpha(n')}p'_{\alpha(n')}) < \varepsilon.$$

Also by (1) we can choose n' such that for any $x \in Q_{\alpha(n')}$ and $y \in B_n$

(8) $$d(q_{n\alpha(n')}(x), y) < \varepsilon \text{ implies } y \in Q_n.$$

Note that (8) is true for all $\ell \geqq n'$. Let $x \in F'_{\alpha(n')}$. Then by (7),

$$d(q_{n\alpha(n')}p'_{\alpha(n')}(x), p_n r_{nn'}f_{n'}(x)) < \varepsilon.$$

By (8), $p_n r_{nn'}f_{n'}(x) \in Q_n$. Since $p_n^{-1}(Q_n) \subset F_n$, $r_{nn'}f_{n'}(x) \in F_n$ for any $x \in F'_{\alpha(n')}$. Hence $r_{nn'}f_{n'}(F'_{\alpha(n')}) \subseteqq F_n$. Define a function $\beta: N \to N$ by $\beta(n) = n'$. Note that β is an increasing function. Let $f'_n = r_{nn'}f_{n'}$. Then $\mathbf{f}' = (\beta, f'_n): (\mathbf{E}', \mathbf{F}') \to (\mathbf{E}, \mathbf{F})$ is a map of ANR-sequences of pairs.

It is easy to see that another choice of $\varepsilon > 0$ gives an equivalent map of ANR-sequences of pairs.

Now for each n, f'_n induces a homomorphism $f'_{n^*}: \pi_q(E'_{\beta(n)}, F'_{\beta(n)}, e'_{\beta(n)}) \to \pi_q(E_n, F_n, f'_n(e'_{\beta(n)})$. Since $\mathbf{f}'$ is a fiber shape map, for $\ell \geqq n$ there is a relative homotopy $L^{n\ell}: (E'_{\beta(\ell)}, F'_{\beta(\ell)}) \times I \to (E_n, F_n)$. Hence there is a path $\omega(L)$ from $r_{n\ell}f'_\ell(e'_{\beta(\ell)})$ to $f'_n(e'_{\beta(n)})$ which induces an isomorphism $\omega(L)_*: \pi_q(E_n, F_n, r_{n\ell}f'_\ell(e'_{\beta(\ell)})) \to \pi_q(E_n, F_n, f'_n(e'_{\beta(n)}))$.

Now for each n, the sequence of points $\{r_{n\ell}f'_\ell(e'_{\beta(\ell)})\}_{\ell \geqq n}$ converges to a point $e_n \in F_n$ such that $r_{nm}(e_m) = e_n$. Hence the sequence of points $\{e_n\}$ determines a point $e \in F \subset E$ such that for each n, $r_n(e) = e_n$. Also notice that in the sequence of points $\{r_{n\ell}f'_\ell(e'_{\beta(\ell)})\}$, any two consecutive points, and so any two points, are connected by a path in F_n. Also, F_n being an ANR, for every ε-ball containing e_n there is a contractible ball containing e_n in the ε-ball which contains all but finitely many points of the sequence. Therefore there is a path ω_n joining e_n and $f'_n(e'_{\beta(n)})$ in F_n. Let $h_{[\omega_n]}$ be the induced isomorphism $h_{[\omega_n]}: \pi_q(E_n, F_n, e_n) \to \pi_q(E_n, F_n, f'_n(e'_{\beta(n)})$. Finally, let $\hat{r}_{nm^*}: \pi_q(E_m, F_m, e_m) \to \pi_q(E_n, F_n, e_n)$ be the homomorphism $\hat{r}_{nm^*} = h^{-1}_{[\omega_n]}L^{nm}_{e'\ell^*}r_{nm^*}h_{[\omega_m]}$ for all $m \geqq n$. Clearly $\hat{r}_{n\ell^*} = \hat{r}_{nm^*}\hat{r}_{m\ell^*}$. Hence $(\pi_q(\mathbf{E}, \mathbf{F}, \mathbf{e}), \hat{r}_{nm^*})$ is an inverse sequence of groups or in other words is an object of pro-$\mathcal{G}$. Define a morphism of pro-$\mathcal{G}$ $\mathbf{g}_* = (g_{n^*}): (\pi_q(\mathbf{E}', \mathbf{F}', \mathbf{e}'), r'_{nm^*}) \to (\pi_q(\mathbf{E}, \mathbf{F}, \mathbf{e}), \hat{r}_{nm^*})$ by $g_{n^*} = h^{-1}_{[\omega_n]}f'_{n^*}$ for every n. Clearly $\mathbf{g}_*$ is well defined, and if $\mathbf{f}'' \sim \mathbf{f}'$ and $\mathbf{f}''$ induces $\mathbf{g}''_*$, then $\mathbf{g}''_* \sim \mathbf{g}_*$ in pro-$\mathcal{G}$.

