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CONNECTIVITY AND THE L-FUZZY UNIT INTERVAL

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ABSTRACT. The L-fuzzy unit interval has been shown by Bruce Hutton to be connected if L is orthocomplemented. In this paper we show that the L-fuzzy unit interval, the L-fuzzy open unit interval, and the L-fuzzy real line are connected if $0 \in L^b$, a condition holding if L is a chain.

1. Introduction. In [4] Bruce Hutton constructed the L-fuzzy unit interval $I(L)$ and began the study of the fuzzy topological properties of $I(L)$. One interesting result of [4] states that if L is orthocomplemented ($\alpha \vee \alpha' = 1$ for each $\alpha \in L$), then the fuzzy open sets of $I(L)$ and the usual open sets of $I = [0, 1]$ are in a one-to-one correspondence which preserves unions (suprema) and interesctions (infima)—from this result or its proof it follows that $I(L)$ has many fuzzy topological properties such as connectedness (as defined in §3). Infortunately this result of [4] does not include the case when L is a chain. In the non-orthocomplemented case, the study of the fuzzy topological properties of $I(L)$ entails several open questions (see [2] and [5]–[8]).

It is the main purpose of this paper to show that $I(L)$, the L-fuzzy open unit interval $(0, 1)$ (L), and the L-fuzzy real line $\mathbf{R}(L)$ are connected (as defined in §3) if $0 \in L^b$ (a condition satisfied by chains). In §2 preliminaries are discussed, in §3 connectivity is discussed generally, and in §4 the main results are presented.

2. Preliminaries. The definitions of L-fuzzy sets, L-fuzzy topologies, and related concepts are found in [1]–[4], [9], [10]. (X, T) is an L-fuzzy topological space (abbreviated L-fts). Each lattice L in this paper is completely distributive, possesses infimum 0 and supremum 1, and is equipped with an order reversing involution $\alpha \to \alpha'$. If $\alpha \in L$, then α is *nonsup* (*in* L) [*noninf* (*in* L)] if α is not the supremum [infimum] of any nonempty subset of $L - \{\alpha\}$. Note α is noninf if and only if α' is nonsup. The following subsets of L are useful (cf. [2] and [5]–[7]):

$L^c = \{\alpha \in L\colon \alpha \text{ is comparable to each } \beta \in L\}$,

$L^b = \{\alpha \in L\colon \alpha < \beta \text{ and } \alpha < \gamma \text{ imply } \alpha < \beta \wedge \gamma\}$,

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$L_b = \{\alpha \in L: \alpha' \in L^b\}$, $L^a = L^b \cap L^c$,
$L_a = L_b \cap L^c$, and
$L^d = \{\alpha \in L^c$: there is $\beta \in L^c$ such that $\beta > \alpha$ and $[\alpha, \beta]$ is a chain$\}$.
Clearly, $L^d \subset L^a \subset L^c$, $L^b \supset L^a$, etc., with these inclusions generally not reversible. If L is a chain, then each of these subsets equals L. L is *ortho-complemented* if $\alpha \vee \alpha' = 1$ for $\alpha \in L$.

Let (X, T) be an L-fts and let $A \subset X$. The *α-closure* [*α^*-closure*] of A, $\mathrm{Cl}_\alpha(A)$ [$\mathrm{Cl}_{\alpha^*}(A)$], is defined to be $\{x \in X: u \in T$ and $u(x) > \alpha$ $[\geqq \alpha]$ imply $u|A \neq 0\}$ (cf. [**5**], [**6**]). The set A is *α-closed* [*α^*-closed*] if $\mathrm{Cl}_\alpha(A)$ [$\mathrm{Cl}_{\alpha^*}(A)$] $\subset A$. For further discussion see [**5**] and [**6**].

For the constructions of the L-fuzzy unit interval $I(L)$, open unit interval $(0, 1)$ (L), and real line $\mathbf{R}(L)$, see [**2**] and [**4**]. The following result of [**6**] and [7] is useful in the study of $I(L)$, $(0, 1)$ (L), and $\mathbf{R}(L)$.

