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OSCILLATION AND ASYMPTOTIC BEHAVIOR IN CERTAIN DIFFERENTIAL EQUATIONS OF ODD ORDER

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Introduction. The purpose of this paper is to examine the relationship between oscillation and asymptotic behavior for solutions of the equations

$$y^{(2n+1)} - p(t)y = 0 \tag{1}$$

and

$$z^{(2n+1)} + p(t)z = 0 \tag{1*}$$

where $p(t)$ is a positive continuous function defined on $[0, \infty)$. Equation (1*) is the adjoint equation of (1).

It is well-known that (1) has a solution $u = u(t)$ satisfying

$$u(t) > 0, u'(t) > 0, \ldots, u^{(2n)}(t) > 0 \tag{2}$$

for all t on some half-line, $[a, \infty)$, $a \geqq 0$. Such solutions are said to be *strongly increasing* on $[a, \infty)$. We let I denote the set of solutions of (1) which are eventually strongly increasing. The equation (1*) has a solution w satisfying

$$w(t) > 0, w'(t) > 0, \ldots, (-1)^k w^{(k)}(t) > 0, k = 0, 1, \ldots, 2n, \tag{2*}$$

for all t on $[0, \infty)$. These solutions are termed strongly decreasing.

Recall that a nontrivial solution y of (1) or (1*) is said to be *oscillatory* if $\sup\{t \geqq 0: y(t) = 0\} = \infty$. Clearly this implies that y has infinitely many zeros on $[0, \infty)$. Whenever (1) has an oscillatory solution we say that (1) is *oscillatory*. We will show that (1) is oscillatory if and only if (1*) is oscillatory. A solution which is not oscillatory is called *nonoscillatory* and due to the linearity of (1) and (1*), we assume without loss of generality that all nonoscillatory solutions are *eventually positive*. Hereafter the term "solution" shall be interpreted to mean "nontrivial solution" unless otherwise stated.

For $n = 1$, (1) is oscillatory if and only if nonoscillatory solutions of (1) belong to I. When $n > 1$, this equivalence no longer holds. Lovelady, however, has established the following result in **[6]**.

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THEOREM 1. *If each nonoscillatory solution y of* (1) *belongs to I, then* (1) *has an oscillatory solution. Moreover, in this case* (1) *has $2n$ linearly independent oscillatory solutions.*

In light of Theorem 1 and preceding remarks, a relevant question is: How does the asymptotic behavior of the nonoscillatory solutions of (1) affect the existence and number of (independent) oscillatory solutions? In this work we only consider the existence part of the question. Note that if $u(t)$ is a nonoscillatory solution of (1) and $u \notin I$, then $u(t)u^{(2n)}(t) < 0$ for all t on some ray $[b, \infty)$. Furthermore, it has been established in [6] that (1) has a nonoscillatory solution $u \notin I$ only if $p(t)$ is "small" in the sense that

$$\int_0^\infty t^{(2n-1)}\, p(t)dt < \infty. \tag{3}$$

Even though (3) is a necessary condition for the existence of a nonoscillatory solution $u \notin I$, it is not sufficient.

EXAMPLE. The fifth order Euler differential equation

$$y^{(5)} - \frac{\beta}{x^5} y = 0 \tag{E}$$

is oscillatory and has a nonoscillatory solution $u \notin I$ when $0.259\sqrt{30} < \beta \leqq 0.66\sqrt{30}$, but for $\beta > 0.663\sqrt{30}$ all nonoscillatory solutions are strongly increasing.

To insure that (1) has at least two types of nonoscillatory solutions we have the following theorem.

