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# EXTENSIONS OF MAPS DEFINED ON CONVERGENCE SPACES

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ABSTRACT. This is a foundational study of the extendibility of various continuous type maps defined on a dense subspace $X$ of a preconvergence space $Z$. A minimal property (weak-admissibility) for such extensions is established and is applied for the case where the remainder $R = Z - X$ is $U$-principal. Major results include necessary and sufficient conditions for the extendibility of continuous [resp. weakly-continuous] mappings, a general Taĭmanov type characterization for extendibility and a general result which shows that a weakly-admissible map defined on $X$ can be extended to a weak-$n$-continuous map on $Z$ where $Z$ is any extension of $X$. Finally, numerous examples are given which show that the major results obtained are nontrivial and have many well-known propositions as corollaries.

**1. Introduction.** In their paper [**24**], the Steiners make the following remark relative to topological spaces, "Our point of view is that one of the most important kinds of information a structure on a space $X$ provides, besides a topology for $X$, is a topological extension of $X$." The Steiners' philosophy can obviously be applied to the more general concept of the convergence structure on a set $X$ as introduced by D. Kent in his foundational papers [**10**], [**11**] and [**12**].

Various types of extensions for a convergence space $X$ have been recently investigated. The majority of these extensions are compactifications of one type or another (see [**18**], [**19**], [**20**], [**21**] and [**27**]). Once these extensions have been obtained, then, as in the case for topological spaces, the most useful investigation appears to be in the general area of the extendibility of continuous maps onto these space extensions. Indeed, with respect to the Steiners' paper [**24**], D. F. Wooten initiated such a research project for semi-uniform spaces in his paper [**28**]. Other results relative to the extensions of maps on semi-uniform spaces can also be found in [**6**].

The major goal of this present investigation is to add to what is an apparent void in the theory of extensions for maps on convergence spaces. This present research is a foundational study of such concepts, where the maps we wish to extend have a minimal property necessary for extendibil-

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ity. Most of the major results apply readily to space extensions of a very simple type which we have termed "$U$-principal" and "discrete." Indeed, throughout general topology many "$U$-principal" extensions such as $n$-point compactifications, near-compactifications and the famous Katětov type extension have been vigorously investigated (see **[4]**, [5], [7], **[9]**, **[14]**, **[15]**, **[16]**, [17], **[22]** and **[23]**).

As a second goal, we also launch a foundational study of map extensions, where the space extension is completely general in character and the basic map once again satisfies only the minimal condition necessary for extendibility.

**2. Preliminaries.** Throughout this paper $F(X)$ [resp. $U(X)$] denotes the set of all filters [resp. ultrafilters] on a set $X$, while $\mathscr{P}(X)$ denotes the power set. In 1964, D. Kent **[10]** introduced a generalization of Fischer's **[3]** "limitierung" in the following manner, where for each $x \in X$, $[X]$ denotes the principal filter generated by $\{x\}$.

Definition 1.1. A function $q$: $F(X) \to \mathscr{P}(X)$ is a *convergence function* if
$CS(1)$: for each $x \in X$, $x \in q([x])$ and
$CS(2)$: whenever $\mathscr{F}, \mathscr{G} \in F(X)$, $\mathscr{F} \subset \mathscr{G}$, then $q(\mathscr{F}) \subset q(\mathscr{G})$.

In his 1967 paper **[11]**, Kent adjoins to $CS(1)$ and $CS(2)$ the following additional axiom.

$$CS(3)\text{: If } x \in q(\mathscr{F}), (\mathscr{F}) \in F(x), \text{ then } x \in q(\mathscr{F} \cap [x]).$$

The pair $(X, q)$ where $q$ satisfies axioms $CS(1)$, $CS(2)$ and $CS(3)$ is called a *convergence space* and $q$ is a *convergence structure.* Many results in this paper only depend upon axioms $CS(1)$ and $CS(2)$. Indeed, many only depend on $CS(2)$. Hence for this reason, we call a pair $(X, q)$ where $q$ satisfes $CS(1)$ and $CS(2)$ a *preconvergence space* and $q$ is a *convergence function.* We shall assume that the reader is familiar with the basic concepts associated with convergence space, many of which can be located in the cited references. However, we shall modify the concepts to the extent that in most cases $(X, q)$ will be assumed to be a preconvergence space. Thus, for example, the preconvergence space $(X, q)$ is *compact* if and only if every ultrafilter $q$-converges. Moreover, unless otherwise indicated, $q$, $q'$, $q''$, $p$ denote convergence functions and $(X, q') = X$, $(Y, p) = Y$, $(Z, q) = Z$, etc. denote preconvergence spaces. If $\mathscr{F} \in F(x)$, then $\mathscr{F} \to x$ means that $x \in q(\mathscr{F})$ or we say something like $\mathscr{F}$ is *q-convergent to x.*

The following additional notation will also be helpful. Let $A \subset X$. Then

$$U(X, A) = \{x | [x \in U(X)] \wedge \exists y[[y \in A] \wedge [x \to y]]\}$$

[resp. $F(X, A) = \{x | [x \in F(x)] \ldots\}$]. If $A = \{x\}$, then we simply write

$U(X, \{x\})$ [resp. $F(X, \{x\})$] as $U(X, x)$ [resp. $F(X, x)$]. For $\mathscr{F} \in F(X)$, $A \subset X$, let $\mathscr{F} \sqcap A$ mean that $F \cap A \neq \varnothing$ for each $F \in \mathscr{F}$ and $\mathscr{F} \perp A$ means that there exists some $F \in \mathscr{F}$ such that $F \cap A = \varnothing$. If $\mathscr{F} \sqcap A$, $A \subset X$, then let $\mathscr{F}_A$ denote the filter generated on $A$ by $\{F \cap A | F \in \mathscr{F}\}$. If $\mathscr{A} \subset \mathscr{P}(X)$ and $\mathscr{A}$ has the finite intersection property, then $[\mathscr{A}]_X$ or simply $[\mathscr{A}]$ denotes the filter on $X$ generated by $\mathscr{A}$ and if $A \subset X$, then $[A]$ denotes the principal filter generated by $A$. For $A, B \subset X$ let

$$U_B(X, A) = \{\mathscr{U}_B | [\mathscr{U} \in U(X)] \wedge [\mathscr{U} \sqcap B] \wedge \exists y[[y \in A] \wedge [\mathscr{U} \to y]]\}.$$

[resp. $F_B(X, A) = \{\mathscr{F}_B | [\mathscr{F} \in F(X)] \ldots\}$]. Let $\mathscr{G}$ be a collection of filter bases on $X$ and any map $g: (X, q) \to (Y, p)$. Then $g(\mathscr{G}) = \{g(\mathscr{F}) | \mathscr{F} \in \mathscr{G}\}$, where $g(\mathscr{F}) = [\{g[F] | F \in \mathscr{F}\}]$. For $\mathscr{F} \in \mathscr{G}$, let $C(\mathscr{F}) = \{x | [x \in X] \wedge [\mathscr{F} \to x]\}$, where $\mathscr{F} \to x$ if and only if $[\mathscr{F}] \to x$ and $I(\mathscr{G}) = \bigcap\{C(\mathscr{F}) | \mathscr{F} \in \mathscr{G}\}$. Notice that if $\varnothing \neq \mathscr{G} \subset U(X)$, then $I(\mathscr{G}) = \bigcap\{a_q(\mathscr{F}) | \mathscr{F} \in \mathscr{G}\}$, where $a_q(\mathscr{F})$ is the adherence of $\mathscr{F}$ and if $(X, q)$ is Hausdorff, then the cardinality of $I(\mathscr{G})$, $|I(\mathscr{G})|$, is less than or equal to one.