PROPOSITION 2.1. *Let* $[\lambda]$ *be a member of* $\mathbf{R}(L)$. *The following statements hold.*

(1) *Let* $\alpha \in L^c$. *There is* $a(\lambda, \alpha) \in [-\infty, +\infty]$ *such that for some representative, say* λ, $\lambda(t) < \alpha'$ *if and only if* $t > a(\lambda, \alpha)$. *There is* $b(\lambda, \alpha) \in [-\infty, +\infty]$ *such that for some* (*other*) *representative, say* λ, $\lambda(t) > \alpha$ *if and only if* $t < b(\lambda, \alpha)$.

(2) *Let* $\alpha \in L^c$. *There is* $a^*(\lambda, \alpha) \in [-\infty, +\infty]$ *such that for some representive, say* λ, $\lambda(t) \leqq \alpha'$ *if and only if* $t \geqq a^*(\lambda, \alpha)$. *There is* $b^*(\lambda, \alpha) \in [-\infty, +\infty]$ *such that for some* (*other*) *representative, say* λ, $\lambda(t) \geqq \alpha$ *if and only if* $t \leqq b^*(\lambda, \alpha)$.

3. Connectivity generally.

DEFINITION 3.1. Let $\alpha \in L$. We say (X, T) is *α-connected* [*α^*-connected*] if there do not exist $u, v \in T - \{0, 1\}$ such that on X, $u \vee v > \alpha'$ $[\geqq \alpha']$ and $u \wedge v = 0$. (X, T) is *α-disconnected* [*α^*-disconnected*] if there are $u, v \in T - \{0, 1\}$ such that on X, $u \vee v > \alpha$ $[\geqq \alpha]$ and $u \wedge v = 0$. (X, T) is *connected* if it is 1-connected and *disconnected* if it is 1*-disconnected.

REMARK 3.1. (1) In keeping with the definitions of α-compactness and α^*-compactness (see [**2**]) and α-Hausdorff and α^*-Hausdorff (see [**6**]), we have allowed for degrees of connectivity and disconnectivity. These may be related; if $\alpha < \beta$, then β-connected implies α-connected, and β-disconnected implies α-disconnected. Similarly for the α^* case if $\alpha \leqq \beta$, then β^*-connected implies α-connected and if $\alpha > \beta$, then α-connected implies β^*-connected. See Corollary 4.2 of §4 for examples.

(2) We have recognized that a greater degree of connectivity should be associated with a lesser degree of disconnectivity, e.g., (X, T) is α-connected if and only if it fails to be α'-disconnected (recall $\alpha \to 1$ if and only if $\alpha' \to 0$).

(3) It can be shown using functors of [**6**] that Definition 3.1 is a true

generalization of the connectivity and disconnectivity of ordinary topology.

(4) Definition 3.1 may be simplified by replacing "$(u \vee v)(x) >$" with "$u(x) >$ or $v(x) >$" if $\alpha \in L^c$ (recall $\alpha \in L^c$ if and only if $\alpha' \in L^c$), "$(u \vee v)(x) \geqq$" with "$u(x) \geqq$ or $v(x) \geqq$" if $\alpha \in L^a$ in the α^*-connected case ($\alpha \in L_a$ in the α^*-disconnected case), and "$(u \wedge v)(x) =$" with "$u(x) =$ or $v(x) =$" if $0 \in L^b$.

If $B \subset X$ and (X, T) is an L-fts, then B is *α-connected* if B is α-connected in the fuzzy subspace topology [9]. An *α-component* of (X, T) is a maximal (with respect to inclusion) α-connected subset of X. Similar conventions are adopted for the α^*-case.

PROPOSITION 3.1. *Let (X, T) be an L-fts.* (1) *Unions of pairwise intersecting α-connected sets are α-connected.* (2) *(X, T) is α-connected implies there is not a non-empty proper subset A of X such that A and $X - A$ are α'-closed. If* $0 \in L^b$ *and* $\alpha \in L^c$, *then* (3) *the converse of* (2) *holds,* (4) *B is α-connected if $A \subset B \subset \mathrm{Cl}_{\alpha'}(A)$ and A is α-connected, and* (5) *each α-component is α'-closed. Similar statements hold in the α^*-case if $\alpha < 1$ in* (1) *and* (2), *and* $0 \in L^b$ *and* $\alpha \in L^a$ *in* (3), (4), *and* (5).

