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BOREL EXCEPTIONAL VALUES IN THE UNIT DISK

L. R. SONS

1. Introduction. Some time ago E. Borel [**1**] introduced for entire functions the idea of a Borel exceptional value. The analogous idea for functions meromorphic in the unit disk **D** has been considered by R. Nevanlinna [**4**, p. 144] (indirectly) and by M. Tsuji [**6**, p. 293]. Below we extend some of Tsuji's results and consider analogues of results of G. Valiron [**7**, p. 71–78] and S. Singh and H. Gopalakrishna [**5**] as well as some relations between Borel exceptional values and other types of exceptional values studied in value distribution theory. Contrary to the situation for entire functions, an analytic function in the unit disk may have a Borel exceptional value and not have regular growth. We shall use the notation of Nevanlinna theory (see, for example, W. Hayman [**3**]).

For our purposes we define the *order* ρ of a meromorphic function f defined in **D** by

$$\rho = \limsup_{r\to 1} \frac{\log^+ T(r,f)}{-\log(1-r)},$$

and the *lower order* λ by

$$\lambda = \liminf_{r\to 1} \frac{\log^+ T(r,f)}{-\log(1-r)}.$$

Let $\{a_n\}$ be the zeros of $f(z) - a$ for z in **D**. Define the *convergence exponent* $\mu_a \geqq 0$ of $\{a_n\}$ as follows:

(i) If $\sum_n(1 - |a_n|) < \infty$, then $\mu_a = 0$.

(ii) If $\sum_n(1 - |a_n|) = \infty$, then $\mu_a = \mu$ is that number such that for any $\varepsilon > 0$

$$\sum_n (1 - |a_n|)^{\mu+1-\varepsilon} = \infty \text{ and } \sum_n (1 - |a_n|)^{\mu+1+\varepsilon} < \infty.$$

Tsuji [**6**, p. 204] notes that

$$\int_{r_0}^1 N(r, a)(1 - r)^{\lambda-1} dr \text{ and } \sum_n (1 - |a_n|)^{\lambda+1},\ \lambda > 0,$$

converge or diverge simultaneously. So

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$$\limsup_{r \to 1} \frac{\log^+ N(r, a)}{-\log(1 - r)} = \mu_a.$$

Since $N(r, a) \leqq T(r, f) + O(1)$, $(r \to 1)$, we know that $0 \leqq \mu_a \leqq \rho$. μ_∞ is defined similarly.

If $\rho < \infty$, we say a is a *Borel exceptional value* for f if $\mu_a < \rho$. If $\rho = \infty$, we say a is a *Borel exceptional value* for f if $\mu_a < \infty$.

If the case $\rho < \infty$ the following theorem is in Tsuji [6, p. 293].

THEOREM 1. *Let f be a meromorphic function in* **D** *such that f has order $\rho \leqq \infty$. Then for every a in* **C** $\cup \{\infty\}$ *with at most two exceptions, $\mu_a = \rho$.*

Suppose we define $\bar{N}(r, a)$ in a similar manner to $N(r, a)$ where we consider only the distinct a-values of f. Let

$$\limsup_{r \to 1} \frac{\log^+ \bar{N}(r, a)}{-\log(1 - r)} = \bar{\mu}_a.$$

Then we have the following corollary.

COROLLARY 1. *Let f be a meromorphic function in* **D** *such that f has order $\rho \leqq \infty$. Then for every a in* **C** $\cup \{\infty\}$ *with at most two exceptions, $\bar{\mu}_a = \rho$.*

The best possible nature of these two theorems is seen by considering the function f defined in **D** by $f(z) = \exp((1 - z)^{-2})$.

Turning to simple zeros, we define $N_s(r, a)$ in a similar manner to $N(r, a)$ where we consider only simple a-values of f. Let

$$\limsup_{r \to 1} \frac{\log^+ N_s(r, a)}{-\log(1 - r)} = \mu_{as}.$$

Then the following theorems hold.

THEOREM 2. *Let f be a meromorphic function in* **D** *such that f has order $\rho \leqq \infty$. Suppose there are distinct elements a and b in* **C** $\cup \{\infty\}$ *which are Borel exceptional values for f. Then for any c in* **C** *with $c \neq a$ and $c \neq b$, $\mu_{cs} = \rho$.*

THEOREM 3. *Let f be a meromorphic function in* **D** *such that f has finite order ρ. If there exists an element a in* **C** $\cup \{\infty\}$ *such that $\bar{\mu}_a < \rho$, then $\mu_{bs} = \rho$ except for at most two distinct elements b in* **C** $\cup \{\infty\} - \{a\}$.

