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# ON THE EXISTENCE OF UNIQUE EIGENSETS OF MONOTONE PROCESSES

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ABSTRACT. A Sufficient condition is given to guarantee the existence of a unique eigenset of a monotone process. Then, a special class of monotone processes is proved to have unique eigensets through this condition and the Perron-Frobenius Theorem.

**1. Introduction.** Rockafellar [**4**, p. 69, Theorem 4] proved a theorem which provides necessary and sufficient conditions for the existence of unique eigensets of monotone processes. Since those necessary and sufficient conditions must be satisfied by every pair of non-singular monotone sets in $P_n$ and $P_n^*$, it is almost impossible to verify that a certain monotone process actually satisfies these conditions. In this paper, a sufficient condition in a simpler form is given to guarantee the existence of a unique eigenset. This sufficient condition in fact is a modification of Rockafellar's conditions. Then, a special class of monotone processes is proved to have unique eigensets through this modified condition and the Perron-Frobenius Theorem [**2**].

We shall only give the definitions of monotone sets, monotone processes, and eigensets of a monotone process. For more detailed definitions (e.g., positively homogeneous, sub-additive, non-singular, etc.), examples, and properties of monotone processes see [**3**], [**4**], and the references therein.

DEFINITION 1.1. [**4**, p. 11]. A monotone set of concave type in $P_n$, the nonnegative orthant of $R^n$, is a non-empty closed bounded convex set $C$ such that $0 \leqq y_1 \leqq y_2 \in C$ implies $y_1 \in C$. A monotone set of convex type is a non-empty closed convex set such that $y_1 \geqq y_2 \in C$ implies $y_1 \in C$.

DEFINITION 1.2. [**4**, p. 9]. A monotone process of concave type from $P_n$ to $P_m$ is a nonnegative process $T$ which is positively homogeneous, sub-additive, closed, and satisfies

(a) $T(x)$ is a monotone set of concave type for all $x \in P_n$, and
(b) $0 \leqq x_1 \leqq x_2$ implies $T(x_1) \subseteqq T(x_2)$.

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Dually, $T$ is a monotone process of convex type if conditions (a) and (b) are replaced by

(a′) $T(x)$ is a monotone set of convex type for all $x \in P_n$, and

(b′) $x_1 \geqq x_2 \geqq 0$ implies $T(x_1) \subseteqq T(x_2)$.

DEFINITION 1.3. [**4**, p. 58]. Let $T$ be a non-singular monotone process from $P_n$ to $P_n$. A non-singular monotone set $C$ in $P_n$ (of the same type as $T$) will be termed an eigenset of $T$ if, for some $\lambda > 0$, $T(C) = \lambda C$.

Since this paper is mainly an extension of Rockafellar's result, we will adopt his notation and terminology freely.

**2. Existence of unique eigensets of monotone processes.** Let $C$ and $D$ be two monotone sets in $P_n$. We say $C \leqq D$ if and only if $\langle C, x^*\rangle \leqq \langle D, x^*\rangle$ for all $x^* \in P_n^*$, where $P_n^*$ is the set of all nonegative linear functional on $R^n$ [**4**, p. 16]. It is known that if $C$ and $D$ are monotone of concave type, then $C \leqq D$ if and only if $C \subseteqq D$; and if both are of convex type, then $C \leqq D$ if and only if $C \supseteqq D$.

We now define uniform convergence of a sequence of monotone sets. It is essentially the same as Rockafellar's Definition [**4**, p. 69], but it covers both concave and convex types.

DEFINITION 2.1. A sequence $C_1, C_2, \ldots$ of non-singular monotone sets of the same type in $P_n$ converges uniformly to a set $C_0$ of the same type as each $C_k$ if for every $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon)$ such that

$$(1) \qquad (1 + \varepsilon)^{-1}C_0 \leqq C_k \leqq (1 + \varepsilon)C_0$$

for all $k \geqq k_0$.

If the sets $C_k$ are of concave type, then (1) is equivalent to $(1 + \varepsilon)^{-1} C_0 \subseteqq C_k \subseteqq (1 + \varepsilon)C_0$ for all $k \geqq k_0$ as given by Rockafellar [**4**, p. 69]. If the sets $C_k$ are of convex type, then (1) is equivalent to $(1 + \varepsilon)C_0 \subseteqq C_k \subseteqq (1 + \varepsilon)^{-1}C_0$ for all $k \geqq k_0$. In either case, we shall write $\lim_{k\to\infty} C_k = C_0$.

LEMMA 2.1. *If* $\lim_{k\to\infty} C_k = C_0$, *then* $\lim_{k\to\infty}\langle C_k, y^*\rangle = \langle C_0, y^*\rangle$ *for all* $y^* \in P_n^*$.