**3. U-principal extensions.** Throughout the remainder of this paper the space $(X, q') = X$ is a dense subspace of $(Z, q) = Z$ and $Z - X = R \neq \varnothing$. Moreover, $(R, q'')$ is the remainder subspace of $Z$ with the induced convergence function $q''$. Recall that $X$ is *dense in* $Z$ if for each $z \in Z$, there exists some $\mathscr{U} \in U(Z, z)$ such that $\mathscr{U} \sqcap X$ (i.e., $X \in \mathscr{U}$). A map $g: (X, q) \to (Y, p)$ is *weakly-continuous* [resp. *continuous*] if for each $x \in X$ and each $\mathscr{U} \in U(X, x)$ [resp. $\mathscr{F} \in F(X, x)$] it follows that $g(\mathscr{U})$ [resp. $g(\mathscr{F})$] converges to $g(x)$. Observe that if $Y$ is pseudotopological and $g: X \to Y$ is weakly-continuous, then $g$ is continuous. Obviously, in general, continuity implies weak-continuity. We now give the fundamental minimal property a map must possess in order that it be extendible.

DEFINITION 3.1. A map $g: X \to Y$ is *weakly-admissible* if for each $r \in R$, the set $I(g(U_X(Z, r))) \neq \varnothing$. The map $g$ is *admissible* if for each $r \in R$ there exists $y \in Y$ such that $g(F_X(Z, r)) \subset F(Y, y)$.

The following three properties are easily established:
(i) admissiblity implies weak-admissibility;
(ii) since $X$ is dense in $Z$, then for each $r \in R$, $U_X(Z, r) \neq \varnothing$; and
(iii) if $Y$ is Hausdorff and $g$ weakly-admissible, then $|I(g(U_X(Z, r)))| = 1$ for each $r \in R$.

THEOREM 3.1. *Let $g: X \to Y$. If there exists an extension $G: Z \to Y$ which is weakly-continuous* [*resp. continuous*] *at $r \in R$, then $I(g(U_X(Z, r))) \neq \varnothing$* [*resp. $g(F_X(Z, r)) \subset F(Y, G(r))$*].

PROOF. Assume that $G$ extends $g$ and arbitrary $\mathscr{U} \in U(Z, r)$, $r \in R$ and $\mathscr{U} \sqcap X$. Weak-continuity at $r$ implies that $G(\mathscr{U}) \to G(r)$. Now $G(\mathscr{U}) =$

$g(\mathscr{U}_X)$, since $\mathscr{U}_X \in U(X)$, $G(\mathscr{U}) \in U(Y)$, $g(\mathscr{U}_X) \in U(Y)$ and $g(\mathscr{U}_X) \subset G(\mathscr{U})$. Thus $g(\mathscr{U}_X) \to G(r)$, Since $\mathscr{U}_X$ is an arbitrary member of $U_X(Z, r)$, then $G(r) \in I(g(U_X(Z, r)))$.

For continuity, let $\mathscr{F} \in F(Z, r)$, $r \in R$ and $\mathscr{F} \sqcap X$. Continuity at $r$ implies that $G(\mathscr{F}) \to G(r)$. Now $\mathscr{F} \subset [\mathscr{F}_X]$ implies that $G(\mathscr{F}) \subset G([\mathscr{F}_X]) = [g(\mathscr{F}_X)] = g(\mathscr{F}_X)$. Consequently, $g(\mathscr{F}_X) \to G(r)$ implies that $g(\mathscr{F}_X) \in F(Y, G(r))$.

COROLLARY 3.1.1. *Let $g$: $X \to Y$. If there exists an extension $G$: $Z \to Y$ of $g$ which is weakly-continuous* [*resp. continuous*] *on $R$, then $g$ is weakly-admissible* [*resp. admissible*].

The major investigation in this section is an attempt to obtain a converse or partial converse to Theorem 3.1. In order to accomplish this in the most expedient manner, we restrict the extension space to various proper subclasses of the class of all extensions of $X$. The remainder space $(R, q'')$ is said to be *U-principal* if the only $q''$-convergent ultrafilters on $R$ are the principal ultrafilters. The space $(R, q'')$ is *discrete* if the only $q''$-convergent filters in $F(R)$ are the principal ultrafilters. Thus $(R, q'')$ is a discrete space if and only if $q''$ is equivalent to convergence for the discrete topology on $R$. Recall that a convergence function $q''$ with the property that $r \in q''(\mathscr{F})$ if and only if $\mathscr{F} \subset [r]$ is called a *principal* convergence function. Clearly, a principal or discrete space is a $U$-principal space.

DEFINITION 3.2. An extension $Z$ of $X$ is *U-principal* [resp. *discrete*] if $R$ is $U$-principal [resp. discrete]. The space $R$ is *separated from $X$* if for each $r \in R$, $\mathscr{U} \in U(Z, r)$ implies that $\mathscr{U} \notin U(Z, x)$ for any $x \in X$.

As mentioned in the introduction many $U$-principal extensions have been investigated for topological spaces. Moreover, the "Hausdorff except for $X$" topological extension **[14]** as well as any Hausdorff preconvergence space extension has a remainder which is separated from $X$. For a preconvergence space the one-point compactification, which shall be shortly defined, is a discrete extension. The following theorem is a simple but useful characterization for $U$-principal extensions.

THEOREM 3.2. *The following statements are equivalent for any extension $Z$ of $X$.*

(i) *The space $(R, q'')$ is a U-principal subspace.*

(ii) *If nonprincipal $\mathscr{U} \in U(Z, r)$, $r \in R$, then $\mathscr{U} \sqcap X$.*

PROOF. (i) $\to$ (ii). Assume that for nonprincipal $\mathscr{U} \in U(Z, r)$, $r \in R$, we have that $\mathscr{U} \perp X$. Then $X \notin \mathscr{U}$ implies that $R \in \mathscr{U}$. Thus $\mathscr{U}_R \in U(R)$. By definition of subspace convergence, $\mathscr{U}_R$ is $q''$-convergent to $r$. Hence $\{r\} \in \mathscr{U}_R$ implies that $\{r\} \in \mathscr{U}$. This contradiction yields the result.

(ii) $\to$ (i). This is obvious since $\mathscr{U} \sqcap X$ if and only if $\mathscr{U} \perp R$. Hence the only $q''$-convergent ultrafilters are the principal ones and the proof is complete.