COROLLARY 2. *Let f be an analytic function in* **D** *with finite order ρ. Then $\mu_{as} = \rho$ except for at most two distinct numbers a in* **C**.

We may also define $\bar{N}_{12}(r, a)$ in a similar manner to $\bar{N}(r, a)$ where we consider each simple and each double zero of $f(z) - a$ counted once only. Let

$$\limsup_{r \to 1} \frac{\log^+ \bar{N}_{12}(r, a)}{-\log(1 - r)} = \bar{\mu}_{a12}.$$

Following a line of reasoning much like that used in the proof of Theorem 3, we can prove Theorem 4.

THEOREM 4. *Let f be a meromorphic function in* **D** *such that f has finite order ρ. Let a and b be distinct elements of* $\mathbf{C} \cup \{\infty\}$ *such that $\bar{\mu}_a < \rho$ and $\bar{\mu}_{b12} < \rho$. Then for every c in* $\mathbf{C} \cup \{\infty\}$ *such that $c \neq a$ and $c \neq b$, we have $\mu_{cs} = \rho$.*

An entire function with finite order which has a Borel exceptional value a in **C** must have its order an integer. If f is defined in **D** by $f(z) = \exp((1 - z)^{-3/2})$, we see f has zero as a Borel exceptional value and order 1/2. Further, if we define

$$\lim_{r\to 1} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log^{+}\log^{+}M(r, f)}{-\log(1 - r)} = \begin{Bmatrix} \rho^* \\ \lambda^*, \end{Bmatrix}$$

where $M(r, f) = \max|f(z)|$, $(|z| = r)$, then f has $\rho^* = 3/2$ which is also not an integer.

If an entire function of finite order has a Borel exceptional value a in **C**, the function must have regular growth (i.e., order and lower order are equal). The following theorem shows this need not be the case in the disk and also shows there are functions of irregular growth in the disk which need not take every value (i.e., no analogue exists of the result of J. M. Whittaker in [**8**]).

THEOREM 5. *There exist analytic functions f in* **D** *such that $f(z) = \exp(g(z))$ for z in* **D**, *$\lambda^* \neq \rho^*$, and $\lambda \neq \rho$.*

Finally we give a theorem relating Borel exceptional values to other types of exceptional values studied in value distribution theory.

THEOREM 6. *Let f be a meromorphic function in* **D** *such that f has a as a Borel exceptional value.*

(i) *If f has finite order ρ in* **D**, *then $\Delta(a, f)$, the Valiron deficiency of a, satisfies $\Delta(a, f) = 1$.*

(ii) *If f has $\rho = \lambda$, then $\delta(a, f)$, the Nevanlinna deficiency of a, satisfies $\delta(a, f) = 1$.*

2. Proofs of Theorem 1 and Corollary 1. Assume first that $\rho < \infty$.

Suppose there exist three distinct elements a_1, a_2, and a_3 in $\mathbf{C} \cup \{\infty\}$, with $\mu_{a_i} < \rho$. Let λ be a number such that $\mu_{a_i} < \lambda < \rho$ for $i = 1, 2, 3$. Then

$$\int_{r_0}^{1} N(r, a_i)(1 - r)^{\lambda - 1}\, dr < \infty, \quad (i = 1, 2, 3).$$

By the Second Fundamental Theorem of Nevanlinna theory (cf. [**4**, p. 143–144]), we see

$$T(r,f) < \sum_{i=1}^{3} N(r, a_i) - N_1(r) + S(r) \tag{2.1}$$

where $N_1(r) = (2N(r,f) - N(r,f')) + N(r, 1/f') \geqq 0$ and

$$\int_{r_0}^{r} S(t)(1 - t)^{\lambda-1}dt = O\left(\int_{r_0}^{r} (\log^+ T(t))(1 - t)^{\lambda-1}dt),\right)$$