PROOF. For (1) we show $C = A \cup B$ is α-connected if A and B are α-connected and $A \cap B \neq \emptyset$—the general proof is very similar. Suppose not; there are $u, v \in T(C) - \{0_C, 1_C\}$ such that $u \vee v > \alpha'$ and $u \wedge v = 0$ on C. It follows by case work that either each of $u|A$, $v|A$ is not in $\{0_A, 1_A\}$ or each of $u|B$, $v|B$ is not in $\{0_B, 1_B\}$. If, say, the latter, $(u|B \vee v|B) > \alpha'$ and $u|B \wedge v|B = 0$ on B yields a contradiction.

To show (2), suppose not, and observe A and $X - A$ are α'-closed implies that for each $x \in A$, $y \in X - A$, there are $u_x, v_y \in T$ such that $u_x(x) > \alpha'$, $u_x|X - A = 0$, $v_y(y) > \alpha'$, and $v_y|A = 0$. Let $u = \vee_A u_x$ and $v = \vee_{X-A} v_y$. It follows that $u, v \in T - \{0, 1\}$ and $u \vee v > \alpha'$, $u \wedge v = 0$ on X, a contradiction.

To show (3), if u and v are the α'-disconnection of X, then, using the $0 \in L^b$ and $\alpha \in L^c$ hypotheses, $A = \{x: u(x) > \alpha'\}$ and $B = \{x: v(x) > \alpha'\}$ are non-empty proper α'-closed sets such that $B = X - A$, a contradiction.

To show (4), we need only consider $A \neq \emptyset$ and $A \subsetneqq B$. If B is not α-connected, it follows there exist $u, v \in T$ such that $A \subset \{x: u(x) > \alpha'\}$ and $(B - A) \cap \{x: v(x) > \alpha'\} \neq \emptyset$. This contradicts the assumption that $B \subset \mathrm{Cl}_{\alpha'}(A)$.

Note (5) is immediate from (4). The proofs for the α^*-case are similar and omitted.

PROPOSITION 3.2. *Fuzzy continuity preserves α-connectivity. A fuzzy homeomorphism maps α-components onto α-components and does so fuzzy* *homeomorphically. Similar statements hold in the α^*-case.*

PROOF. The first assertion follows from the fact that if $f: X \to Y$ and b is a fuzzy set in Y, then $f^{-1}(b)(x) = b(f(x))$. The second assertion follows from the first.

THEOREM 3.1. *Let $\{(X_\gamma, T_\gamma): \gamma \in \Gamma\}$ be a collection of L-fts, and let P be the L-fuzzy product topology on $\Pi_{\gamma \in \Gamma} X_\gamma$(see* **[10]**). *Then $(\Pi_{\gamma \in \Gamma} X_\gamma, P)$ is α-connected implies each (X_γ, T_γ) is α-connected. The converse holds if $0 \in L^b$ and $\alpha \in L^c$. Similar statements hold in the α^*-case, where for the converse $0 \in L^b$ and $\alpha \in L^c - \{1\}$ is noninf.*

PROOF. The first assertion is immediate since fuzzy projections are fuzzy continuous **[10]**. For the converse, if $x = \{x_\gamma\}$ and $y = \{y_\gamma\}$ differ by at most finitely many coordinates, then x and y lie in an α-connected fuzzy subspace of $(\Pi_{\gamma \in \Gamma} X_\gamma, P)$; this follows by induction on Proposition 3.1(1) and the fact that fuzzy injections (though not fuzzy continuous) preserve α-connectivity. Furthermore, given $x = \{x_\gamma\}$, $\mathrm{Cl}_{\alpha'}(D) = \Pi_{\gamma \in \Gamma} X_\gamma$ where $D = \{y = \{y_\gamma\}: x \text{ and } y \text{ differ by at most finitely many coordinates}\}$. To see this, let $z \in \Pi_{\gamma \in \Gamma} X_\gamma$ and let $u \in P$ such that $u(z) > \alpha'$. Since $\alpha' \in L^c$, there are $\gamma_1, \ldots, \gamma_n$ such that