For $U$-principal extensions, we have the following partial converse to corollary 3.1.1.

THEOREM 3.3. *Let $g$: $X \to Y$ be weakly-admissible [resp. admissible and $Y$ a convergence space] $Z$ a $U$-principal [resp. discrete] extension of $X$. Then there exists an extension $G$: $Z \to Y$ of $g$ which is weakly-continuous [resp. continuous] on $R$.*

PROOF. For the weakly-admissible case, define $G$: $Z \to Y$ as follows: for each $r \in R$, let $G(r) \in I(g(U_X(Z, r)))$ and for each $x \in X$, let $G(x) = g(x)$. Let nonprincipal $\mathscr{U} \in U(Z, r)$, $r \in R$. Then $\mathscr{U} \sqcap X$ by Theorem 3.2. Hence $\mathscr{U}_X \in U_X(Z, r)$ implies that $G(\mathscr{U}) = g(\mathscr{U}_X) \to G(r)$. Since $G([r]) = [G(r)] \to G(r)$, then $G$ is weakly-continuous on $R$. $G$ is obviously an extension of $g$.

For the admissible case, define $G$: $Z \to Y$ in the following manner. For each $r \in R$, let

$$M_r = \{y | [y \in Y] \wedge [g(F_X(Z, r)) \subset F(Y, y)]\}.$$

Then for each $r \in R$, let $G(r) \in M_r$ and for each $x \in X$, let $G(x) = g(x)$. Observe that it must be the case that for $\mathscr{F} \in F(Z)$, either $\mathscr{F} \sqcap X$ or $\mathscr{F} \sqcap R$. Assume that $\mathscr{F} \to z \in Z$. If $\mathscr{F} \sqcap X$, then it follows that $[\mathscr{F}_X]$ $q$-converges to $z$ since $\mathscr{F} \subset [\mathscr{F}_X]$. Let $z = r \in R$. If $\mathscr{F} \sqcap R$, then we have that $[\mathscr{F}_R]$ is $q$-convergent to $r$. Hence $\mathscr{F}_R$ is $q''$-convergent to $r$ since $(R, q'')$ is a subspace. Therefore, if $\mathscr{F} \sqcap R$, then $\mathscr{F}_R = [r]_R$. Assume that $\mathscr{F} \perp X$ and $\mathscr{F} \sqcap R$. Then it follows that $R \in \mathscr{F}$. Hence $\mathscr{F} = [\mathscr{F}_R] = [[r]_R]_Z = [r]_Z$. Consequently in this case $G(\mathscr{F}) = G([\mathscr{F}_R]) = G([r]) = [G(r)] \to G(r)$ . Now assume that $\mathscr{F} \sqcap X$ and $\mathscr{F} \sqcap R$. Then $\mathscr{F} = [\mathscr{F}_X] \cap [\mathscr{F}_R]$ implies that

$$G(\mathscr{F}) = G([\mathscr{F}_X]) \cap G([\mathscr{F}_R]) = g(\mathscr{F}_X) \cap G(\mathscr{F}_R)$$

since $\mathscr{F}_X$ and $\mathscr{F}_R$ are filter bases for $[\mathscr{F}_X]$ and $[\mathscr{F}_R]$ respectively. Since $g(\mathscr{F}_X) \to G(r)$ by admissibility and the definition of $G$, then axiom $CS(3)$ implies that $G(\mathscr{F}) \to G(r)$ and the proof is complete.

COROLLARY 3.3.1. *Let $g$: $X \to Y$ be weakly-admissible, $Z$ a $U$-principal extension of $X$ and $Y$ pseudotopological. Then there exists an extension $G$: $Z \to Y$ of $g$ which is continuous on $R$.*

Recall that a subspace $(X, q')$ of $(Z, q)$ is *open* [resp. *weakly-open*] if whenever $\mathscr{F} \in F(Z)$ [resp. $\mathscr{U} \in U(Z)$] $q$-converges to $x \in X$, then $X \in \mathscr{F}$ [resp. $X \in \mathscr{U}$]. Hence $X$ is open in $Z$ [resp. weakly-open] if and only if for each $x \in X$ if $\mathscr{F} \in F(Z)$ [resp. $\mathscr{U} \in U(Z)$] is $q$-convergent to $x \in X$, then $\mathscr{F} = [\mathscr{F}_X]$ [resp. $[\mathscr{U}_X] = \mathscr{U}$]. Observe that $[\mathscr{U}_X] = \mathscr{U}$ if and only if $\mathscr{U} \cap X$, where $\mathscr{U} \in U(Z)$. (See note added in proof.)

THEOREM 3.4. *Let $g: X \to Y$ be weakly-continuous and weakly-admissible. If $Z$ is a U-principal extension of weakly-open $X$, then there exists a weakly-continuous extension $G: Z \to Y$* of $g$.

PROOF. Let $G$ be defined as in Theorem 3.3. Let $\mathscr{U}$ be $q$-convergent to $x \in X$. Then $\mathscr{U}_Y$ $q'$-converges to $x$. Thus $G(\mathscr{U}) = g(\mathscr{U}_X)$ $p$-converges to $g(x) = G(x)$.

COROLLARY 3.4.1 . *Let $g$: $X \to Y$ be weakly-continuous and weakly-admissible. If $Z$ is a U-principal extension of weakly-open $X$ and $Y$ is pseudotopological, then there exists a continuous extension $G$: $Z \to Y$* of $g$.

The following example shows a pseudotopological extension of $X$ such that $R$ is separated from $X$, $Z$ is $U$-principal, but $X$ is not open in $Z$.

EXAMPLE 3.1. Let $X$ and $R$ be infinite disjoint sets. Assume that $Z = X \cup R$. Let $c \in X$, $\mathscr{U}$ be a nonprincipal ultrafilter on $X$ and $\mathscr{V}$ a nonprincipal ultrafilter on $R$. Define convergence on $Z$ as follows: for $\mathscr{F} \in F(Z)$ let $\mathscr{F}$ be $q$-convergent to $x \in X$, $x \neq c$ if and only if $\mathscr{F} = [x]$; let $\mathscr{F}$ be $q$-convergent to $c$ if and only if $[c] \cap [\mathscr{V}] \subset \mathscr{F}$. Finally let $\mathscr{F}$ be $q$-convergent to $r \in R$ if and only if $[r] \cap [\mathscr{U}] \subset \mathscr{F}$. This pseudotopological structure has the properties that $X$ is not open in $Z$, $Z$ is separated from $X$, and $Z$ is a $U$-principal extension.