By modifying the arguments of Mardešić's theorem in **[12]** we can prove that with these new bonding maps $\mathbf{p}_* = (p_{n^*}): (\pi_q(E_n, F_n, e_n), \hat{r}_{nm^*}) \to (\pi_q(B_n, C_n, b_n), q_{nm^*})$ is still an isomorphism of pro-$\mathcal{G}$. It suffices to show that for each n there is an index $\ell > n$ and a homomorphism $K\colon \pi_q(B_\ell, C_\ell, b_\ell) \to \pi_q(E_n, F_n, e_n)$ such that the Figure 5 commutes.

$$\begin{tikzcd}
\pi_q(E_\ell, F_\ell, e_\ell) \arrow[r, "\hat{r}_{n\ell *}"] \arrow[d, "p_{\ell *}"] & \pi_q(E_n, F_n, e_n) \arrow[d, "p_{n*}"] \\
\pi_q(B_\ell, C_\ell, b_\ell) \arrow[r, "q_{n\ell *}"] & \pi_q(B_n, C_n, b_n)
\end{tikzcd}$$

Figure 5

Let ℓ be the lifting index for m and m be the lifting index for n. Consider $[\phi] \in \pi_q(B_\ell, C_\ell, b_\ell)$ where $\phi\colon (I^q, \partial I^q, J^{q-1}) \to (B_\ell, C_\ell, b_\ell)$ is the map and $J^{q-1} = (\partial I^{q-1} \times I) \cup (I^{q-1} \times 1)$. Let $h_{[\mu_\ell]}[\phi] = [\phi'] \in \pi_q(B_\ell, C_\ell, p_\ell f'_\ell(e'_{\beta(\ell)}))$. Let $\bar{e}'\colon J^{q-1} \to E_\ell$ be the constant map $f'_\ell(e'_{\beta(\ell)})$. Then $p_\ell \bar{e}' = \phi'|J^{q-1}$. Since $(I^q, J^{q-1}) \approx (I^q, I^{q-1} \times 0)$ and ℓ is a lifting index for m, there is a map $\tilde{\phi}\colon I^q \to E_m$ such that $\tilde{\phi}|J^{q-1} = r_{m\ell}\bar{e}'$ and $p_m\tilde{\phi} = q_{m\ell}\phi'$. See Figure 6.

$$\begin{tikzcd}
I^{q-1}\times 0 \arrow[r, "\bar{e}'"] \arrow[d, hook] & E \arrow[r, "r_{m\ell}"] \arrow[d, "p_\ell"] & E_m \arrow[d, "p_m"] \\
I^{q-1}\times 0 \arrow[r, "\phi'"'] \arrow[urr, dashed, "\tilde{\phi}"] & B \arrow[r, "q_{m\ell}"'] & B_m
\end{tikzcd}$$

Figure 6

Since $p_m\tilde{\phi}(\partial I^q) \subset Q_m$, $\tilde{\phi}(\partial I^q) \subset F_m$. Hence $[\tilde{\phi}] \in \pi_q(E_m, F_m, r_{m\ell} f'_\ell(e'_{\beta(\ell)}))$. Define $K\colon \pi_q(B_\ell, C_\ell, b_\ell) \to \pi_q(E_n, F_n, e_n)$ by $K[\phi] = \hat{r}_{nm*} h^{-1}_{[\omega_m]} L^{m\ell}_{\bar{e}'*}[\tilde{\phi}]$. By straightforward arguments one can show that K is a well-defined homomorphism and that $p_{n*}K = q_{n\ell *}$ and $Kp_{\ell *} = \hat{r}_{n\ell *}$.