$$u \geqq \bigwedge_{i=1}^{n} p_{\gamma_i}^{-1}(u_{\gamma_i}) \text{ and } \bigwedge_{i=1}^{n} p_{\gamma_i}^{-1}(u_{\gamma_i})(z) > \alpha'$$

where $u_{\gamma_i} \in T_{\gamma_i}$ for each i. Let $y = \{y_\gamma\}$ be chosen such that $y_{\gamma_i} = z_{\gamma_i}$ for each i and $y_\gamma = x_\gamma$ otherwise. Then $y \in D$ and

$$u(y) \geqq \bigwedge_{i=1}^{n} p_{\gamma_i}^{-1}(u_{\gamma_i})(y) = \bigwedge_{i=1}^{n} u_{\gamma_i}(p_{\gamma_i}(y))$$

$$= \bigwedge_{i=1}^{n} u_{\gamma_i}(z_{\gamma_i}) = \bigwedge_{i=1}^{n} p_{\gamma_i}^{-1}(u_{\gamma_i})(z) > \alpha' \geqq 0.$$

The claim follows. The theorem follows from Proposition 3.1(4).

4. Connectivity in I(L), (0, 1) (L), and R(L). The following definition uses notation developed in Proposition 2.1.

DEFINITION 4.1. Let $X \subset \mathbf{R}(L)$ and $\alpha \in L^c$. We say X is $\mathbf{C}(\alpha, L)$ if there does not exist a nonempty, proper subset A of X such that for each $[\lambda_1] \in A$ and $[\mu_1] \in X - A$, there exist A_1, A_2, B_1, B_2 such that $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, and the following hold:

(i) $a(\mu_1, \alpha') < \wedge_{A_1} a(\lambda, 0)$ and $\vee_{A_2} b(\lambda, 0) < b(\mu_1, \alpha')$; and

(ii) $a(\lambda_1, \alpha') < \wedge_{B_1} a(\mu, 0)$ and $\vee_{B_2} b(\mu, 0) < b(\lambda_1, \alpha')$.

We adopt the convention that $\wedge_\varnothing = +\infty$ and $\vee_\varnothing = -\infty$; for example A_1 could be empty. We say X is $C^*(\alpha, L)$ if in the above we replace $a(\ , \alpha')$ by $a^*(\ , \alpha')$, $b(\ , \alpha')$ by $b^*(\ , \alpha')$, and $<$ by $\leqq$.

THEOREM 4.1. *Let* (X, T) *be a fuzzy subspace of* $\mathbf{R}(L)$. *Consider the following statements*:

(1) (X, T) *is* α*-connected*;

(2) X is $C(\alpha, L)$;

(3) (X, T) *is* α^**-connected*; *and*

(4) X *is* $C^*(\alpha, L)$.

Then the following statements hold: (1) *implies* (2) *if* $\alpha \in L_a$; (3) *implies* (4) *if* $\alpha \in L^c - \{1\}$; (2) *implies* (1) *if* $0 \in L^b$ *and* $\alpha \in L^c$; *and* (4) *implies* (3) *if* $0 \in L^b$ *and* $\alpha \in L^c$ *is noninf.*