THEOREM 3.5. *Let $g$: $X \to Y$ be a weakly-admissible [resp. admissible and $Y$ a convergence space] and continuous map with $X$ open in the U-principal [resp. discrete] extension $Z$. Then there exists an extension $G$: $(Z, q) \to (Y, p)$ of $g$ which is $q$-continuous on $X$ and $q$-weakly-continuous [resp. $q$-continuous] on $R$.*

PROOF. Let $\mathscr{F} \in F(Z, x)$, $x \in X$. Then $\mathscr{F} = [\mathscr{F}_X]$. Now $\mathscr{F}_X$ $q'$-converges to $x$. Hence $g(\mathscr{F}_X)$ $p$-converges to $g(x)$. Since $G(\mathscr{F}) = G([\mathscr{F}_X]) = [g(\mathscr{F}_X)] = g(\mathscr{F}_X)$, then $G(\mathscr{F})$ is $p$-convergent to $g(x) = G(x)$. Application of Theorem 3.3 completes the proof.

COROLLARY 3.5.1. *Let $g$: $X \to Y$ be weakly-admissible, weakly-continuous with $X$ open in the U-principal extension $Z$. If $Y$ is pseudotopological, then there exists a continuous extension $G$: $Z \to Y$* of $g$.

REMARK 3.1. Since for Hausdorff $Y$ and weakly-admissible [resp. admissible] $g$: $X \to Y$, $|I(g(U_X(x, r)))| = 1$ [resp. $|M_r| = 1$] for each $r \in R$, then it follows that for Hausdorff $Y$ the map $G$ in Theorems 3.3, 3.4, 3.5 and their corollaries is unique.

Let $(X, q')$ be noncompact, $r \notin X$ and $X^+ = X \cup \{r\}$. Define the following map $q: F(X^+) \to \mathscr{P}(X^+)$. Let $\mathscr{F} \in F(X^+)$. If $x \in X$, then $x \in q(\mathscr{F})$ if and only if $X \in \mathscr{F}$ and $\mathscr{F}_X$ $q'$-converges to $x$. Now let $r \in q(\mathscr{F})$ if and only if there exists some $\mathscr{G} \in F(X^+)$ such that $\mathscr{G} \sqcap X$, $a_{q'}(\mathscr{G}_X) = \varnothing$ and $\mathscr{G} \sqcap [r] \subset \mathscr{F}$. Note that $a_{q'}(\mathscr{G}_X)$ is the set of $q'$-adherence points of $\mathscr{G}_X$. It is straightforward to show that if $(X, q')$ is a preconvergence [resp. convergence] space, then $X^+$ is a preconvergence [resp. convergence] space. Moreover, $X^+$ is compact, $\{r\}$ is separated from $X$ and $X^+$ is a discrete extension. We call $X^+$ a *one-point compactification* of $X$. It is shown in [8] that $X^+$ is pseudotopological if and only if $X$ is pseudotopological and if $X$ is pretopological Hausdorff, then $X^+$ is pretopological if and only if $X^+$ is a projective minimum in the class of all pretopological compactifications of $X$. Obviously if $g$: $X \to Y$ is weakly-continuous [resp. continuous], then $g: X \to Y^+$ is weakly-continuous [resp. continuous] for noncompact $Y$. The following result is an immediate consequence of our previous results.

THEOREM 3.6. *Let $g$: $X \to Y$ be weakly-continuous [resp. continuous and $Y$ a convergence space] and $X$, $Y$ noncompact. Then there exists a weakly-continuous [resp. continuous] extension $G$: $X^+ \to Y^+$ of $g$ if and only if $g$ is weakly-admissible [resp. admissible].*

A monoid $(H, \cdot)$ which is not necessarily Abelian is a *preconvergence* [resp. *convergence*] monoid if there exists a convergence function [resp. structure] $q$ such that the monoid operation is $q$-continuous. Every noncompact preconvergence monoid can be imbedded into a one-point compactification.

THEOREM 3.7. *Let $(H, \cdot, q')$ be a noncompact preconvergence monoid and $(K, q)$ any compactification of $H$. Then the monoid operation cannot be extended to a continuous monoid operation on $K$ onto $H$ such that $H$ is a submonoid of $K$.*

PROOF. Let $(H, \cdot, q')$ be a proper submonoid and subspace of the compact preconvergence monoid $(K, \cdot, q)$. Since $H$ is noncompact, then there exists some $\mathscr{U} \in U(H)$ such that $\mathscr{U}$ does not $q'$-converge to any $h \in H$. Since $K$ is compact, then $[\mathscr{U}]$ is $q$-convergent to some $k \in K - H$. Assume that the monoid operator $g$: $(H \times H) \to H$ has a continuous extension to $G$: $(K \times K) \to H$. Let $X = H \times H$ and $e$ be the identity in $K$. Then $G([e] \times [\mathscr{U}])$ is $q \times q$-convergent to $e \cdot k = k \in K$ and $([e] \times [\mathscr{U}]) \sqcap X$. Clearly, $([e] \times [\mathscr{U}])_X = [e]_H \times [\mathscr{U}]_H = [e]_H \times \mathscr{U} \in U_X(K \times K, (e, k))$. Since $g$ is weakly-admissible, then $g(([e] \times \mathscr{U})_X) = g([e]_H \times \mathscr{U})$ is $q'$-convergent to some $h_0 \in H$ and $h_0 = G(e, k)) = e \cdot k = k$. The result now follows from this contradiction.

Let $Z_0$ be any extension of $X$ such that $Z_0 - X$ is separated from $X$.

THEOREM 3.8. *Assume that $X$ is noncompact and* [*resp. weakly*] *open in $Z_0$. Then there exists a* [*resp. weakly*]*-continuous surjection $G\colon Z_0 \to X^+$ such that $G|X$ is the identity on $X$.*

PROOF. We only establish this result for the continuity case. Define $G\colon Z_0 \to X^+$ as follows: let $G(Z_0 - X) = \{r\}$, where $X^+ = X \cup \{r\}$, and for each $x \in X$ set $G(x) = x$. Assume that $\mathscr{F} \in F(Z_0)$ and $\mathscr{F} \to x \in X$. Then $X \in \mathscr{F}$. Consequently, $\mathscr{F} = [\mathscr{F}_X]$. Hence $G(\mathscr{F}) = [\mathscr{F}_X]_{X^+}$ will $q^+$-converge to $x$. Now let $\mathscr{F} \in F(Z_0)$ and $\mathscr{F} \to z \in Z_0 - X$. Assume that $\mathscr{F} \perp X$. Then $Z_0 - X \in \mathscr{F}$ implies that $\{r\} \in G(\mathscr{F})$. Consequently, $G[\mathscr{F}] = [r]_{X^+}$. Now assume that $\mathscr{F} \sqcap X$. Observe that for any $\mathscr{U} \in U(Z_0, z)$, the fact that $\mathscr{U} \not\to x$ for any $x \in X$ implies that $a_{q'}(\mathscr{F}_X) = \emptyset$. If $R = Z_0 - X$ and $\mathscr{F} \sqcap R$, then

$$\begin{aligned} G(\mathscr{F}) &= G(([R] \vee \mathscr{F}) \cap ([X] \vee \mathscr{F})) \\ &= G([R] \vee \mathscr{F}) \cap G([X] \vee \mathscr{F}) \\ &= G(\mathscr{F}_R) \cap G(\mathscr{F}_X) = [r]_{X^+} \cap [\mathscr{F}_X]_{X^+}. \end{aligned}$$

If $\mathscr{F} \perp N$, then $G(\mathscr{F}) = [\mathscr{F}_X]_{X^+}$. In either case $G(\mathscr{F})$ is $q^+$-convergent to $r$.

