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CONVERGENCE TO PUSHED FRONTS

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ABSTRACT. We study the convergence to the stationary state for the parabolic equation $u_t = u_{xx} + f(u)$ with $f'(0) > 0$ in case there exist front-type solutions $U(x + ct)$ for a continuum of velocities $c \geqq c(f)$ and $c^2(f) > 4f'(0)$.

The initial data are only restricted in their asymptotic behavior for $x \to \infty$. We prove strict uniform convergence to a front with velocity $c(f)$ (or a pair of diverging fronts, respectively) with an exponential rate.

Introduction. This paper is concerned with the asymptotic behavior as $t \to \infty$ for solutions $u(x, t)$ of the pure initial value problem for the nonlinear diffusion equation

$$u_t - u_{xx} - f(u) = 0 \tag{1}$$

for $(x, t) \in \mathbf{R} \times (0, \infty)$ with initial value being

$$u(x, 0) = \varphi(x) \qquad \text{for } x \in \mathbf{R}. \tag{2}$$

It is always assumed that $f \in C^1[0, 1]$, f' is Lipschitz-continuous, and

$$f(0) = f(1) = 0. \tag{$f0$}$$

The purpose of the investigation is to show the development of an advancing wave-front from a variety of initial conditions. By a wave-front we mean a solution of equation (1) of the form $u(x, t) = U(x + xt)$ which satisfies the side conditions $0 \leqq U(z) \leqq 1$ for all $z \in \mathbf{R}$, $\lim_{z\to-\infty} U(z) = 0$ and $\lim_{z\to+\infty} U(z) = 1$. The parameter c is called the front velocity. These problems have been studied thoroughly in recent years (see **[1, 4, 5, 7–13]** and there for further references).

In **[4]**, Fife and McLeod are concerned with the so-called bistable case, where

$$f'(0) < 0 \text{ and } f'(1) < 0 \tag{$f1$}$$

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and there exists $\alpha \in (0, 1)$ such that

$$(f2) \qquad f(u) < 0 \text{ for } u \in (0, \alpha) \text{ and } f(u) > 0 \text{ for } u \in (\alpha, 1).$$

In this case there exists a travelling front unique up to translations with a unique velocity. If the initial data satisfy

$$(\varphi 1) \qquad \limsup_{x \to \infty} \varphi(x) < \alpha \text{ and } \liminf_{x \to \infty} \varphi(x) > \alpha,$$

then for some x_0 the solution of the initial value problem (1), (2) approaches uniformly in x and with exponential rate the front $U(x + ct - x_0)$.

In this paper we apply the methods of Fife and McLeod to the much more delicate case

$$(f3) \qquad f'(0) > 0 \quad \text{and} \quad f'(1) < 0,$$

and

$$(f4) \qquad f(u) \geqq 0 \text{ for all } u \in (0, 1).$$

As was pointed out in **[1]** **[8]** **[11]**, we have the following situation for existence of travelling wave-fronts. Under the conditions ($f0$), ($f3$) and ($f4$), there exists a minimal front velocity $c(f) > 0$. For every velocity $c \geqq c(f)$ there exists a front which is unique up to translations, whereas there are no fronts for velocities less than $c(f)$. The minimal velocity satisfies the inequality

$$(3) \qquad f'(0) \leqq \frac{c(f)^2}{4} \leqq \max_{0 \leqq u \leqq 1} \frac{f(u)}{u}.$$

The front is monotone and $U'(z) > 0$ for all $z \in \mathbf{R}$. The asymptotic behavior of U for $x \to -\infty$ (equivalently $U \to 0$) is given by

$$(4) \qquad U(x) \cong k \exp \lambda(c)x$$

where

$$(5) \qquad \lambda(c) = \begin{cases} c/2 + (c^2/4 - f'(0))^{1/2} \equiv \lambda_+ \text{ for } c = c(f) \\ c/2 - (c^2/4 - f'(0))^{1/2} \qquad \text{for } c > c(f). \end{cases}$$

The front with the minimal velocity $c(f)$ turns out to be of special interest. We call it the minimal front.