PROOF. Assume that the sets $C_k$ and $C_0$ are of concave type. Then, given any $\varepsilon < 0$, there exists $k_0 = k_0(\varepsilon)$ such that $(1 + \varepsilon)^{-1}C_0 \subseteqq C_k \subseteqq (1 + \varepsilon)C_0$, for all $k \geqq k_0$. Therefore, for any $y^* \in P_n^*$, and for all $k \geqq k_0$,

$$(1 + \varepsilon)^{-1} \sup_{y \in C_0} \langle y, y^*\rangle \leqq \sup_{y \in C_k} \langle y, y^*\rangle \leqq (1 + \varepsilon) \sup_{y \in C_0} \langle y, y^*\rangle.$$

Hence, for all $y^* \in P_n^*$ and all $k \geqq k_0$, we have

$$(1 + \varepsilon)^{-1} \langle C_0, y^*\rangle \leqq \langle C_k, y^*\rangle \leqq (1 + \varepsilon) \langle C_0, y^*\rangle.$$

Thus, $\lim_{k\to\infty}\langle C_k, y^*\rangle = \langle C_0, y^*\rangle$ for all $y^* \in P_n^*$.

The same argument holds with the change of the direction of inequalities if $C_k$ and $C_0$ are of convex type.

DEFINITION 2.2. A sequence $T_1, T_2, \ldots$ of monotone processes of the same type from $P_n$ to $P_n$ is said to converge uniformly to a monotone process $T_0$ of the same type if, given any $\varepsilon > 0$, there exists a $k_0 = k_0(\varepsilon)$ such that

$$(1 + \varepsilon)^{-1}T_0 \leqq T_k \leqq (1 + \varepsilon)T_0, \quad \text{for all } k \geqq k_0,$$

where $T_i \leqq T_j$ if and only if $T_i(x) \leqq T_j(x)$ [4, p. 17] for all $x \in P_n$.

In this event we write $\lim_{k\to\infty} T_k = T_0$. It is to be noted that in Definition 2.2, $k_0$ is independent of $x \in P_n$.

Let $T$ be a non-singular monotone process of either type from $P_n$ to $P_n$. Let $\bar{C}$ be a non-singular monotone set of the same type as $T$ and $\bar{D}^*$ be a non-singular monotone set of type opposite to $\bar{C}$. Define a process $T_0$ from $P_n$ to $P_n$ by $T_0(x) = \langle x, \bar{D}^*\rangle\bar{C}$; then $T_0$ is a non-singular monotone process of the same type as $T$ [4].

LEMMA 2.2. *Let $T$, $T_0$, $\bar{C}$, $\bar{D}^*$ be given as above. Let $T^k$ be defined inductively by $T^k(x) = \bigcup_{y\in T^{k-1}(x)} T(y)$. If $\lim_{k\to\infty} T^k = T_0$, then*

(a) $\lim_{k\to\infty} T^k(x) = \langle x, \bar{D}^*\rangle\, \bar{C}$, *for all $x \in P_n$, and*

(b) $\lim_{k\to\infty} \langle T^k(C), D^*\rangle = \langle \bar{C}, D^*\rangle \cdot \langle C, \bar{D}^*\rangle$,

*for all non-singular monotone sets $C$ and $D^*$ of types the same as $\bar{C}$ and $\bar{D}^*$, respectively.*

PROOF. Given any $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that for all $x \in P_n$ and for all $k \geqq k_0$, we have

$$(1 + \varepsilon)^{-1}\, T_0(x) \leqq T^k(x) \leqq (1 + \varepsilon)T_0(x). \tag{2}$$

By Definition 2.1, (2) implies $\lim_{k\to\infty} T^k(x) = T_0(x) = \langle x, \bar{D}^*\rangle\, \bar{C}$, for all $x \in P_n$.

Let $C$ be any non-singular monotone set in $P_n$ of the same type as $\bar{C}$. Then from (2), we have

$$(1 + \varepsilon)^{-1} \bigcup_{x\in C} T_0(x) \leqq \bigcup_{x\in C} T^k(x) \leqq (1 + \varepsilon) \bigcup_{x\in C} T_0(x),$$

for all $k \geqq k_0$. This implies $(1 + \varepsilon)^{-1}T_0(C) \leqq T^k(C) \leqq (1 + \varepsilon)\, T_0(C)$, for all $k \geqq k_0$. Hence, $\lim_{k\to\infty} T^k(C) = T_0(C)$. But, in [4, p. 69], it is shown that $T_0(C) = \langle C, \bar{D}^*\rangle \cdot \bar{C}$. Therefore,

$$\lim_{k\to\infty} T^k(C) = \langle C, \bar{D}^*\rangle \cdot \bar{C}. \tag{3}$$