COROLLARY 3.8.1. *Let $Y$ be noncompact, $g\colon X \to Y$ be weakly-continuous* [*resp. continuous and $Y$ a convergence space*] *and $X$ weakly-open* [*resp. open*] *in $Z_0$. Then there exists a weakly-continuous* [*resp. continuous extension $G\colon Z_0 \to Y^+$ if and only if $g$ is weakly-admissible* [*resp. admissible*].

REMARK 3.2. In Corollary 3.8.1, if $X$ is compact, then there does not exist a proper extension $Z$ of $X$ such that $Z - X$ is separated from $X$.

Let $\mathscr{E}(X)$ [resp. $\mathscr{E}_w(X)$] be any class of extensions of $X$ such that if $Z \in \mathscr{E}(X)$ [resp. $Z \in \mathscr{E}_w(X)$], then $X$ is open [resp. weakly-open] in $Z$ and $Z - X$ is separated from $X$. A space $Z_m \in \mathscr{E}(X)$ [resp. $Z_m \in \mathscr{E}_w(X)$] is said to be the [resp. *weak*] *projective minimum* if for each $Z \in \mathscr{E}(X)$ [resp. $Z \in \mathscr{E}_w(X)$] there exists a [resp. weak] continuous surjection $G\colon Z \to Z_m$ such that $G|X$ is the identity on $X$.

THEOREM 3.9. *If $X$ is noncompact and any $X^+ \in \mathscr{E}(X)$* [*resp. $X^+ \in \mathscr{E}_w(X)$*], *then $X^+$ is a* [*resp. weak*] *projective minimum.*

PROOF. This follows easily since the identity map from $X$ into $X^+$ is clearly admissible.

The following example shows that theorem 3.9 is a considerable improvement over theorem 1.3 in **[18]**.

EXAMPLE 3.2. This is an example of a pseudotopological convergence space extension $Z$ of $X$ such that $X$ is open in $Z$ and $Z - X$ is separated from $Z$. However, $Z - X$ is not discrete and $Z$ is not compact. Let $X$ and $R$ be infinite disjoint sets. Let $Z = X \cup R$, $\mathscr{U}$ be a nonprincipal ultrafilter on $X$ and $\mathscr{V}$ a nonprincipal ultrafilter on $R$. Define $q$-convergence in $Z$ as follows: for $\mathscr{F} \in F(Z)$, let $\mathscr{F}$ $q$-converge to $x \in X$ if and only if $\mathscr{F} = [x]_Z$. Let $r_0$ be a fixed point in $R$. Then $\mathscr{F}$ $q$-converges to $r \in R - \{r_0\}$ if and only if $[r]_Z \cap [\mathscr{U}]_Z \subset \mathscr{F}$; finally, $\mathscr{F}$ $q$-converges to $r_0$ if and only if $[\mathscr{U}]_Z \cap [r_0]_Z \cap [\mathscr{V}] \subset \mathscr{F}$. The space $Z$ has the indicated properties.

In **[25]**, Taĭmanov gives the following famous characterization for continuously extending a continuous map $f$: $X \to Y$ for topological spaces $X$ and $Y$ where $Y$ is compact. There exists a continuous $F$: $X \to Y$ if and only if for each pair of disjoint closed subsets $U, V$ of $Y$, $\mathrm{cl}_Z(f^{-1}[U]) \cap \mathrm{cl}_Z(f^{-1}[V]) = \varnothing$. Various authors (see **[2]**, **[23]**, **[26]** **[28]**) have studied and generalized this result and have given similar characterizations for other types of structures. We now briefly consider a Taĭmanov type characterization for preconvergence spaces.

Recall that for $\mathscr{F} \in F(X)$, $\mathrm{cl}_X\mathscr{F} = [\{\mathrm{cl}_X(F) \mid F \in \mathscr{F}\}]$ where "$\mathrm{cl}_X$" is the $q$-closure in $X$ and $\{\mathrm{cl}_X(F) \mid F \in \mathscr{F}\}$ is a base for $\mathrm{cl}_X \mathscr{F}$. If $\mathscr{B}$. is a base for $\mathscr{F} \in F(X)$, then it follows that $\{\mathrm{cl}_X(B) \mid B \in \mathscr{B}\}$ is a base for $\mathrm{cl}_X \mathscr{F}$.

THEOREM 3.10. *Let $Y$ be compact Hausdorff and $g$: $X \to Y$. If whenever $\mathscr{V}$, $\mathscr{V}' \in U(Y)$, $\mathscr{V} \cap g[X]$, $\mathscr{V}' \cap g[X]$, $\mathscr{V} \to y$, $\mathscr{V}' \to y'$ and $y \neq y'$ it follows that* $\mathrm{cl}_Z(g^{-1}(\mathscr{V})) \perp \mathrm{cl}_Z(g^{-1}(\mathscr{V}'))$, *then $g$ is weakly-admissible.*

PROOF. Let $\mathscr{U}_X, \mathscr{U}'_X \in U_X(Z, r)$, $r \in R$. Then $g(\mathscr{U}_X) \to y$ and $g(\mathscr{U}'_X) \to y'$ by compactness since $g(\mathscr{U}_X) = \mathscr{V} \in U(Y)$ and $g(\mathscr{U}'_X) = \mathscr{V}' \in U(Y)$. Assume that $y \neq y'$. Hausdorffness implies that $\mathscr{V} \perp \mathscr{V}'$. If $V \in \mathscr{V}$ and $V' \in \mathscr{V}'$ such that $V \cap V' \neq \varnothing$, then $g^{-1}[V] \cap g^{-1}[V'] \neq \varnothing$ implies that $\mathrm{cl}_Z(g^{-1}[V]) \cap \mathrm{cl}_Z(g^{-1}[V']) \neq \varnothing$. Now assume that $V \cap V' = \varnothing$. Clearly $U \cap g^{-1}[V] \neq \varnothing$ for each $U \in \mathscr{U}_X$ and $U' \cap g^{-1}[V'] \neq \varnothing$ for each $U' \in \mathscr{U}'_X$ imply that $W \cap g^{-1}[V] \neq \varnothing$ for each $W \in \mathscr{U}$ and $W' \cap g^{-1}[V'] \neq \varnothing$ for each $W' \in \mathscr{U}'$. Consequently, $g^{-1}[(V)] \in \mathscr{U}$ and $g^{-1}[V'] \in \mathscr{U}'$ imply that $r \in \mathrm{cl}_Z(g^{-1}[V]) \cap \mathrm{cl}_Z(g^{-1}[V'])$. Thus $\mathrm{cl}_Z(g^{-1}(\mathscr{V})) \cap \mathrm{cl}_Z(g^{-1}(\mathscr{V}'))$. This contradiction implies that $y = y'$. Hence $I(g(U_X)(Z, r)) = \{y\}$ and the result follows.