REMARKS. 1) The condition $f'(0) > 0$ implies that all fronts have velocities $c > 0$. This can easily be seen by analysing the singular points in the phase plane (u, u_x). For $c = 0$ the point (0. 1) is a center and for $c < 0$ a stable equilibrium, which proves nonexistence of fronts.
2) The assumption ($f4$) can be relaxed, but it is difficult to see exactly how. The condition

$$(f5) \qquad \int_u^1 f(v)dv > 0 \text{ for all } u \in [0, 1)$$

instead of $(f4)$ is necessary but not sufficient. (This point is stated incorrectly in Stokes [13]). Necessity of $(f5)$ follows from the identity

$$\int_{U(y)}^1 f(v)dv = c \int_\infty^y U_x^2\, dx + \frac{1}{2} U_x^2(y) > 0.$$

In their classical paper, Kolmogorov et. al. [9] showed that from an initial unit step function

$$(\varphi 2) \qquad \varphi(x) = \begin{cases} 0 & x \leqq 0 \\ 1 & x > 0 \end{cases}$$

a moving front originates which approaches in shape and velocity the minimum front. But it remained an open question whether $u(x, t)$ converges uniformly for all x to $U(x + c(f)t - x_0)$ for some x_0. The answer to this question can be positive or negative. For example, if f satisfies the condition

$$(f6) \qquad f(u) \leqq f'(0)u$$

(which by (3) implies $c(f)^2 = 4f'(0)$), the answer is negative as shown by Hadeler [7] and Bramson [2] [3]. (The last reference gives very accurate asymptotics for $u(x, t)$). On the other hand, if f and the minimal front velocity $c(f)$ satisfy the condition

$$(cf) \qquad c(f)^2 > 4f'(0),$$

the answer is positive as Stokes proved in [12] (His proof is in some respects inaccurate but the result can be shown as a special case of our theorem 1 below).

Stokes [12] calls the minimal fronts satisfying

$$(cf1) \qquad c(f)^2 = 4f'(0)$$

"pulled fronts", as their speed of propagation is determined by the leading edge of the population distribution. On the other hand, fronts satisfying (cf) are called "pushed fronts" because the velocity of propagation is determined not by the behavior of the leading edge of the distribution, but by the whole wave-front. This notion is very useful in dealing the question posed above, but does not lead to a complete solution, as can be seen from the following example.

In [8], Hadeler and Rothe calculated the minimal velocity for the special case $f(u) = u(1 - u)(1 + 2au)$. In this case

$$c(f) = \begin{cases} 2 & \text{for } -1/2 \leqq a \leqq 1 \\ \sqrt{a} + 1/\sqrt{a} & \text{for } 1 \leqq a. \end{cases}$$

Thus for $-1/2 \leqq a \leqq 1/2$ conditions $(f6)$ and hence $(cf1)$ are satisfied, and we have a pulled wave which lags behind the linear propagation front $U(x + c(f)t - x_0)$. For $1/2 < a \leqq 1$ we still have a pulled front, but can not determine the exact asymptotic behavior, because $(f6)$ does not hold. For $a > 1$ we have a pushed wave, which converges uniformly to $U(x + c(f)t - x_0)$ for some x_0 by the results of Stokes.

The asymptotics for more general initial data than the step function $(\varphi 2)$ were investigated by Rothe **[10]**. There is was shown that monotonely increasing initial data, which satisfy $\lim_{x\to-\infty} u(x, 0) = 0$, $\lim_{x\to+\infty} u(x, 0) = 1$ and decay exponentially for $x \to -\infty$, evolve to travelling fronts. Initial conditions decaying faster than the minimal front evolve to the minimal front. Initial conditions decaying with the same exponential rate as a front with higher velocity evolve to the corresponding front.

As the previous review showed, a number of questions remain open, namely concerning the convergence to pushed fronts for more general initial data. This is the aim of the present paper. The main results, summarized by theorem 1 below, consist in establishing a number of sufficient conditions for asymptotic convergence. It will turn out that the monotonocity condition for the initial data, which played a key role in **[10]**, can be removed and the "essential" condition is a decay property for $x \to -\infty$ of the initial data. Theorem 2 deals with pairs of diverging fronts, and shows that convergence holds for initial data decaying fast enough at $|x| \to \infty$. Convergence—only in form—was proved in a similar situation, without assuming (cf) by Stokes **[13]**. For related results, valid under the hypotheses $(f1)$, $(f2)$, see Fife **(5)**, Fife and McLeod **[4]**.

The methods of our proofs are inspired by those of **[4]**, which we will refer to in many technical points. The important tools are sub- and supersolutions and a Lyapunov functional. After this work was written, the author heard of the paper **[14]** of Uchiyama. That paper is very thorough and contains nearly all known results about the Fisher equation. There is also a more complicated proof of Theorem 1, without exponential rate of convergence, but no analog of our Theorem 2.