$(r \to 1)$. Using (2.1) we then obtain

$$\begin{aligned}\int_{r_0}^{r} T(t,f)(1 - t)^{\lambda-1}dt &\leqq \sum_{i=1}^{3}\int_{r_0}^{r} N(t, a_i)(1 - t)^{\lambda-1}dt \\ &\quad + \int_{r_0}^{r} S(t)(1 - t)^{\lambda-1}dt \\ &\leqq C + o\left(\int_{r_0}^{r} T(t,f)(1 - t)^{\lambda-1}dt\right),\end{aligned} \tag{2.2}$$

$(r \to 1)$, where C is a positive constant. Hence,

$$(1 + o(1))\int_{r_0}^{r} T(t,f)(1 - t)^{\lambda-1}dt < \infty$$

which implies $T(t)(1 - t)^{\lambda} \to 0$ as $t \to 1$. However, since f has order ρ, there exists an increasing sequence $\{r_k\}$ which converges to one such that $T(r_k)(1 - r_k)^{\rho-\varepsilon} > 1$ for $k = 1, 2, \ldots$ where $\rho - \varepsilon > \lambda$. We have a contradiction. Slight modifications of the above give the theorem when $\rho = \infty$.

To prove Corollary 1 we observe that $N_1(r) \geqq N(r, 1/f')$, and

$$\sum_{i=1}^{3} N(r, a_i) - N(r, 1/f') \leqq \sum_{i=1}^{3} \bar{N}(r, a_i).$$

So (2.1) implies that $T(r,f) \leqq \sum_{i=1}^{3} \bar{N}(r, a_i) + S(r)$, and we may proceed as above to reach a contradiction if we assume $\bar{\mu}_{a_i} < \rho$ for three distinct elements a_1, a_2, and a_3 in $\mathbf{C} \cup \{\infty\}$.

3. Proof of Theorem 2. We may assume $a \neq \infty$. Since f' and $f - a$ have the same order (cf. Tsuji [**6**, p. 228]), we know that zero is also a Borel exceptional value for f'. By Theorem 1 we know $\mu_c = \rho$, but there is a number $\lambda < \rho$ for which

$$\sum(1 - |a_n|)^{\lambda+1+\varepsilon} < \infty$$

where $\{a_n\}$ is the sequence of multiple zeros of $f - c$ (since each value a_n is a zero of f'). Thus $\mu_{cs} = \rho$.

4. Proof of Theorem 3. We shall use the following lemma.

LEMMA. *Let f be a meromorphic function in* **D** *such that f has finite order ρ.*

(i) *If a is in* **C** *then* $\mu_a < \rho$ *if and only if there exists a meromorphic function* ϕ *defined in* **D** *such that no zero of* $f - a$ *is a pole of* ϕ *and the order of* $(f - a)\phi$ *is less than* ρ.

(ii) $\mu_\infty < \rho$ *if and only if there exists a meromorphic function* ϕ *defined in* **D** *such that no pole of* f *is a pole of* ϕ *and the order of* ϕ/f *is less than* ρ.

PROOF OF THE LEMMA.

(i) Suppose there is a number a with $\mu_a < \rho$. We may write the values of $f - a$ as $f(z) - a = z^m e^{g(z)}(P_1(z))/(P_2(z))$ where the P_i are Tsuji products (cf. [3, p. 222 and 227]) with the order of P_1 less than ρ and P_2 containing all the poles of $f - a$. We define ϕ by $\phi(z) = e^{-g(z)}P_2(z)$.

On the other hand, if there is a meromorphic function ϕ defined in **D** such that $(f - a)\phi$ has order $\rho' < \rho$, we observe that $N(r, a)$ for f is less than or equal to $N(r, 0)$ for $(f - a)\phi$. It follows that $\mu_a < \rho$.

(ii) If $\mu_\infty < \rho$ then zero is a Borel exceptional value for $1/f$. So by (i) above there exists a function ϕ meromorphic in **D** for which no zero of $1/f$ is a pole of ϕ and the order of $\phi(1/f)$ is less than ρ.

Finally, suppose there is a function ϕ which is meromorphic in **D** such that no pole of f is a pole of ϕ and the order of ϕ/f is less than ρ. Thus no zero of $1/f$ is a pole of ϕ, so (i) implies zero is a Borel exceptional value for $1/f$. Hence $\mu_\infty < \rho$.