We now weaken the definition of a Urysohn space as it appears in **[13]** and call $(X, q)$ *weakly-Urysohn* if whenever $\mathscr{F} \to x$, $\mathscr{G} \to x'$ with $\mathscr{F}, \mathscr{G} \in F(X)$ and $x \neq x'$, then $\mathrm{cl}_X \mathscr{F} \perp \mathrm{cl}_X\mathscr{G}$. Clearly a wealky-Urysohn space is Hausdorff.

THEOREM 3.11. *Let $Y$ be compact weakly-Urysohn and $g$: $X \to Y$. If* *whenever* $\mathscr{V}$, $\mathscr{V}' \in U(Y)$, $\mathscr{V} \sqcap g[X]$, $\mathscr{V}' \sqcap g[X]$, $\mathscr{V} \to y$, and $\mathscr{V}' \to y'$ *and* $y \neq y'$ *it follows that*

$$\mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y\mathscr{V})) \perp \mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y\mathscr{V}')),$$

*then g is weakly-admissible.*

PROOF. Let $\mathscr{V} \to y$, $\mathscr{V}' \to y'$ and $y \neq y'$ as defined in the proof of 3.10. Then $\mathrm{cl}_Y\mathscr{V} \perp \mathrm{cl}_Y\mathscr{V}'$. It now follows as the proof of 3.10 that

$$\mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y\mathscr{V})) \sqcap \mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y\mathscr{V})).$$

The conclusion follows from this contracidtion.

THEOREM 3.12. *Let* $G: Z \to Y$ *be a weakly-continuous extension of* $g: X \to Y$. *If* $\mathscr{F}$, $\mathscr{G} \in F(Y)$, $\mathscr{F} \sqcap g[X]$, $\mathscr{G} \sqcap g[X]$, *and* $\mathrm{cl}_Y\mathscr{F} \perp \mathrm{cl}_Y\mathscr{G}$, *then*

$$\mathrm{cl}_Z(g^{-1}(\mathscr{F})) \perp \mathrm{cl}_Z(g^{-1}(\mathscr{G})).$$

PROOF. Assume that $\mathscr{F}$, $\mathscr{G} \in F(X)$, $\mathscr{F} \sqcap g[X]$, $\mathscr{G} \sqcap g[X]$, $\mathrm{cl}_Y(\mathscr{F}) \perp \mathrm{cl}_Y(\mathscr{G})$, but $\mathrm{cl}_Z(g^{-1}(\mathscr{F})) \sqcap \mathrm{cl}_Z(g^{-1}(\mathscr{G}))$. Then there exist $F \in \mathscr{F}$, $H \in \mathscr{G}$ such that $\mathrm{cl}_Y(F) \cap \mathrm{cl}_Y(H) = \emptyset$ and some $z \in Z$ such that $z \in \mathrm{cl}_Z(g^{-1}[F]) \cap \mathrm{cl}_Z(g^{-1}[H])$. Then there exist $\mathscr{U}$, $\mathscr{U}' \in U(Z)$ such that $\mathscr{U} \to z$, $\mathscr{U}' \to z$, $g^{-1}[F] \in \mathscr{U}$, and $g^{-1}[H] \in \mathscr{U}'$. Hence $G[g^{-1}[F]] = gg^{-1}[F] \in g(\mathscr{U}_X) = G(\mathscr{U})$, and $G[g^{-1}[H]] = gg^{-1}[H] \in g((\mathscr{U}')_X) = G(\mathscr{U}')$. Thus $F \in G(\mathscr{U}) \in U(Y)$ and $H \in G(\mathscr{U}') \in U(Y)$. However, $G(\mathscr{U}) \to G(z)$ and $G(\mathscr{U}') \to G(z)$ imply that $\mathrm{cl}_Y(F) \cap \mathrm{cl}_Y(H) \neq \emptyset$. The result follows from this contradiction.

COROLLARY 3.12.1. *Let* $G: Z \to Y$ *be a weakly-continuous extension of* $g: X \to Y$ *and* $Y$ *is weakly-Urysohn. If* $\mathscr{F}$, $\mathscr{G} \in F(X)$, $\mathscr{F} \sqcap G[X]$, $\mathscr{G} \sqcap g[X]$, $\mathscr{F} \to y$, $\mathscr{G} \to y'$ *and* $y \neq y'$, *then* $\mathrm{cl}_Z(g^{-1}(\mathscr{F})) \perp \mathrm{cl}_Z(g^{-1}(\mathscr{G}))$.

If we now strengthen the structure of the codomain $Y$, then we obtain a result which is strongly similar to Taĭmonov's fundamental theorem. Recall that $(X, q)$ is *regular* if $\mathscr{F} \in F(X)$ and $\mathscr{F} \to x$, then $\mathrm{cl}_X(\mathscr{F}) \to x$. Notice that a regular Hausdorff space is Urysohn.

THEOREM 3.13. *Let* $G: Z \to Y$ *be a weakly-continuous extension of* $g: X \to Y$ *and* $Y$ *be regular Hausdorff. If* $\mathscr{F}$, $\mathscr{G} \in F[Y]$, $\mathscr{F} \sqcap g[X]$, $\mathscr{G} \sqcap g[X]$, $\mathscr{F} \to y$, $\mathscr{G} \to y'$ *and* $y \neq y'$, *then* $\mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y(\mathscr{F})) \perp \mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y(\mathscr{G}))$.

PROOF. Assume the hypothesis and that

$$\mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y\mathscr{F})) \sqcap \mathrm{cl}_Z(g^{-1}(\mathrm{cl}_Y\mathscr{G})).$$

Let arbitrary $F \in \mathscr{F}$ and $H \in \mathscr{G}$. Then

$$\emptyset \neq \mathrm{cl}_Z(g^{-1}[\mathrm{cl}_Y F]) \cap \mathrm{cl}_Z(g^{-1}[\mathrm{cl}_Y H]).$$

Since $G$ is weakly-continuous, then

$$\begin{aligned}\emptyset \neq\ & G[\mathrm{cl}_Z(g^{-1}[\mathrm{cl}_Y F])] \cap G[\mathrm{cl}_Z(g^{-1}[\mathrm{cl}_Y H])]\\ &\subset \mathrm{cl}_Y(G(g^{-1}[\mathrm{cl}_Y F]) \cap \mathrm{cl}_Y(G[g^{-1}[\mathrm{cl}_Y H]])\\ &\subset \mathrm{cl}_Y(\mathrm{cl}_Y F) \cap \mathrm{cl}_Y(\mathrm{cl}_Y H).\end{aligned}$$

Now regularity implies that $\mathrm{cl}_Y(\mathrm{cl}_Y\mathscr{F})) \to y$ and $\mathrm{cl}_Y(\mathrm{cl}_Y(\mathscr{G}) \to y'$. Since $\mathrm{cl}_Y(\mathrm{cl}_Y F)$and $\mathrm{cl}_Y(\mathrm{cl}_Y H)$ are base elements for $\mathrm{cl}_Y(\mathrm{cl}_Y\mathscr{F}))$ and $\mathrm{cl}_Y(\mathrm{cl}_Y(\mathscr{G}))$ respectively, then there exists an ultrafilter $\mathscr{U} \in U(Y)$ such that $\mathrm{cl}_Y(\mathrm{cl}_Y\mathscr{F}) \subset \mathscr{U}$ and $\mathrm{cl}_Y(\mathrm{cl}_Y \mathscr{G}) \subset \mathscr{U}$. Hausdorffness implies that $y = y'$, since $\mathscr{U} \to y$ and $\mathscr{U} \to y'$. The result follows from this contradiction.