2. Statement of the results.

THEOREM 1. *Let $f \in C^1[0, 1]$, f' Lipschitz-continuous satisfy*

$(f0)$ $$f(0) = f(1) = 0$$

$(f3)$ $$f'(0) > 0, \quad f'(1) < 0, \quad \textit{and}$$

$(f7)$ $$f(u) > 0 \text{ for } \alpha_1 < u < 1.$$

Assume that there exists a minimal front U(with velocity $c = c(f)$) and that *this is a "pushed front", i.e.*

(*cf*) $$c(f)^2 > 4f'(0).$$

Let the initial data satisfy

(φ0) $$0 \leqq \varphi(x) \leqq 1 \text{ for all } x \in \mathbf{R},$$

(φ3) $$\liminf_{x\to\infty} \varphi(x) > \alpha_1, \text{ and}$$

(φ4) $$\varphi(x) \leqq K_1 \exp \lambda x \quad \text{for all} \quad x \in \mathbf{R}$$

where $K_1 > 0$ *and* $\lambda > \lambda_- \equiv c(f)/2 - (c(f)^2/4 - f'(0))^{1/2} > 0$. *Then for some constants* z_0, K_2 *and* ω, *the last two positive, the solution* $u(x, t)$ *of* (1), (2) *satisfies*

(6) $$\sup_{x\in\mathbf{R}} |u(x, t) - U(x + ct - z_0)| < K_2 e^{-\omega t}$$

for all $t \geqq 0$.

REMARK. Notice that in the above convergence theorem the hypotheses are quite general, and do not guarantee, by themselves, the existence of a front. Therefore it was necessary to assume explicitly that U exists.

THEOREM 2. *Let* f *satisfy the hypotheses of theorem* 1 *and let the initial data satisfy*

(φ0) $0 \leqq \varphi(x) \leqq 1$ *for all* $x \in \mathbf{R}$,

(φ5) $\varphi(x) \leqq K_1 \exp - \lambda|x|$ *for all* $x \in \mathbf{R}$ where $\lambda > \lambda_-$, *and*

(φ6) $\varphi(x) > \alpha_1 + \eta$ *for* $|x| \leqq L$

where η, L *are some positive numbers. Then, if* L *is sufficiently large (depending on* f *and* η), *there exist constants* x_0, x_1, K *and* ω (*the last two positive*) *such that*

(7) $$|u(x, t) - U(x + ct - x_0)| < Ke^{-\omega t} \quad \text{for} \quad x < 0, \quad \text{and}$$

(8) $$|u(x, t) - U(-x + ct - x_1)| < Ke^{-\omega t} \quad \text{for} \quad x > 0.$$

COROLLARY. *If* f *satisfies*

(f8) $$f(u) > 0 \text{ for all } u \in (0, 1)$$

condition (φ6) *is superfluous. We only need the condition*

(φ7) $$\varphi(x) > 0 \text{ for some } x \in \mathbf{R}.$$

3. Proofs. The proof of theorem 1 relies on a number of auxilary lemmas. For the purpose of orientations, we present a short resumé of their content. Lemma 1 establishes (using super- and sub-solutions) estimates from above and below for the solution of the initial value problem (1), (2) in terms of travelling fronts. Lemma 2 proves that travelling fronts are stable ("from above and from below") with respect to (1), (2). Lemma 3 states a priori estimates for the solution of (1), (2), together with their space-time derivatives. Lemma 4—a straight-forward consequence of lemma 3—establishes compactness property for the trajectories of (1), (2). Lemma 5 and 6 prove the decaying properties of a defined Lyapunov-functional. Lemma 7 is devoted to the ω-limit set (for locally uniform convergence) and shows that it consists of translated travelling fronts. Lemma 8 uses the stability to show that the ω-limit set contains only one element and then proves convergence to a travelling front. Lemma 9 sharpens the convergence result using spectral properties of the linearized stability equation and proves the final claim of theorem 1.