PROOF. To show that (2) implies (1), suppose (X, T) is not α-connected. By Proposition 3.1(3) there is a nonempty, proper subset A of X such that A and $X - A$ are α'-closed. Fix $[\lambda] \in A$ and let $[\mu] \in X - A$. Since $X - A$ is α'-closed, there is $u \in T$ such that $u([\lambda]) > \alpha'$ and $u|X - A = 0$, hence $u([\mu]) = 0$. Since $\alpha' \in L^c$, we may suppose that u is a basic fuzzy open set, in particular, that $u = L_t \wedge R_s$(see [2]) where $t, s \in [-\infty, +\infty]$ and $L_t[\nu] = (\wedge_{r<t} \nu(r))'$, $R_s[\nu] = \vee_{r>s} \nu(r)$ (see [2] or [4]). Since $0 \in L^b$ and $\alpha' \in L^c$, then $L_t([\lambda]) > \alpha'$ and $R_s([\lambda]) > \alpha'$, $L_t([\mu]) = 0$ or $R_s([\mu]) = 0$. This is equivalent to (see [6] or [7]) $a(\lambda, \alpha') < t$ and $s < b(\lambda, \alpha')$, and $t \leqq a(\mu, 0)$ or $b(\mu, 0) \leqq s$. Define B_1, B_2 by $B_1 = \{[\mu] \in X - A: t \leqq a(\mu, 0)\}$ and $B_2 = \{[\mu] \in X - A: b(\mu, 0) \leqq s\}$. Then $X - A = B_1 \cup B_2$, $a(\lambda, \alpha') < \wedge_{B_1} a(\mu, 0)$, and $\vee_{B_2} b(\mu, 0) < b(\lambda, \alpha')$. A similar proof shows that given $[\mu] \in X - A$, there are A_1, A_2 such that $A = A_1 \cup A_2$, $a(\mu, \alpha') < \wedge_{A_1} a(\lambda, 0)$, and $\vee_{A_2} b(\lambda, 0) < b(\mu, \alpha')$. Hence the denial of (2) is established, so (2) implies (1). If we assume that $\alpha \in L_b$, then the denial of (2) allows us to reverse the steps of the above proof and pick for each $[\lambda] \in A$ and $[\mu] \in (X - A)$, t_1, s_1, t_2 and s_2 such that $u = L_{t_1} \wedge R_{s_1}$ and $v = L_{t_2} \wedge R_{s_2}$ suffice, i.e., $u([\lambda])$, $v([\mu]) > \alpha'$ and $u|X - A$, $v|A = 0$. Hence A and $X - A$ are α'-closed and the denial of (1) follows from Proposition 3.1 (2). The proof of the α^*-case is similar to the above.

THEOREM 4.2. $I(L)$, $(0, 1)$ (L), *and* $\mathbf{R}(L)$ *are connected if* $0 \in L^b$.

PROOF. We may assume $\{0, 1\} \subsetneqq L$, for otherwise $I(L) = [0, 1]$, $(0, 1)(L) = (0, 1)$, and $\mathbf{R}(L) = \mathbf{R}$. Let (X, T) be any of these spaces. Suppose X is not $C(1, L)$. Then X possesses a nonempty, proper subset A such that given $[\lambda_1] \in A$ and $[\mu_1] \in X - A$, $A = A_1 \cup A_2$, $X - A = B_1 \cup B_2$ where $a(\mu_1, 0) < \wedge_{A_1} a(\lambda, 0)$, $\vee_{A_2} b(\lambda, 0) < b(\mu_1, 0)$, $a(\lambda_1, 0) < \wedge_{B_1} a(\mu, 0)$ and $\vee_{B_2} b(\mu, 0) < b(\lambda_1, 0)$.

Several cases are to be considered—the proof is similar in each. We give the proof for the case when each A_i and B_i are nonempty. We may assume without loss of generality that $[\mu_1] \in B_1$. Then $[\lambda_1] \in A_2$ and the following inequalities hold:

$$a(\lambda_1, 0) < \wedge_{B_1} a(\mu, 0) \leqq a(\mu_1, 0) < \wedge_{A_1} a(\lambda, 0),$$

and

$$\vee_{B_2} b(\mu, 0) < b(\lambda_1, 0) \leqq \vee_{A_2} b(\lambda, 0) < b(\mu_1, 0).$$

Let $\alpha \in L - \{0, 1\}$, let $a \in (a(\lambda_1, 0), \wedge_{B_1} a(\mu, 0))$, let $b \in (\vee_{A_2} b(\lambda, 0), b(\mu_1, 0))$, and define $\lambda_2 \colon \mathbf{R} \to L$ by

$$\lambda_2(t) = \begin{cases} 1, & t < a \\ \alpha, & a < t < b \\ 0, & t > b \end{cases}$$

(the definition of λ_2 on $\{a, b\}$ is inconsequential). Then $a(\lambda_2, 0) = a$ and $b(\lambda_2, 0) = b$. It follows that $[\lambda_2]$ is not in any of A_1, A_2, B_1 or B_2, a contradiction. Hence X is $C(1, L)$ and the theorem follows from Theorem 4.1.