THEOREM 2. *If the second order equation*

$$z'' + \frac{1}{(2n-1)!} t^{(2n-1)}\, p(t)z = 0 \tag{4}$$

is nonoscillatory, then (1) *has a nonoscillatory solution $u \notin I$.*

PROOF. Suppose (4) is nonoscillatory and let z be a nonoscillatory solution of (4). Choose $c > 0$ such that $z(t)z'(t) \neq 0$ on $[c, \infty)$. Note that $z'(t) > 0$ on $[c, \infty)$. Suppose $c \leqq t \leqq \alpha$, then

$$\begin{aligned} z'(t) &= z'(\alpha) + \frac{1}{(2n-1)!} \int_t^\alpha s^{2n-1}\, p(s)z(s)ds \\ &\geqq \frac{1}{(2n-1)!} \int_t^\alpha s^{2n-1}\, p(s)z(s)ds. \end{aligned}$$

Consequently,

$$z'(t) \geqq \frac{1}{(2n-1)!}\int_t^\infty s^{2n-1}p(s)z(s)ds$$
$$\geqq \frac{1}{(2n-1)!}\int_t^\infty (s-t)^{2n-1}p(s)z(s)ds.$$

Using standard iteration techniques, we can construct a continuously differentiable function v from $[c, \infty)$ into $[z, (c), \infty)$ such that $v(c) = z(c)$, $v(t) \leqq z(t)$, whenever $c \leqq t$, and such that

$$v'(t) = \frac{1}{(2n-1)!}\int_t^\infty (s-t)^{2n-1}p(s)v(s)ds, \tag{5}$$

for $t \geqq C$. Clearly $2n - 1$ differentiations of (5) yield

$$v^{(2n)} = -\int_t^\infty p(s)v(s)ds$$

and then (1). Thus v solves (1) on $[c, \infty)$ and satisfies $v(t)v^{(2n)}(t) < 0$ on $[c, \infty)$. To complete the theorem we need only the remark that v can be extended to a solution of (1) on $[0, \infty)$ and this solution will satisfy the requirements of the theorem.

Before continuing our investigation of (1) we should note that inequality (3) has a long history. It has appeared in many papers concerning oscillation properties of differential equations, some of these include the works of Wong [9], Komkov [3] and Onose [7].

To facilitate our study of (1) we will utilize the following functional

$$F[y(t)] = 2\sum_{r=0}^{n-1}(-1)^r y^{(2n-r)}(t)y^{(r)}(t) + (-1)^n[y^{(n)}(t)]^2.$$

When $y(t)$ is a solution of (1), this functional is increasing. To see this note that $F'[y(t)] = 2p(t)y^2(t)$.

Main results. A solution y of (1) is called *minimally increasing* if $y > 0$, $y' > 0$, $y'' < 0, \ldots, (-1)^k y^{(k)} < 0$ on $[a, \infty)$ for some $a \geqq 0$ and $i = 2, 3, \ldots, 2n$. In this section we show that for $n > 1$, minimally increasing solutions can be "introduced" into the solution space of (1) without "forcing out" all of the oscillatory solutions, and moreover, these solutions are the "first" nonoscillatory solutions different from those in I that can occur in the solution space. We shall denote the set of minimally increasing solutions by $I(m)$.

THEOREM 3. *Suppose* $y \in I(m)$. *Then*

(a) $$\lim_{t\to\infty} y^{(k)}(t) = 0, \quad k = 2, 3, \ldots, 2n$$

and

(b) $$F[y(t)] < 0, \textit{for all } t \geqq 0.$$

PROOF. Condition (a) is immediate. To prove (b), suppose that y is a minimally increasing solution of (1) with $F[y(t)] > 0$ on $[b, \infty)$ for some $b \geqq 0$. Assume that b is so large that $y, y', \ldots, y^{(2n)}$ do not change sign on $[b, \infty)$. For n odd, note that

$$(N)\qquad \begin{aligned} F[y] - yy^{(2n)} + 2y'''y^{(2n-3)} + \cdots + 2y^{(n-2)}\, y^{(n+2)} + [y^{(n)}]^2 \\ = 2y'\, y^{(2n-2)} + 2y^{(4)}\, y^{(2n-4)} + \cdots + 2y^{(n-1)}\, y^{(n+1)}. \end{aligned}$$

Since $F[y(t)]$ is increasing the left side of (N) is positive and bounded away from zero, while the right side of (N) tends to zero as $t \to \infty$, a contradiction. The case for n even is proved similarly. Thus $F[y[t)] < 0$ on $[0, \infty)$.