To start with introduce, as usual, a moving coordinate frame. Putting $z = x + ct$, we write the solution of (1), (2) as $v(z, t) = u(x, t) = u(z - ct, t)$. The function v satisfies

(9) $$N(v) \equiv v_t - v_{zz} + cv_z - f(v) = 0 \text{ for all } z \in \mathbf{R},\ t > 0, \text{ and}$$

(10) $$v(z, 0) = \varphi(z) \text{ for all } z \in \mathbf{R}.$$

Define $\lambda_\pm = c/2 \pm (c^2/4 - f'(0))^{1/2}$ and choose $\lambda_1 \in (\lambda_-, \lambda_+)$ such that $\lambda_1 \leqq \lambda$, where λ is taken from assumption (φ4). Define the function

$$\min \exp x = \begin{cases} \exp x & \text{for } x < 0 \\ 1 & \text{for } x \geqq 0. \end{cases}$$

LEMMA 1. *Under the assumptions of theorem* 1 *there exist constants* z_1, z_2, z_3, q_0 *and* μ *(the last two positive) such that*

(11) $$U(z - z_1) - q_0 e^{-\mu t} \min \exp \lambda_1(z - z_3) \leqq v(z, t)$$

and

(12) $$v(z, t) \leqq U(z - z_2) + q_0 e^{-\mu t} \min \exp \lambda_1(z - z_3)$$

for all $z \in \mathbf{R}$.

To prove this lemma, observe first that by comparison arguments $0 \leqq v(z, t) \leqq 1$. Hence the behavior of $f(u)$ for $u \in (-\infty, 0) \cup (1, \infty)$ is unessential and we can choose $f(u)$ linear in this domain and of class C^1 in $\mathbf{R}$.

To establish the inequality (11), we shall construct a subsolution by the Ansatz

(13) $$\underline{v}(z, t) = U(z - \xi(t)) - q(z, t)$$

where

(14) $$q(z, t) = q_0 e^{-\mu t} \min \exp \lambda_1(z - z_4 - \xi(0)).$$

The relevant numbers q_0, μ, z_4 and the function $\xi(t)$ are determined by the following procedure:
1) Choose q_0 such that $\alpha_1 < 1 - q_0 < \lim_{x\to\infty} \inf \varphi(x)$.
2) Choose μ, $\delta > 0$ such that
   a) $U \leqq 2\delta$ implies $f(U) - f(U - q) + (\mu + \lambda_1^2 - c\lambda_1)q \leqq 0$ for any $q \geqq 0$, and
   b) $1 - \delta \leqq U$ and $q \leqq q_0$ imply $f(U) - f(U - q) + \mu q \leqq 0$.
3) Choose $\chi > 0$ such that $f(U) - f(U - q) + \mu q \leqq \chi q$.
4) Choose $\beta > 0$ such that $\delta \leqq U \leqq 1 - \delta$ implies $U' \geqq \beta$.
5) $\xi'(t)$ is determined by $-\beta\xi'(t) + \chi q_0 e^{-\mu t} = 0$.
6) Choose z_4 such that $U(z) \leqq \delta$ implies $z \leqq z_4 + \xi(0) - \xi(\infty)$.
7) Choose $\xi(0)$ such that $v(z, 0) \leqq \varphi(z)$ for all $z \in \mathbf{R}$.

Since $\lambda_1 \in (\lambda_-, \lambda_+)$ implies $f'(0) + \lambda_1^2 - c\lambda_1 < 0$, choice 2a) is possible. For 2b) see [4]. Since $U(2 - \xi(0)) \cong k \exp \lambda_+ z$ for $z \to -\infty$ and $\lambda_+ > \lambda_1$, there exists z_5 ($\leqq z_4$) such that

$$\underline{v}(z, 0) = U(z - \xi(0)) - q_0 \exp \lambda_1(z - z_4 - \xi(0)) \leqq 0 \leqq \varphi(z)$$

for $z - \xi(0) \leqq z_5$. On the other hand the choice 1) and ($\varphi 3$) imply $v(z, 0) \leqq 1 - q_0 \leqq \varphi(z)$ for large z. Thus choice 7) is possible. The rest is evident.

To prove that $\underline{v}(z, t)$ is a subsolution, it remains to verify $N(\underline{v}) \leqq 0$. A straightforward calculation shows

$$N(\underline{v}) = -\xi'(t)U'(z - \xi(t)) + f(U) - f(U - q) + \mu q$$
$$-\lambda_1 q e^{-\mu t}\delta(z - z_4 - \xi(t)) + \begin{cases} (\lambda_1^2 - c\lambda_1)q & \text{for } z \leqq z_4 + \xi(0) \\ 0 & \text{for } z \geqq z_4 + \xi(0) \end{cases}$$

where δ is the Dirac-distribution. We cut the half plane $H = \mathbf{R} \times [0, \infty)$ into three regions A, B, C, where

$$A = \{(z, t) \in H \mid U(z - \xi(t)) \leqq \delta\}$$
$$B = \{(z, t) \in H \mid \delta < U(z - \xi(t)) < 1 - \delta\}$$
$$C = \{(z, t) \in H \mid 1 - \delta \leqq U(z - \xi(t))\}.$$