At the end we have to show that the two isomorphisms $\mathbf{g}_*$ and $(\mathbf{p}_*)^{-1}(\mathbf{p}'_*)$ from $(\pi_q(\mathbf{E}', \mathbf{F}', \mathbf{e}'), r'_{nm*})$ to $(\pi_q(\mathbf{E}, \mathbf{F}, \mathbf{e}), \hat{r}_{nm*})$ are equal. It suffices to show that for each n there is an index $\ell' \geqq \ell$ such that Figure 7 commutes.

$$
\begin{tikzcd}[column sep=small]
 & \pi_q(E'_{\beta(\ell')},F'_{\beta(\ell')},e'_{\beta(\ell')}) \arrow[dl] \arrow[dr] & \\
\pi_q(E_{\ell'},F_{\ell'},e_{\ell'}) \arrow[ddr] & & \pi_q(B_{\beta(\ell')},C_{\beta(\ell')},b_{\beta(\ell')}) \arrow[d] \\
 & & \pi_q(B_\ell,C_\ell,b_\ell) \arrow[dl] \\
 & \pi_q(E_n,F_n,e_n) &
\end{tikzcd}
$$

Figure 7

where the index ℓ and the homomorphism K are as defined above.

Let ε' be a positive number such that $\varepsilon' <$ diameter of C_ℓ. Choose $\delta \in \Gamma(B_\ell, \varepsilon')$. Since $\mathbf{f}' = (\beta, f'_n)\colon (\mathbf{E}', \mathbf{F}') \to (\mathbf{E}, \mathbf{F})$ is a fiber shape map, for ℓ and δ there is an index ℓ^* such that for all $\ell' \geqq \ell^*$

$$d(q_{\ell\beta(\ell')}p'_{\beta(\ell')}, q_{\ell\ell'}p'_{\ell'}f'_{\ell'}) < \delta. \tag{9}$$

See Figure 8.

$$
\begin{tikzcd}
E'_{\beta(\ell')} \arrow[r, "f'_{\ell'}"] \arrow[d, "p'_{\beta(\ell')}"] & E_{\ell''} \arrow[r, "r_{\ell\ell'}"] \arrow[d, "p_{\ell'}"] & E_\ell \arrow[d, "p_\ell"] \\
B_{\beta(\ell')} \arrow[r, "q_{\ell'\beta(\ell')}"] & B_{\ell'} \arrow[r, "q_{\ell\ell'}"] & B_\ell
\end{tikzcd}
$$

Figure 8

By choice δ is so small that any two δ-close maps are connected by an ε'-homotopy. Let $\Psi \in \pi_q(\mathbf{E}', \mathbf{F}', \mathbf{e}')$ and $[\Psi] \in \pi_q(E'_{\beta(\ell')}, F'_{\beta(\ell')}, e'_{\beta(\ell')})$ which is represented by a continuous function $\Psi\colon (I^q, \partial I^q, J^{q-1}) \to (E'_{\beta(\ell')}, F'_{\beta(\ell')}, e'_{\beta(\ell')})$. Note that $h_{[\mu_{\ell'}]}p_{\ell'*}h^{-1}_{[\omega_{\ell'}]} = p_{\ell'*}$ where $\omega_{\ell'}\colon I \to E_{\ell'}$ is a path from $e_{\ell'}$ to $f'_{\ell'}(e'_{\beta(\ell')})$ and $\mu_{\ell'} = p_{\ell'}\omega_{\ell'}$. Since $q_{\ell\ell'}p_{\ell'} = p_\ell r_{\ell\ell'}$, $q_{\ell\ell'*}p_{\ell'*} = p_{\ell*}r_{\ell\ell'*}$. Now by the choice of ℓ' and $\delta > 0$, (9) implies that there is a homotopy $G'\colon (E'_{\beta(\ell')}, F'_{\beta(\ell')}) \times I \to (B_\ell, C_\ell)$ such that $G'_0 = p_\ell r_{\ell\ell'} f'_{\ell'}$ and $G'_1 = q_{\ell\beta(\ell')}p'_{\beta(\ell')}$. Hence $G = G'\cdot(\Psi \times 1_I)\colon (I^q, \partial I^q) \to (B_\ell, C_\ell)$ is a homotopy such that $G_0 = G'_0\Psi = p_\ell r_{\ell\ell'} f'_{\ell'}\Psi$ and $G_1 = G'_1\Psi = q_{\ell\beta(\ell')}p'_{\beta(\ell')}\Psi$. Therefore