**4. General extension theory.** In his paper "Extending maps from dense subspaces" **[23]** Rudolf discusses the philosophy of extensions of topologically continuous maps. He writes, "The classical notion of (topological) continuity does not seem to be natural when nonregular spaces are in question, and it has often been replaced either from necessity or for convenience by the notion of $\theta$-continuity." The same type of replacement appears to have merit for general extension theory on preconvergence spaces. This is especially true if we require only the minimal condition for extendibility of a map. There are, however, some special difficulties associated with preconvergences spaces. In particular, if $(R, q'')$ is non-$U$-principal, then there exists $r \in R$ and $\mathscr{U} \in U(Z, r)$ such that $\mathscr{U} \perp X$. Since there are important extensions of $(X, q)$ such as the Richardson compactification **[20]** which are not $U$-principal extensions, then this difficulty should in some manner be effectively overcome. One natural method to eliminate this difficulty and which is similar in content to the introduction of $\theta$-continuity or weak-$\theta$-continuity for topological spaces is to introduce the following concept of "$n$-continuous" and "weakly-$n$-continuous" extensions. First, recall that for each $x \in X$ the neighborhood filter of $x$ is

$$\mathscr{N}(x) = \bigcap\{\mathscr{U} \mid [\mathscr{U} \in U(X)] \wedge [\mathscr{U} \to x]\}.$$

For any extension $Z$ of $X$ notice that for each $z \in Z$, $\mathscr{N}(z) \cap X$.

DEFINITION 4.1. A map $g\colon X \to Y$ is *n-continuous* if for each $x \in X$, $\mathscr{N}(g(x)) \subset g(\mathscr{N}(x))$.

We now have the following two elementary properties for $n$-continuity.

THEOREM 4.1. *If $g\colon X \to Y$ is weakly-continuous, then $g$ is $n$-continuous.*

PROOF. Let $x \in X$. Then

$$\begin{aligned}g(\mathscr{N}(x)) &= g(\bigcap \{\mathscr{U} \mid [\mathscr{U} \in U(X)] \wedge [\mathscr{U} \to x]\})\\ &= \bigcap\{g(\mathscr{U}) \mid [\mathscr{U} \in U(X)] \wedge [\mathscr{U} \to x]\}\\ &\supset (\bigcap\{\mathscr{V} \mid [\mathscr{V} \in U(Y)] \wedge [\mathscr{V} \to g(x)]\}).\end{aligned}$$

Hence $\mathcal{N}(g(x)) \subset g(\mathcal{N}(x))$.

THEOREM 4.2. *If $Y$ is pretopological and $g: X \to Y$ is n-continuous, then $g$ is continuous.*

PROOF. Since $Y$ is pseudotopological, we need only show weak-continuity. Let $\mathcal{U} \to x \in X$ and $\mathcal{U} \in U(X)$. Then $\mathcal{N}(x) \subset \mathcal{U}$ implies that $g(\mathcal{N}(x)) \subset g(\mathcal{U})$. By $n$-continuity, $\mathcal{N}(g(x)) \subset g(\mathcal{N}(x)) \subset g(\mathcal{U})$. Since $Y$ is pretopological, then $\mathcal{N}(g(x)) \to g(x)$. Thus $g(\mathcal{U}) \to g(x)$ and the result follows.

Observe that if $(X, q)$ is non-pretopological and $\hat{q}$ is the pretopological modification of $q$ [**10**], then the identity map $I: (X, \hat{q}) \to (X, q)$ is $n$-continuous but not continuous.

DEFINITION 4.2. A map $g: X \to Y$ is weakly-$n$-continuous if for each $x \in X$, $\mathrm{cl}_Y(\mathcal{N}(g(x))) \subset g(\mathcal{N}(x))$.

Clearly, $n$-continuity implies weak-$n$-continuity.

THEOREM 4.3. *If $g: X \to Y$ is weakly-n-continuous and $Y$ is a regular pretopological space, then $g$ is continuous.*

PROOF. The result follows from the simple observation that for each $x \in X$, $\mathrm{cl}_Y(\mathcal{N}(g(x))) = \mathcal{N}(g(x))$.

REMARK 4.1. There are numerous examples of regular pretopological convergence spaces which are not topological [**1**, p. 495].

We now eatablish the major result in this section.

THEOREM 4.4. *Let $g: X \to Y$ be weakly-continuous and weakly-admissible. If $Z$ is any extension of $X$, then $g$ can be extended to a weakly-n-continuous map $G: Z \to T$.* If *$Y$ is Hausdorff, then $G$ is unique.*

PROOF. Let $A, B \subset Z$. Then it is straightforward to show that, in general, if $N \in \mathcal{N}(z)$, $z \in Z$, then $A \cap \mathrm{cl}_Z(B) = A \cap \mathrm{cl}_Z(B \cap N)$. Hence let $A = N$ and $B = X$. It follows that $N \cap \mathrm{cl}_Z(X) = N \cap Z = N = N \cap \mathrm{cl}_Z(X \cap N)$. Therefore, $N \subset \mathrm{cl}_Z(N \cap X)$. Define the map $G$ in the usual manner as follows: if $r \in R$, let $G(r) \in I(g(U_X(Z, r)))$; and if $x \in X$, let $G(x) = g(x)$. Now for $M \subset Y$, assume that $r \in R \cap \mathrm{cl}_Z(g^{-1}[M])$. Consequently, $g^{-1}[M] \neq \emptyset$ and there exists some $\mathcal{U} \in U(Z)$ such that $\mathcal{U} \to r$ and $g^{-1}[M] \in \mathcal{U}$. Thus $\mathcal{U}_X$ exists. Hence $gg^{-1}[M] \subset M \in g(\mathcal{U}_X) \to G(r)$ by weakly-admissibility. Since $g(\mathcal{U}_X) \in U(Y)$, then this implies that $G(r) \in \mathrm{cl}_Y(M)$. Therefore, $G[R \cap \mathrm{cl}_Z(g^{-1}[M])] \subset \mathrm{cl}_Y M$. Now consider $\mathcal{N}(r)$, $r \in R$. Then $(\mathcal{N}(r))_X = \bigcap\{\mathcal{U}_X \mid \mathcal{U}_X \in U_X(Z, r)\}$. Hence $g((\mathcal{N}(r))_X) = \bigcap\{g(\mathcal{U}_X) \mid \mathcal{U}_X \in U_X(Z, r)\}$ and weak-admissibility imply that $\mathcal{N}(G(r)) \subset g((\mathcal{N}(r))_X)$. Let $M \in \mathcal{N}(G(r))$. Then there exists some $N \cap X \in (\mathcal{N}(r))_X$,