First let $(z, t) \in A$. By choice 5) and 6) this implies $z < z_4 + \xi(0)$. Hence by 2a) and $\xi' > 0$, $U' > 0$,

$$N(\underline{v}) \leqq f(U) - f(U - q) + (\mu + \lambda_1^2 - c\lambda_1)\, q \leqq 0.$$

Now let $(z, t) \in B$. Then by 3), 4) and 5) imply

$$N(\underline{v}) \leqq -\xi'(t)\, U' + \chi q \leqq -\beta\xi'(t) + \chi q_0 e^{-\mu t} = 0.$$

Now assume $(z, t) \in C$. By 2b), $\xi' > 0$, $U' > 0$, we have

$$N(\underline{v}) \leqq f(U) - f(U - q) + \mu q \leqq 0.$$

Finally we note that the term involving the Dirac-distribution also has a negative sign. Thus v is a weak subsolution. With $z_1 = \xi(\infty)$ and $z_3 = z_4 + \xi(0)$ this establishes inequality (11).

Let us now prove the inequality (12). The Ansatz for the supersolution is

$$\bar{v}(z, t) = U(z + \xi(t)) + q(z, t) \tag{15}$$

where

$$q(z, t) = q_0 e^{-\mu t} \min \exp \lambda_1(z - z_4 + \xi(0)). \tag{16}$$

The relevant numbers q_0, μ, z_3 and the function $\xi(t)$ are determined by the following procedure.

1*) Choose $\delta, \mu > 0$ such that
  a) $U \leqq \delta$ and $q \leqq \delta$ imply $f(U+q) - f(U) + (\mu + \lambda_1^2 - c\lambda_1)q \leqq 0$, and
  b) $1 - \delta \leqq U$ implies $f(U + q) - f(U) + \mu q \leqq 0$.

2*) Choose $q_0 \leqq \delta$.

3*) Choose $\chi < 0$ such that $f(U + q) - f(U) + \mu q \leqq \chi q$.

4*) Choose $\beta > 0$ such that for the minimal pulse $\delta \leqq U \leqq 1 - \delta$ implies $U' \geqq \beta$.

5*) $\xi'(t)$ is determined by $-\beta\xi'(t) + \chi q_0 e^{-\mu t} = 0$.

6*) Choose z_4 such that $U(z) \leqq \delta$ implies $z \leqq z_4$.

7*) Choose $\xi(0)$ large enough such that
  a) $\min\{1, K_1 \exp z\lambda\} \leqq q_0 + U(z + \xi(0))$ for all $z \in \mathbf{R}$, and
  b) $z_4 - \xi(0) - \ln q_0/\lambda_1 \leqq -\ln K_1/\lambda$.

To show that $\bar{v}(z, t)$ is a supersolution first we prove the 7*) implies $\varphi(z) \leqq \bar{v}(z, 0)$ for all $z \in \mathbf{R}$. Let $z_6 = -\ln K_1/\lambda$. We distinguish the cases i) $z \leqq z_4 - \xi(0)$ and ii) $z \geqq z_4 - \xi(0)$.

i) $z \leqq z_4 - (0) \leqq z_6$ by b). Hence by assumption (φ4), $\lambda \leqq \lambda_1$ and b) we have

$$\begin{aligned}\varphi(z) &\leqq K_1 \exp \lambda z = \exp \lambda(z - z_6) \leqq \exp \lambda_1(z - z_6)\\ &\leqq q_0 \exp \lambda_1(z - z_4 + \xi(0)) \leqq \bar{v}(z, 0).\end{aligned}$$

ii) $z \geqq z_4 - \xi(0)$ implies

$$\varphi(z) \leqq \min\{1, K_1 \exp \lambda z\} \leqq q_0 + U(z + \xi(0)) = \bar{v}(z, 0)$$

It remains to verify $N(\bar{v}) \geqq 0$. A trivial computation shows

$$N(\bar{v}) = \xi'(t)U'(z + \xi(t)) - f(U + q) + f(U) - \mu q$$

$$+ \lambda_1 q_0 e^{-\mu t}\,\delta(z - z_4 + \xi(0)) + \begin{cases}(-\lambda_1^2 + c\lambda_1)q & \text{for } z < z_4 - \xi(0)\\ 0 & \text{for } z > z_4 - \xi(0)\end{cases}$$

where δ is the Dirac-distribution.