Now, let $D^*$ be any non-singular monotone set in $P_n^*$ of the same type as $\bar{D}^*$. Then, (3) implies

$$\langle \lim_{k\to\infty} T^k(C), D^*\rangle = \langle C, \bar{D}^*\rangle \cdot \langle \bar{C}, D^*\rangle.$$

Applying Lemma 2.1 to the left hand side of this equation, we have

$$\lim_{k\to\infty} \langle T^k(C), D^*\rangle = \langle C, \bar{D}^*\rangle \cdot \langle C, D^*\rangle$$

The conclusions (a) and (b) in Lemma 2.2 are equivalent to the necessary and sufficient conditions given by Rockafellar [**4**, p. 69, Theorem 4] for the existence of a unique eigenset of a monotone process. Lemma 2.2 actually shows that uniform convergence of the sequence $T$, $T^2$, ... to $T_0$ guarantees the existence of a unique eigenset for $T$. This conclusion can be rewritten as the following theorem.

THEOREM 2.1. *Let $T$ be a non-singular monotone process of either type from $P_n$ to $P_n$. If there exist non-singular monotone sets $\bar{C}$ and $\bar{D}^*$ of suitable types and there exists a scalar $\lambda > 0$ such that*

$$\lim_{k\to\infty} \left(\frac{1}{\lambda} T\right)^k = T_0, \tag{4}$$

*where $T_0(\cdot) = \langle \cdot, \bar{D}^*\rangle \bar{C}$, then, aside from positive multiples, $\bar{C}$ and $\bar{D}^*$ are the unique non-singular eigensets of $T$ and the adjoint of $T$, $T^*$, respectively. This means $T(\bar{C}) = \lambda \bar{C}$, and $T^*(\bar{D}^*) = \lambda \bar{D}^*$.*

The scalar $\lambda$ is known as the growth rate in the literature (e.g., [**4**]). Since a monotone process is possitively homogeneous, we can replace $[(1/\lambda)T]$ by $T$ and assume $\lambda = 1$. For this reason, (4) is the sufficient condition in Lemma 2.2.

In the next section, we shall describe a special class of monotone processes satisfying (4) whose members therefore have unique eigensets.

**3. Application.** Let $A$ be an $n \times n$ matrix such that none of its rows is identical to the zero vector. Define a process $A^\wedge$ from $P_n$ to $P_n$ by $A^\wedge(x) = (Ax)^\wedge = \{y | 0 \leqq y \leqq Ax\}$. Then $A^\wedge$ is a monotone process of concave type, and the adjoint of $A^\wedge$, $(A^\wedge)^*$, is a monotone process of convex type where $(A^\wedge)^*(x)^* = (A^t x^*)^\vee = \{y^* | y \geqq A^t x^*\}$ [**4**, p. 9].

In this section, we shall prove that the monotone process $A^\wedge$ defined by a nonnegative matrix $A$ has unique eigenset.

First let us cite several known results in matrix theory.

LEMMA 3.1. [**1**]. *Let $P$ be an $n \times n$ irreducible stochastic matrix. Then*

(a) *$P^\infty = \lim_{k\to\infty} P^k$ exists, i.e., for every $\varepsilon > 0$, there exists $k_0 = k^0(\varepsilon)$ such that for all $k \geqq k_0$ and $i, j = 1, \ldots, n$,*

$$(1 - \varepsilon)P_{ij}^\infty \leqq P_{ij}^k \leqq (1 + \varepsilon)P_{ij}^\infty$$

(b) *Furthermore, there exists a vector $\pi = (\pi_1, \ldots, \pi_n)$, where $\sum_{j=1}^n \pi_j = 1$ and $\pi_j > 0$ for $j = 1, \ldots, n$, such that each row of $P^\infty$ is equal to $\pi$.*

THEOREM 3.1. (PERRON-FROBENIUS THEOREM [2]). *Let $A$ be an $n \times n$ irreducible nonnegative matrix. Then $A$ has a "maximal" positive eigenvalue $\lambda_0$ that is a simple root of the characteristic equation such that $|\lambda| \leqq \lambda_0$ for other eigenvalues $\lambda$ of $A$. Furthermore, to this $\lambda_0$, there corresponds an eigenvector $z^0 = (z_1^0, \ldots, z_n^0)$ such that each component of $z^0$ is greater than zero.*

Now, if $A$ is an $n \times n$ irreducible nonnegative matrix, then $A$ is similar to some matrix $(\lambda_0 P)$, where $\lambda_0$ is the "maximal" eigenvalue of $A$ given in Theorem 3.1, and $P$ is an irreducible probability matrix. In fact, c.f. [2], we have

$$A = Z(\lambda_0 P)Z^{-1}, \tag{5}$$

where $Z$ is a diagonal matrix with diagonal elements $z_1^0, \ldots, z_n^0$ and $(z_1^0, \ldots, z_n^0)$ is a positive eigenvector of $A$ corresponding to $\lambda_0$.