Let $0 \in L^b$. We note $I(L)$ is a fuzzy continuum (compact in the sense of [**1**] or [**3**] and connected), and an α-continuum (α-compact in the sense of [**2**] and α-connected) if $\alpha \in L^a$. The L-fuzzy Tychonoff cube, $\Pi_{\gamma \in \Gamma} I_\gamma(L)$, is connected (Theorem 3.1), a fuzzy continuum under the hypotheses of the Tychonoff Theorem of [**3**], and an α-continuum if $\alpha \in L^a$.

Definition 4.2. Let (X, T) be an L-fts. We say (X, T) is *α-suitable* [*α^*-suitable*] if X possesses a proper, nonempty α-closed [α^*-closed] subset. We note from [7] that (X, T) is suitable if and only if it is 1^*-suitable. The results of [7] concerning suitability can be stated, with modification, for α-suitability and α^*-suitability, e.g., the L-fuzzy product is α-suitable if some factor is (since the fuzzy continuous image of a non α-suitable space is not α-suitable), etc. Also $\beta > \alpha$ only if β-suitability implies α-suitability, α-suitability implies α^*-suitability, etc. See Corollary 4.3 below.

The following states the relationship between α-suitability and α-connectivity and follows immediately from Proposition 3.1 (3).

Proposition 4.1. *(X, T) is not α'-suitable [$(\alpha')^*$-suitable] implies it is α-connected [α^*-connected], providing $0 \in L^b$ and $\alpha \in L^c$ [$\alpha \in L^a - \{1\}$].*

That the above implication cannot be reversed is illustrated by comparing Theorem 4.4 below with Theorem 4.2, and by Corollary 4.1 below.

Definition 4.3. Let X be a subset of $\mathbf{R}(L)$, and let $\alpha \in L^c$. We say X is $S(\alpha, L)$ if X possesses a nonempty, proper subset A such that for each $[\lambda] \in A$, $X - A = B_1 \cup B_2$ where $a(\lambda, \alpha) < \wedge_{B_1} a(\mu, 0)$ and $\vee_{B_2} b(\mu, 0) < b(\lambda, \alpha)$. We say X is $S^*(\alpha, L)$ if we replace $a(\lambda, \alpha)$ by $a^*(\lambda, \alpha)$, $b(\lambda, \alpha)$ by $b^*(\lambda, \alpha)$, and $<$ by $\leqq$.

Theorem 4.3. *Let (X, T) be a fuzzy subspace of $\mathbf{R}(L)$. Then (X, T) is α-suitable implies X is $S(\alpha, L)$ if $0 \in L^b$ and $\alpha \in L^c$, and the converse holds if*

$\alpha \in L^a$. (X, T) is α^*-suitable implies X is $S^*(\alpha, L)$ if $0 \in L^b$ and $\alpha \in L^c$ is nonsup, and the converse holds if $\alpha \in L^c$.

PROOF. The proof is similar to that of Theorem 4.1 above or Theorem 4.2 of [7].