To prove the theorem we first consider the case $a = \infty$, so $\bar{\mu}_\infty < \rho$. We assume there are three distinct numbers a_1, a_2, a_3 in **C** such that $\mu_{a_i s} < \rho$ for $i = 1, 2, 3$. Choose ρ' such that $\rho' < \rho$ and $\bar{\mu}_\infty < \rho'$ and $\mu_{a_i s} < \rho'$ for $i = 1, 2, 3$. For $i = 1, 2, 3$, let $P_i(z)$ be the Tsuji product (cf. [**6**, p. 222]) formed with the simple zeros of $f(z) - a_i$. Thus the order of $P_i(z)$ is less than ρ' so

$$m(r, P_i) = T(r, P_i) = O((1 - r)^{-\rho'}), \quad (r \to 1), \tag{4.1}$$

$(i = 1, 2, 3)$. We shall construct a function ϕ which can be employed with the Lemma to show a_1, a_2, and a_3 are Borel exceptional values for f. This contradicts Theorem 1.

Define ϕ in **D** by

$$\phi(z) = \frac{P_1(z)P_2(z)P_3(z)(f'(z))^2}{(f(z) - a_1)(f(z) - a_2)(f(z) - a_3).}$$

Then ϕ is a meromorphic function in **D**, and for $i = 1, 2, 3$, no zero of $f(z) - a_i$ is a pole of $\phi(z)$. We consider the order of $(f - a_1)\phi$. We have

$$(f(z) - a_1)\phi(z) = P_1(z)P_2(z)P_3(z)\left(\frac{f'(z)}{f(z) - a_2}\right)\left(\frac{f'(z)}{f(z) - a_3}\right). \tag{4.2}$$

Since f is of finite order, we have

$$m\left(r, \frac{f'}{f - a_i}\right) = O((1 - r)^{-\rho'}), \quad (r \to 1), \tag{4.3}$$

$(i = 1, 2, 3)$. Using (4.1), (4.2), and (4.3), we then see

$$\begin{aligned} m(r, (f - a_1)\phi) &\leqq m(r, P_1) + m(r, P_2) + m(r, P_3) \\ &\quad + m\left(r, \frac{f'}{f - a_2}\right) + m\left(r, \frac{f'}{f - a_3}\right) \\ &= O((1 - r)^{-\rho'}), \quad (r \to 1). \end{aligned} \tag{4.4}$$

By the definition of ϕ we note that the poles of $(f - a_1)\phi$ can occur only at the poles of f. Further, each pole of f is a simple pole of $(f')/(f - a_i)$, $(i = 1, 2, 3)$. Hence,

$$N(r, (f - a_1)\phi) = 2\bar{N}(r, f) = O((1 - r)^{-\rho'}), \quad (r \to 1), \tag{4.5}$$

using our choice of ρ'. From (4.4) and (4.5) we conclude

$$T(r, (f - a_1)\phi) = O((1 - r)^{-\rho'}), \quad (r \to 1),$$

so the order of $(f - a_1)\phi$ is less than or equal to ρ'. Similarly the order of $(f - a_2)\phi$ and of $(f - a_3)\phi$ is less than or equal to ρ'. By the Lemma, a_1, a_2, a_3 are Borel exceptional values for f, and Theorem 1 is contradicted.

We now consider the case $a \neq \infty$. We define g in **D** by $g(z) = 1/(f(z) - a)$. Then g has order ρ, and we may apply the above result to g. Thus there are at most two distinct values in **C** which are exceptional for simple values of g. If b is in $\mathbf{C} \cup \{\infty\} - \{a\}$, then $1/(b - a)$ is in **C**, and $f(z) = b$ if and only if $g(z) = 1/(b - a)$. Hence $\mu_{bs} = \rho$ except for at most two values of b in $\mathbf{C} \cup \{\infty\} - \{a\}$.

5. Proof of Theorem 5. The construction given is an adaptation of J. Clunie [**2**].

Fix α and β as non-negative numbers for which $\alpha + 2 < \beta$, and let ϕ be an increasing, convex, continuous function defined on $0 \leqq r < 1$ such that

$$\lim_{r \to 1} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\phi(r)}{-\log(1 - r)} = \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix}, \tag{5.1}$$

and $\phi(1/2) < 1$.
Using A. Zygmund [**9**, p. 69] we may assume

$$\phi(r) = \int_{1/2}^{r} \frac{\psi(t)}{t} dt + \phi(1/2) \tag{5.2}$$

where ψ is continuous, strictly increasing, and unbounded.