From (5), we have $[(1/\lambda_0)A]^k = ZP^kZ^{-1}$, for all positive integers $k$. Therefore,

$$\lim_{k\to\infty}\left(\frac{1}{\lambda_0}A\right)^k = \left(\frac{1}{\lambda_0}A\right)^\infty = ZP^\infty Z^{-1} \tag{6}$$

exists.

If we apply (a) of Lemma 3.1 to (6), it is not difficult to prove that, given $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that for all $k \geqq k_0$ and $i, j = 1, \ldots, n$,

$$(1 - \varepsilon)\left(\frac{1}{\lambda_0}A\right)_{ij}^\infty \leqq \left(\frac{1}{\lambda_0}A\right)_{ij}^k \leqq (1 + \varepsilon)\left(\frac{1}{\lambda_0}A\right)_{ij}^\infty \tag{7}$$

If $A$ is an $n \times n$ irreducible nonnegative matrix, then so is $A^t$. By the Perron-Frobenius Theorem, there exists a vector $w^0$ with all components positive such that $A^t w^0 = \lambda_0 w^0$. Therefore $([(1/\lambda_0]A)^k)^t\, w^0 = w^0$, for all integers $k$. Hence, we have

$$\left(\left(\frac{1}{\lambda_0}A\right)^\infty\right)^t w^0 = w^0. \tag{8}$$

Applying (6) in (8), using the representation of $P^\infty$ described in Lemma 3.1 and then equating the components on both sides in (8), we get $\langle z^0, w^0\rangle \cdot \pi_j/z_j^0 = w_j^0$ for $j = 1, \ldots, n$.
Hence,

$$\pi_j = z_j^0 w_j^0/\langle z^0, w^0\rangle, \text{ for } j = 1, \ldots, n. \tag{9}$$

Now, we are ready to prove the main result of this section.

THEOREM 3.2. *Let $A$ be an $n \times n$ irreducible non-negative matrix. Then the monotone process $T = A^\wedge$ defined by $A$ and its adjoint, $T^*$, have unique non-singular eigensets, expect for positive scalar multiples.*

PROOF. Let $z^0$ and $w^0$ be positive eigenvectors of $A$ and $A^t$ corresponding to the "maximal" eigenvalue, $\lambda_0$ of $A$, respectively.

Since $T((z^0)\hat{}) = \{T(y) \mid 0 \leqq y \leqq z\} = T(z^0) = (Az^0)\hat{} = \lambda_0(z^0)\hat{}$, $(z^0)\hat{}$ is an eigenset of $T$. Similarly, $(w^0)\check{}$ is an eigenset of $T^*$. Denote $\langle z^0, w^0\rangle^{1/2}$ by $s$. If we let $\bar{C} = (1/s)(z^0)\hat{}$ and $\bar{D}^* = (1/s)(w^0)\check{}$, then it is easy to see that $\langle \bar{C}, \bar{D}^*\rangle = 1$, and $\bar{C}$ and $\bar{D}^*$ are eigensets of $T$ and $T^*$, respectively.

It is clear that for each integer $k$, $T^k(x) = (A^k x)\hat{}$ for all $x \in P_n$. If we let $T_0(x) = ([(1/\lambda_0)A]^\infty(x))\hat{}$, and use (6) and (9), we have $T_0(x) = (\langle x, w^0\rangle/s^2)(z^0)\hat{}$ for all $x \in P_n$. On the other hand,

$$\begin{aligned}\langle x, \bar{D}^*\rangle\, \bar{C} &= (\langle x, (w^0)\check{}\rangle/s)\cdot(1/s)(z^0)\hat{} \\ &= (\langle x, (w^0)\rangle/s^2)(z^0)\hat{},\end{aligned}$$

for all $x \in P_n$. Hence, we conclude that $T_0(x) = \langle x, \bar{D}^*\rangle\, \bar{C}$ for all $x \in P_n$.

From (7) and the fact that $T^k(x) = (A^k x)\hat{}$, it follows that for any given $\varepsilon > 0$, there exists $k_0 = k_0(\varepsilon)$ such that $(1 - \varepsilon)T_0 \leqq [(1/\lambda_0)\, T]^k \leqq (1 + \varepsilon)T_0$, for all $k \geqq k_0$.

Hence,

$$\lim_{k\to\infty}\left(\frac{1}{\lambda_0}\, T\right)^k = T_0 = \langle \cdot, \bar{D}^*\rangle \bar{C}.$$

From Theorem 2.1, $\bar{C}$, $\bar{D}^*$ are the unique eigensets of $T$ and $T^*$ respectively.

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