The following lemma is known.

LEMMA. *If u is solution of* (1) *such that* $u(c) \geqq 0, u'(c) \geqq 0, \ldots, u^{(2n)}(c) > 0$ *then* $u(t) > 0, u'(t) > 0, \ldots, u^{(2n)}(t) > 0$ *for all* $t > c$.

THEOREM 4. *If $n > 1$ and all nonsocillatory solutions of* (1) *belong either to I or $I(m)$ then* (1) *is oscillatory.*

PROOF. Suppose z_1 and z_2 are independent solutions of (1) which have zeros of multiplicity $2n - 1$ and $2n - 2$ respectively at $x = a$, $a \geqq 0$. For each integer $r > a$, let u_r be a solution of (1) such that $u_r(r) = 0$ and $u_r = c_{1r}z_1 + c_{2r}z_2$, where $c_{1r}^2 + c_{2r}^2 = 1$. Since the sequences $\{c_{1r}\}$ and $\{c_{2r}\}$ are bounded, we can assume without loss of generality that $c_{1r} \to c_1$ and $c_{2r} \to c_2$ as $n \to \infty$. Let $u = c_1z_1 + c_2z_2$. Then $u_r \to u$ uniformly on compact subsets of $[0, \infty)$ and since $F[u_r(t)] > 0$ on (a, ∞) for each r, it follows that $F[u(t)] > 0$ on (a, ∞).

Suppose that $u(t)$ is nonoscillatory, then $u(t) \in I$ or $u(t) \in I(m)$. It follows from Theorem 3 that $u(t) \notin I(m)$ since $F[u(t)] > 0$ on (a, ∞). Thus $u(t)$ is strongly increasing on some ray $[c, \infty)$, $c \geqq a$. Since $u_r^{(k)}(t) \to u^k(t)$ for $k = 0, 1, 2, \ldots, 2n$, there exists N such that for $r > N$, $u_r^{(k)}(d) > 0$, $k = 0, 1, 2, \ldots, 2n$. Let r_0 be an integer greater than $d + N$, then since $u_{r_0}^{(k)}(d) > 0$, for $k = 0, 1, 2, \ldots, 2n$, it follows from the preceding lemma that u_{r_0} is strongly increasing on (d, ∞), but $u_{r_0}(r_0) = 0$, a contradiction. Hence u must be oscillatory.

We now show that the first type of nonoscillatory solution that can appear in the solution space of (1) other than the strongly increasing kind is the minimally increasing type. To prove this, however, the notion of *oscillation type* will be needed. We will say that (a, b) is a $(k, 2n + 1 - k)$ interval of oscillation provided there is a solution of (1) which is positive on (a, b) with zeros of order not less than k and $2n + 1 - k$ at a and b respectively. If for $t \geqq 0$, there is an (i, j) interval of oscillation in $[t, \infty)$,

$i + j = 2n + 1$, we let $r_{ij}(t) = \min\{b > t\colon [t, b]$ contains an (i, j) interval of oscillation$\}$. The number $r_{ij}(t)$, when it exists, is called an *oscillation nomber*. if no such b exists, we write $r_{ij}(t) = \infty$ and say that (1) is (i, j) disconjugate on $[t, \infty)$. It is known [4], that if $r_{ij}(t) < \infty$ for some t, $i + j = 2n + 1$, then j must be even. Recently Jones [2] has shown that oscillation types can be ordered. This is turn means that the oscillation numbers can be ordered. Using the results in [2], we conclude that

$$\begin{aligned} &r_{n(n+1)} \leqq r_{(n-2)(n+3)} \leqq \cdots \leqq r_{1(2n)}, \text{ if } n \text{ is odd, or} \\ &r_{(n+1)n} \leqq r_{(n-1)(n+2)} \leqq \cdots \leqq r_{1(2n)}, \text{ if } n \text{ is even.} \end{aligned} \tag{6}$$

Furthermore, it is known that if k is odd, $r_{k(2n+1-k)} = \infty$ if and only if (1) has a nonoscillatory solution y such that $y > 0, y' > 0, \ldots, y^{(k)} > 0$, $y^{(k+1)} < 0$ on some ray $[c, \infty)$, see Elias [1]. Analogous results hold for (1*). Using these facts we establish our final three results.