(10) $$[G_0] = p_{\ell*}r_{\ell\ell'*}f'_{\ell'*}\,[\Psi]$$

and

(11) $$[G_1] = q_{\ell\beta(\ell')*}p'_{\beta(\ell')*}\,[\Psi].$$

The path $\mu_{\ell'} = p_{\ell'}\omega_{\ell'}\colon I \to B_{\ell'}$ from $b_{\ell'}$ to $p_{\ell'}f_{\ell'}(e'_{\beta(\ell')})$ induces a homotopy $H'\colon (B_{\ell'}, C_{\ell'}) \times I \to (B_{\ell'}, C_{\ell'})$ such that $H'_0 = 1_{(B_{\ell'}, C_{\ell'})}$ and $H'|b_{\ell'} \times I = \mu_{\ell'}$. Then $q_{\ell\ell'}H'\colon (B_{\ell'}, C_{\ell'}) \times I \to (B_\ell, C_\ell)$ is a homotopy such that $q_{\ell\ell'}H'_0 = q_{\ell\ell'}$ and $q_{\ell\ell'}H'|b_{\ell'} \times I = q_{\ell\ell'}\mu_{\ell'}\colon I \to B_\ell$ is a path from b_ℓ to $p_\ell r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell')})$.

Let $H = q_{\ell\ell'}H'(q_{\ell'\beta(\ell')}p'_{\beta(\ell')}\Psi \times 1_I)$. Then $H\colon (I^q, \partial I^q) \to (B_\ell, C_\ell)$ is a homotopy such that

(12) $$H_0 = G_1 \text{ and } H|J^{q-1} \times I = q_{\ell\ell'}H'|b_{\ell'} \times I.$$

Note that

(13) $$h_{[q_{\ell\ell'}\mu_{\ell'}]}\,[H_0] = [H_1] \in \pi_q(B_\ell, C_\ell, p_\ell r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell)})).$$

Define a homotopy $M\colon (I^q, \partial I^q) \times I \to (B_\ell, C_\ell)$ by

$$M(x, s) = \begin{cases} G(x, 2s) & 0 \leqq s \leqq \frac{1}{2} \\ H(x, 2s - 1) & \frac{1}{2} \leqq s \leqq 1 \end{cases}$$

for $x \in I^q$, $s \in I$. For any $x \in J^{q-1}$, $M(x, s)\colon I \to C_\ell$ is a loop at $p_\ell r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell')})$. Since C_ℓ is contractible in itself, this loop is contractible. Hence $[M_0] = [M_1] \in \pi_q(B_\ell, C_\ell, p_\ell\, r_{\ell\ell'} f'_{\ell'}(e'_{\beta(\ell')}))$, i.e., $[G_0] = [H_1]$. By (10), (13), (12) and (11)

(14) $$p_{\ell*}r_{\ell\ell'*}f'_{\ell'*}[\Psi] = h_{[q_{\ell\ell'}\mu_{\ell'}]}q_{\ell\beta(\ell')*}p'_{\beta(\ell')*}.$$

Therefore

(15) $$p_{\ell*}h^{-1}_{[\omega_\ell]}L^{\ell\ell'}_{e'*}\, r_{\ell\ell'*}f'_{\ell'*}[\Psi] = h^{-1}_{[\mu_\ell]}\hat{L}^{\ell\ell'}_{e'*}\, q_{\ell\ell'*}h^{-1}_{[\mu_{\ell'}]}q_{\ell'\beta(\ell')*}p'_{\beta(\ell')*}[\Psi]$$

which implies that

(16) $$p_{\ell*}\hat{r}_{\ell\ell'*}g_{\ell'*}[\Psi] = \hat{q}_{\ell\ell'*}q_{\ell'\beta(\ell')*}p'_{\beta(\ell')*}[\Psi].$$

Since $\hat{q}_{\ell\ell'*} = q_{\ell\ell'*}$

$$p_{\ell*}\hat{r}_{\ell\ell'*}g_{\ell'*}[\Psi] = q_{\ell\ell'*}q_{\ell'\beta(\ell')*}p'_{\beta(\ell')}[\Psi]. \tag{17}$$

which implies that

$$\hat{r}_{n\ell*}\hat{r}_{\ell\ell'*}g_{\ell'*}[\Psi] = Kq_{\ell\beta(\ell')*}p'_{\beta(\ell')*}[\Psi]. \tag{18}$$

Since Ψ is any element of $\pi_q(\mathbf{E}', \mathbf{F}', \mathbf{e}')$, (18) shows that $\mathbf{g}_*$ and $(\mathbf{p}_*)^{-1}(\mathbf{p}'_*)$ are equivalent in pro-$\mathcal{G}$. Hence inj lim $\mathbf{g}_*$ = inj lim $(\mathbf{p}_*)^{-1}(\mathbf{p}'_*)$: $\check{\pi}_q(E, F, e') \to \check{\pi}_q(E, F, e)$.

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The William Paterson College of New Jersey, Wayne, NJ 07470