THEOREM 4.4. *Let (X, T) be either $I(L)$, $(0, 1)(L)$, or $\mathbf{R}(L)$. The following hold.*

(1) *Let $\{0, 1\} \subsetneqq L$ and $0 \in L^b$. (X, T) is not α-suitable [α^*-suitable] if $\alpha \in L^c$ and $\alpha \geqq \alpha'$ [$\alpha > \beta$, where $\beta \in L^c$ and $\beta \geqq \beta'$].*

(2) *Let $\{0, 1\} \subsetneqq L$ and $0 \in L^b$. (X, T) is not α-suitable [α^*-suitable] if $\alpha \in L^a$ and $\alpha' > \alpha > 0$ [$\alpha > \beta$ where $\beta \in L^a$ and $\beta' > \beta > 0$].*

(3) *Let $\alpha \in L$. Then (X, T) is both α-suitable and α^*-suitable if $\alpha = 0$ or $L = \{0, 1\}$.*

PROOF. The α^*-case follows from the α-case in each of (1), (2), and (3). To prove (1) for the α-case, use Theorem 4.3 and the techniques of the proofs of Theorems 4.3 and 4.4 of [7]. To prove (2) for the α-case, note that in [5] it was shown that if $0 \in L^b$, $\alpha \in L^a$, and $0 < \alpha < \alpha'$, then $\mathrm{Cl}_\alpha(A)$ is α-closed if and only if $\mathrm{Cl}_\alpha(A) = X$ for each nonempty $A \subset X$. To prove (3) for the α-case, let $A = \{[\lambda] \in \mathbf{R}(L) : a(\lambda, 0) < b \text{ and } a < b(\lambda, 0)\}$, where $0 < a < b < 1$. The $S(0, L)$ condition is satisfied; apply Theorem 4.3.

REMARK 4.2 (1) Let (X, T) be either $I(L)$, $(0, 1)(L)$, or $\mathbf{R}(L)$. If $0 \in L^b$ and $\{0, 1\} \subsetneqq L^c$, then (X, T) is not suitable (i.e., 1*-suitable). For let $\gamma \in L^c - \{0, 1\}$. Let $\beta = \gamma$ if $\gamma \geqq \gamma'$ or $\beta = \gamma'$ if $\gamma' > \gamma$. Then $\beta \in L^c - \{0, 1\}$ and $\beta \geqq \beta'$. Note $1 > \beta$. By Theorem 4.4(1), (X, T) is not 1*-suitable. Thus Theorem 4.4 extends and strengthens Theorem 4.3 and 4.4 of [7] (which require $\{0\} \subsetneqq L^d$), and Corollaries III.10(ii) and (iii), III.13(ii) and (iii), III.18(ii), and III.20(ii) of [5] (which require $\{0, 1\} \subsetneqq L^a$).

(2) *Is* $\Pi_{\gamma \in \Gamma} I_\gamma(L)$ α-suitable [α^*-suitable] under the hypotheses of Theorem 4.4((1), (2))?

Among the corollaries to this section are the following.

COROLLARY 4.1. *Let X be any of $\{[\lambda_0], [\lambda_1]\} \cup (0, 1)(L)$, $\{[\lambda_0]\} \cup (0, 1)(L)$, or $\{[\lambda_1]\} \cup (0, 1)(L)$ where*

$$\lambda_0(t) = \begin{cases} 1, & t < 0 \\ 0, & t > 0 \end{cases} \quad \text{and } \lambda_1(t) = \begin{cases} 1, & t < 1 \\ 0, & t > 1. \end{cases}$$

Let $0 \in L^a$, and let (X, T) be the resulting fuzzy subspace of $I(L)$ or $\mathbf{R}(L)$. Then (X, T) is both suitable and connected. There are uncountably many fuzzy subspaces of $(0, 1)(L)$ that are both suitable and connected if $0 \in L^b$.

PROOF. That (X, T) is suitable follows trivially if $L = \{0, 1\}$, for then $X = [0, 1]$, $[0, 1)$, or $(0, 1]$. (X, T) is suitable also follows if $L \supsetneqq \{0, 1\}$

from Theorem 4.3 by letting $A = (0, 1)(L)$ in the definition of $S^*(1, L)$. That (X, T) is connected follows from the proof of Theorem 4.2. The second claim follows by observing that the above holds for, say, $(\{[\lambda_a], [\lambda_b]\} \cup (a, b)(L), T)$ where $a, b \in (0, 1)$.