Let $r_1 < r_2 < r_3 < \cdots$ be the sequence in the unit interval for which $\psi(r_n) = n$. We define a function G by $G(z) = \sum_{n=1}^{\infty} a_n z^n$ where $a_n =$

$1/(r_1 r_2 \cdots r_n)$, $(n \geqq 1)$. Then G is an analytic function in **D** and $\mu(r, G) = a_n r^n$, $(r_n \leqq r < r_{n+1})$, where $\mu(r, G) = \max_k a_k r^k$. From (5.2) we see

$$\phi(1/2) + \log \mu(r, G) \leqq \phi(r) \leqq \log \mu(r, G) + \log r - \log 1/2 + \phi(1/2),$$

and hence

$$\log \mu(r, G) \sim \phi(r), \quad (r \to 1). \tag{5.3}$$

Thus the sequence $\{a_n r_n^n\}$ for $n = 1, 2, \ldots$ is strictly increasing and unbounded.

Define now a sequence $\{\lambda_n\}$ as follows: Take $\lambda_1 = 1$ and assume $\lambda_1, \lambda_2, \ldots, \lambda_n$ have been selected. If

$$a_{\lambda_n+1}\, r_{\lambda_n+1}^{\lambda_n+1} > 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n},$$

take $\lambda_{n+1} = \lambda_n + 1$. Otherwise take λ_{n+1} to be the largest integer m for which

$$a_m r_m^m \leqq 2a_{\lambda_n} r_{\lambda_n}^{\lambda_n}.$$

Set

$$g(z) = \sum_{u=1}^{\infty} n^{-2}\, a_{\lambda_n} z^{\lambda_n}.$$

Then as in J. Clunie [2] we may conclude

$$\log M(r, g) \sim \log \mu(r, G), \quad (r \to 1). \tag{5.4}$$

Thus, defining f by $f(z) = \exp g(z)$, we see

$$M(r, f) = \max_{|z|=r} \exp(\operatorname{Re} g(z)) = \exp M(r, g).$$

From (5.1), (5.3), and (5.4) we have $\alpha = \lambda^*$ and $\beta = \rho^*$. Consequently, the inequality

$$T(r, f) \leqq \log^+ M(r, f) \leqq \frac{R + r}{R - r}\, T(R, f), \quad (0 \leqq r < R < 1),$$

(cf. [3, p. 18]) implies $\lambda \leqq \lambda^* \leqq \rho^* \leqq \rho + 1 \leqq \rho^* + 1$, and our choice of $\lambda^* + 2 < \rho^*$ clearly gives $\lambda \neq \rho$.

6. Proof of Theorem 6. To prove (i) we observe that the First Fundamental Theorem of Nevanlinna theory gives

$$T(r, f) = m(r, a) + N(r, a) + O(1), \quad (r \to 1).$$

Hence

$$\limsup_{r \to 1} \frac{\log^+ N(r, a)}{-\log(1 - r)} = \mu_a$$

with $\mu_a < \rho$ implies

$$\Delta(a, f) = 1 - \liminf_{r \to 1} \frac{N(r, a)}{T(r, f)} = 1.$$

In a similar way the conditions in (ii) show

$$\delta(a, f) = 1 - \limsup_{r \to 1} \frac{N(r, a)}{T(r, f)} = 1.$$

References

1. E. Borel, *Sur les zéros des fonctions entières*, Acta Math. **20** (1896), 357–396.

2. J. Clunie, *On integral functions having prescribed asymptotic growth*, Canadian J. Math. **17** (1965), 396–404.

3. W. K. Hayman, *Meromorphic Functions*, London, 1964.

4. R. Nevanlinna, *Le théorème de Picard-Borel et la theorie des fonctions meromorphes*, Paris, 1929.

5. S. K. Singh and H. S. Gopalakrishna, *Exceptional values of entire and meromorphic functions*, Math. Ann. **191** (1971), 121–142.

6. M. Tsuji, *Potential Theory in Modern Function Theory*, Tokyo, 1959.

7. G. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse, 1923.

8. J. M. Whittaker, *Entire functions of irregular growth take every value*, Bull. London Math. Soc. **4** (1972), 130–132.

9. A. Zygmund, *Trigonometrical Series*, 1st ed., Warsaw, 1935.

Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115