$N \in \mathcal{N}(r)$ such that $N \cap X \subset g^{-1}[M]$. Consequently, $\mathrm{cl}_Z(N \cap X) \subset \mathrm{cl}_Z(g^{-1}[M])$. Now

$$\begin{aligned} G[N] &\subset G[\mathrm{cl}_Z(N \cap X)] \subset G[\mathrm{cl}_Z(g^{-1}[M])] \\ &= G[R \cap \mathrm{cl}_Z(g^{-1}[M])] \cup G[X \cap \mathrm{cl}_Z(g^{-1}[M])] \\ &\subset \mathrm{cl}_Y(M) \cup g[X \cap \mathrm{cl}_Z(g^{-1}[M])]. \end{aligned}$$

It is an easy exercise to show that, in general, for each $A \subset X$, $\mathrm{cl}_X A = \mathrm{cl}_Z A \cap X$. Consequently, $g[X \cap \mathrm{cl}_Z(g^{-1}[M])] = g[\mathrm{cl}_X(g^{-1}[M])]$. Since $g$ is weakly-continuous, then

$$g[\mathrm{cl}_X(g^{-1}[M])] \subset \mathrm{cl}_X(gg^{-1}[M]) \subset \mathrm{cl}_Y M.$$

This yields that $G[N] \subset \mathrm{cl}_Y M$. Hence $\mathrm{cl}_Y(\mathcal{N}(G(r))) \subset G(\mathcal{N}(r))$ implies that $G$ is weakly-$n$-cotinuous on $R$.

Let $x \in X$. Thus if $\mathcal{N}_X(x)$ is a neighborhood filter of $x$ in $(X, q')$, then $\mathcal{N}_X(x) = ((\mathcal{N}(x))_X)$ since $\mathcal{U} \in U(X, x)$ if and only if $\mathcal{U} \in U_X(Z, x)$. The weak-continuity of $g$ implies that $g$ is $n$-continuous on $X$. Consequently, $\mathcal{N}(g(x)) = \mathcal{N}(G(x)) \subset g((\mathcal{N}(x))_X)$. Now letting $M \in \mathcal{N}(G(x))$, we have that there exists some $N \in \mathcal{N}(x)$ such that $N \cap X \subset g^{-1}[M]$. The same procedure as followed above for weak-$n$-continuity on $R$ yields that $G[N] \subset \mathrm{cl}_Y M$. Hence $\mathrm{cl}_Y((G\mathcal{N}(x))) \subset G(\mathcal{N}(x))$. Finally the uniqueness condition is obvious and this completes the proof.

REMARK 4.2. Note that in theorem 4.4, $G$ is weakly-$n$-continuous when restricted to $X$ and $G|X = g$. However this restriction is only shown to be weakly-$n$-continuous with respect to $q$ and not weakly-continuous.

COROLLARY 4.4.1. *Let $g: X \to Y$ be weakly-continuous and $Y$ a regular pretopological space. Then $g$ has a continuous extension onto any extension $Z$ of $X$ if and only if $g$ is weakly-admissible.*

COROLLARY 4.4.2. *Let $Y$ be a compact regular Hausdorff pretopological convergence space. Then a weakly-continuous map $g: X \to Y$ hax a unique continuous extension to any extension $Z$ of $X$ if and only if whenever $\mathcal{F}, \mathcal{G} \in F(X)$, $\mathcal{F} \cap g[X]$, $\mathcal{G} \cap g[X]$, $\mathcal{F} \to y$, $\mathcal{G} \to y'$ and $y \neq y'$, then*

$$\mathrm{cl}_Z(g^{-1}[\mathrm{cl}_Y\mathcal{F}]) \perp \mathrm{cl}_Z(g^{-1}[\mathrm{cl}_Y\mathcal{G}]).$$

REMARK 4.2. As shown in **[21]** a compact regular Hausdorff pretopological space is topological.

Three of the major extension results for topological spaces are Taĭmanov's theorem, as previously stated; Veličko's result **[26]**, which states that for topological spaces a map $g: (X, T) \to (Y, \tau)$ which is weakly-$\theta$-

continuous into an $H$-closed Urysohn space has a weakly-$\theta$-continuous extension onto a topological extension $Z$ of $X$ if and only if for each pair $F_1$, $F_2$ of disjoint $\theta$-closed subsets of $Y$, we have that $\mathrm{cl}_Z(g^{-1}[F_1]) \cap \mathrm{cl}_Z(g^{-1}F_2]) = \varnothing$ and D'Aristotle's result **[2]**, which states that a map $g$: $(X, T) \to (Y, \tau)$ which is a $c$-map into a Stone-Weierstrass space $Y$ has a $c$-continuous extension onto a topological extension $Z$ of $X$ if and only if for each pair $F_1$, $F_2$ of disjoint zero-sets in $Y$, we have that $\mathrm{cl}_Z(g^{-1}[F_1]) \cap \mathrm{cl}_Z(g^{-1}[F_2]) = \varnothing$. It is known that a space $(Y, \tau)$ is $H$-closed Urysohn if and only if $(Y, \tau_s)$, the semiregularization of $Y$, is compact Hausdorff and letting $\tau_w$ be the weak topology generated by $C(Y)$, that a space $(Y, \tau)$ is Stone-Weierstrass if and only if $(Y, \tau_w)$ is compact Hausdorff. Consequently, Veličko's and D'Aristotle's results follow as immediate corollaries to the basic Taĭmonov extension theorem. On the other hand, we now show that corollary 4.4.2 is a nontrivial extension of Taĭmonov's basic result for convergence space domains and topological codomains.

Example 4.1. Let $(X, T)$ be a compact Hausdorff topological space and $X$ an infinite set. Let $q$ by any non-topological Hausdorff convergence structure on $X$ and $(X^*, q^*)$ the Richardson compactification of $X$ (see and **[20]** **[27]**). The identity map $I$: $(X, q) \to (X, T)$ is continuous. Since $(X, T)$ is regular, there exists a unique continuous extension $I^*$: $(X^*, q^*) \to (X, T)$. The class of domains for which the Taĭmonov type characterization of Corollary 4.4.2 holds is thus strictly larger than the class of topological domains.

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(Added in proof.) I have recently realized the almost trivial fact that weakly-open is equivalent to open. This fact may be utilized to improve upon Theorems and Corollaries 3.4, 3.4.1, 3.8, 3.8.1, and 3.9 Also, 3.7 has an obvious, trivial proof.