THEOREM 5. *If* $r_{1(2n)}(t) < \infty$ *for each* $t \geqq 0$, *then all nonoscillatory solutions of* (1) *belong to* I. *Furthermore, if* (1) *has a nonoscillatory solution* $y \notin I$, *then* (1) *has a minimally increasing solution.*

PROOF. We prove only the latter part of the theorem. If y is minimally increasing, we are done. So suppose $y \notin I(m)$ and that none of $y, y' \ldots, y^{(2n)}$ vanish on $[c, \infty)$. There are exactly $(n - 1)$ other possible sign combinations with $y > 0$. Assume $y > 0, \ldots, y^{(i)} > 0, y^{(i+1)} < 0$ for some $1 < i < 2n + 1$, then i is odd and $r_{ij} = \infty$ where $j = 2n + 1 - i$. From (6) it follows that $r_{ij} \leqq r_{1(2n)}$, consequently we see that $r_{1(2n)} = \infty$, which implies that (1) has a minimally increasing solution.

Using (6) and (7) we also obtain some nonoscillation criteria involving asymptotic behavior of solutions.

THEOREM 6. *Equation* (1) *is nonoscillatory if and only if* (1) *has a nonoscillatory solution satisfying* $y > 0, y' > 0, \ldots, y^{(n)} > 0, y^{(n+1)} < 0$, *if* n *is odd, or* $y > 0, y' > 0, \ldots, y^{(n+1)} > 0, y^{(n+2)} < 0$, *if* n *is even.*

Turning to the adjoint of (1)

$$z^{(2n+1)} + p(t)z = 0, \tag{1*}$$

we know that (1*) has a solution w satisfying $w > 0$, $w' < 0, \ldots,$ $(-1)^k w^{(k)} < 0$, $k = 1, 2, \ldots, 2n$. Such solutions are called strongly decreasing. Let D denote the set of strongly decreasing solutions of (1*).

Finally we denote the oscillation numbers associated with (1*) by r^*_{ij}. It is known (see [4]) that $r_{km} = r^*_{mk}$, $m + k = 2n + 1$. Consequently for (1*) we now have

$$(6^*)\quad \begin{aligned} &r^*_{n(n+1)} \leqq r^*_{(n+3)(n-1)} \leqq \cdots \leqq r^*_{(2n-1)2} \leqq r^*_{(2n)1}, \quad \text{if } n \text{ is odd, or}\\ &r^*_{n(n+1)} \leqq r^*_{(n-2)(n+3)} \leqq \cdots \leqq r^*_{(2n-2)3} \leqq r^*_{(2n)1}, \quad \text{if } n \text{ is even.}\end{aligned}$$

Our final theorem shows that there exists a strong connection between the results in [5] and [6]. In fact, because of the next theorem, most of the results that appeared in [5] are easy consequences of the work in [6].

THEOREM 7.
(i) *Equation* (1) *is nonoscillatory if and only if* (1*) *is nonoscillatory.*
(ii) *All nonoscillatory solutions of* (1) *belong to* I *if and only if all nonoscillatory solutions of* (1*) *belong to* D.
(iii) *The first nonoscillatory solution* Z *that appears in the solution space of* (1*) *other than the strongly descreasing type must satisfy* $Z > 0, Z' > 0, \ldots, Z^{(2n)} > 0$ *on some ray* $[b, \infty)$.

The proofs of (i), (ii), and (iii) follow immediately from Theorem 5, Theorem 6, (6), (6*) and (7).

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