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MINIMAL H^P INTERPOLATION IN THE CARATHEODORY CLASS

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ABSTRACT. For $C = (c_1, c_2, \cdots, c_n)$ a vector in $\mathbf{C}^n$, let $C(c_1, \ldots, c_n)$ denote the class of analytic functions with Taylor expansion

$$f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^{\infty} a_k z^k$$

and $\operatorname{Re} f(z) > 0$ in the unit disc. It is shown that for p fixed in $[1, \infty)$ there is a unique function of least H^p-norm in $C(c_1, \ldots, c_n)$.

1. Introduction. In this paper we give a new and shorter proof a result of Beller and Pinchuk [1] and extend their result to the general case of H^p, $1 \leqq p < \infty$. We consider a minimal interpolation problem at the origin of the unit disc D for the class $H^p \cap C$. H^p is the usual Hardy space of functions analytic in D with p-th integral means bounded. The class C is the Caratheodory class of functions

$$f(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

analytic in D with $\operatorname{Re} f(z) > 0$ in D. If n complex numbers $c_1, \ldots, c_n$ are given, we wish to prove that there is a unique function f in $H^p \cap C$ of the form

$$f(z) = 1 + c_1 z + \cdots + c_n z^n + \sum_{k=n+1}^{\infty} a_k z^k$$

where $\|f\|_p$ is minimal among such functions.

It is well known that the mapping ν_n of C into $\mathbf{C}^n$ by $\nu_n : f \rightarrow (c_1, \ldots, c_n)$ has range C_n, which is a compact convex subset of C^n. The following result of C. Caratheodory and 0. Toeplitz appears in [4].

THEOREM. *To each point of* $(C_n)^0$ = *interior* C_n *there correspond infinitely many functions in* C. *Each boundary point of* C_n *corresponds to only one* f *in* C. *The preimages of boundary points are functions of the form*

$$f(z) = \sum_{k=1}^{m} \mu_k \left[\frac{1 + \alpha_k z}{1 - \alpha_k z} \right], \tag{1.1}$$

where $1 \leqq m \leqq n$; $|\alpha_k| = 1$, $\mu_k > 0$ *and* $\sum_{k=1}^{m} \mu_k = 1$.

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In [1] E. Beller and B. Pinchuk prove that there is an extremal function of minimal norm for the problem $C(c_1, \ldots, c_n)$ in the Hilbert space H^2. Their technique is to solve a minimal (integral) extremal problem. Since functions f with positive real part in P are in H^p for $p < 1$, it is perhaps more natural to consider the problem of finding a unique function f in H^1 solving the $C(c_1, \ldots, c_n)$ condition.

Also note that the functions (1.1) are not in H^1.

2. The interpolation in $\mathbf{H^P}$. We begin by elaborating further on the proof of Beller and Pinchuk. For the H^2 case they note that extremal functions exist and then in a sequence of computations they prove its uniqueness. They show moreover that the Herglotz representation of the extremal f is of the form

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \, d\mu(t)$$

where $\mu(t) = \max(0, P(t))$ and

$$P(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt),$$

a_j and b_j are real numbers. We see that u is a Lip 1 function, so its conjugate will be Lip α for $\alpha < 1$. Hence, their f is not only in H^2 but in the disc algebra.

THEOREM. *For each $(c_1, \ldots, c_n)$ in the interior of C_n, there exists a unique function f with least H^p norm in $C(c_1, \ldots, c_n)$.*

PROOF. Consider first p fixed in $(1, \infty)$. The set $H^p \cap C(c_1, \ldots, c_n)$ is nonempty and is a normal family. Indeed, it is a closed convex subset of H^p. In reflexive Banach spaces we know (see [3]) that such sets have a unique element of minimal norm.

For $p = 1$ we still observe that $H^1 \cap C(c_1, \ldots, c_n)$ is convex and closed in H^1. Also one can show that elements of minimal norm exist. Hence, it remains only to prove the uniqueness. The set $H^1 \cap C(c_1, \ldots, c_n)$ consists only of outer functions. Assume that there are two functions of minimal norm, say F and G. Then $H = (F + G)/2$ is also in $H^1 \cap C(c_1, \ldots, c_n)$ and hence $\|H\| = \|F\|$. But this contradicts the known fact [2] that the extreme points of the unit ball in H^1 are the outer functions of norm one.

The following result is a consequence of the Beller-Pinchuk solution. Let F be the mapping of $(C_n)^0 \to H^2$ given by $F(p) = f_p$, where $P = (c_1, \ldots, c_n) \in (C_n)^0$ and f_p is the unique function in $H^2 \cap C(c_1, \ldots, c_n)$ of minimal norm.

PROPOSITION. *The mapping F is one-to-one and continuous from $(C_n)^0$ into H^2.*

PROOF. The multipliers λ_j in the lemma of [1] are continuous functions of $p = (c_1, \ldots, c_n)$. Hence, the solution

$$u_0(t, p) = \max(0, \ -\frac{1}{2}\lambda_1 - \frac{1}{2}\sum_{k=1}^{n} \lambda_{2k} \sin kt + \lambda_{2k+1} \cos kt)$$

varies continuously in $(c_1, \ldots, c_n)$. That is if $p^* \in (C_n)^0$ and $\varepsilon > 0$, there is a $\delta > 0$ such that if $|p - p^*| < \delta$, then $\|u_0(t, p) - u_0(t, p^*)\|_\infty < \varepsilon$. This implies by the Riesz theorem that the conjugates are continuous in the L^2-norm. Hence, for $f_p(z) = u_0(z, p) + i\tilde{u}_0(z, p)$, the analytic completion of $u_0(z, p)$, we have that the mapping $p \to F(p) = f_p$ is continuous into H^2.

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