We note the L-fuzzy "almost" Tychonoff cube, $\Pi_{\gamma \in \Gamma} I_\gamma(L) \times (\{[\lambda_0], [\lambda_1]\} \cup (0, 1)(L), T)$, is connected and suitable.

COROLLARY 4.2. *Let $0 \in L^b$, $\alpha \in L^c$, $\beta \in L^c$ such that $\beta' < \alpha \leqq \beta < 1$. There are uncountably many fuzzy subspaces of* $(0, 1)(L)$ *that are α-connected but not β^*-connected. If $\alpha < \beta \in L_b$ [as well as $\alpha' > \alpha$ and α is noninf] is also assumed, there are uncountably many subspaces that are α-connected [α^*-connected] but not β-connected.*

PROOF. Note $0 < \beta' < \alpha \leqq \beta < 1$ and let $a, b, c, d \in (0, 1)$ such that $a < b < c < d$. Define λ and μ as follows:

$$\lambda(t) = \begin{cases} 1, & t < a \\ \alpha, & a < t < c \\ 0, & t > c \end{cases}$$

and

$$\mu(t) = \begin{cases} 1, & t < b \\ \alpha, & b < t < d. \\ 0, & t > d \end{cases}$$

Let $X = \{[\lambda], [\mu]\}$. Note $a = a(\lambda, 0) = a(\lambda, \beta')$; $b = a(\mu, 0) = a^*(\mu, \alpha')$; $c = b(\lambda, 0) = a(\lambda, \alpha')$; $d = b(\mu, 0) = b(\mu, \beta')$; $b(\lambda, \alpha') = a$ if $\alpha' \geq \alpha$, $b(\lambda, \alpha') = c$ if $\alpha > \alpha'$; and $b^*(\mu, \alpha') = b$ if $\alpha' > \alpha$. It follows that X is neither $C^*(\beta, L)$ nor $C(\beta, L)$ (let $A = \{[\lambda]\}$), but X is $C(\alpha, L)$ and, if $\alpha' > \alpha$, $C^*(\alpha, L)$. The proofs of the claims of the corollary now follow from Theorem 4.1.

COROLLARY 4.3. *Let $0 \in L^b$. If $\alpha \in L^a$ and $\beta \in L^c$ such that $\alpha \vee \beta' < \beta < 1$, there are uncountably many α-suitable but not β-suitable fuzzy subspaces of* $(0, 1)(L)$; *each of these subspaces is α^*-suitable but not β-suitable. If $\alpha \in L^a$* and *$\beta \in L^c$ is nonsup such that $\alpha \vee \beta' < \gamma < \beta < 1$, there are uncountably many α-suitable but not β^*-suitable fuzzy subspaces of* $(0, 1)(L)$; *each of these subspaces is α^*-suitable but not β^*-suitable. If $\alpha \in L^c$ such that $\alpha' < \alpha < 1$, there are uncountably many α^*-suitable but not α-suitable fuzzy subspaces of* $(0, 1)(L)$; *each of these subspaces is α^*-suitable but not β^*-suitable if $\beta > \alpha$.*

PROOF. We prove the second statement, where $\alpha \vee \beta' < \gamma < \beta < 1$. Let $a, b, c, d \in (0, 1)$ such that $a < b < c < d$ and let $X = \{[\lambda], [\mu]\}$, where

$$\lambda(t) = \begin{cases} 1, & t < b \\ \gamma, & b < t < d \\ 0, & t > d \end{cases}$$

and

$$\mu(t) = \begin{cases} 1, & t < a \\ \gamma, & a < t < c\,. \\ 0, & t > c \end{cases}$$

Then $a = a(\mu, 0) = b^*(\mu, \beta)$, $b = b^*(\lambda, \beta) = a(\lambda, 0)$, $c = b(\mu, 0) = a^*(\mu, \beta)$, and $d = b(\lambda, 0) = b(\lambda, \alpha) = a^*(\lambda, \beta)$. It follows that X is $S(\alpha, L)$ (let $A = \{[\lambda]\}$) but not $S^*(\beta, L)$. Hence (X, T) is α-suitable but not β^*-suitable (Theorem 4.3). The other proofs are similar.

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