Again cut the halfplane H into the regions A, B and C. By choice 6) and since $\xi'(t) > 0$, $(z, t) \in A$ implies $z < z_4 - \xi(0)$. The rest of the proof is analogous to the proof above. Choosing $z_2 = \xi(\infty)$ and $z_3 = z_4 - \xi(0)$ establishes inequality (12).

We define the weighted norm

$$\|v\| = \max_{z \in \mathbf{R}} \frac{|v(z)|}{\min \exp \lambda_1 z}$$

Our next objective is to show that the minimal front is stable with respect to this weighted norm. It will also be convenient to use "upper stability".

LEMMA 2. *There exist constants K_3 and $\varepsilon_0 > 0$ such that $\varepsilon \leqq \varepsilon_0$ and*

(17) $\quad v(z, 0) \leqq U(z - z_0) + \varepsilon \min \exp \lambda_1 z$ *for all* $z \in \mathbf{R}$

implies

(18) $\quad v(z, t) \leqq U(z - z_0) + K_3 \min \exp \lambda_1 z$ *for all* $z \in \mathbf{R}$ *and all* $t \geqq 0$.

Analogous results hold for lower bounds and hence $\|v(\cdot, 0) - U(\cdot - z_0)\| \leqq \varepsilon$ implies $\|v(\cdot, t) - U(\cdot - z_0)\| \leqq K_3\, \varepsilon$ for all $t \geqq 0$.

PROOF. We only consider upper bounds and choose $q_0 = \varepsilon$, $z_0 = -\xi(0)$ in the super-solution (15), (16) of lemma 1. Notice that the numbers μ, δ, β, χ and z_4 can be chosen independent of ε. Because of translation invariance it suffices to consider the case $z_0 + z_4 = 0$. Since by 5*) $\xi(\infty) - \xi(0) = \varepsilon\chi/\mu\beta$, lemma 1 implies

(19) $\quad v(z, t) \leqq U(z - z_0 + \varepsilon\chi/\mu\beta) + \varepsilon \min \exp \lambda_1 z$ for all $t \geqq 0$.

Since $U(z)$ and $U'(z)$ are of the order of magnitude $\exp \lambda_+ z$ for $z \to -\infty$ and $\lambda_1 < \lambda_+$, we get

(20) $\quad \|U(\cdot - z_0) - U(\cdot - z_0 + \varepsilon\chi/\mu\beta)\| \leqq K_4 \varepsilon\chi/\mu\beta.$

(19) and (20) imply the assertion with $K_3 = 1 + K_4\chi/\mu\beta$.

LEMMA 3. *For each $\delta > 0$ there exists a constant C such that*

a)
$$|v|,\ |v_z|\ |v_{zz}|\ |v_t| \leqq C$$
$$|v_{zt}|\ |v_{zzt}|\ |v_{tt}| \leqq C$$

for all $z \in \mathbf{R}$, $t \geqq \delta$, and

b)
$$|v|,\ |v_z|\ |v_{zz}|\ |v_t|$$
$$|v_{zt}|\ |v_{zzt}|\ |v_{tt}| \leqq C(e^{\lambda_1 z} + e^{-\mu t + \lambda_1 z})$$

for all $z \in \mathbf{R}$, $t \geqq \delta$.

PROOF. Because $f'(v)$ is Hölder continuous, the derivative v_{zt}, v_{zzt}, v_{tt} exist by [6]. The estimates for v, v_z, v_{zz}, v_t are proved in [4]. Applying the same arguments to the differential equation $b_t = b_{zz} - cb_z + f'(v)b$ for $b = v_z$ yields the estimates for v_{zt}, v_{zzt} and v_{tt}.

LEMMA 4. *For each $\delta > 0$ and $L > 0$ the orbit set $\{v(\cdot, t) | t \geqq \delta\}$ considered as a subset of $C^2([-L, L])$ is relatively compact.*

LET $m > 0$ be a number satisfying $2\mu + m(2\lambda_1 - c) > 0$ and $\zeta(\cdot) \in C^\infty$ a function satisfying $\zeta(x) = 1$ for $x \geqq 0$ and $\zeta(x) = 0$ for $x \leqq -1$. Define $w(z, t) = v(z, t)\zeta(z + mt)$. This is a left truncation of v since

$$\begin{cases} w(z, t) = v(z, t) & \text{for } z \geqq -mt \\ w(z, t) = 0 & \text{for } z \leqq -mt - 1 \end{cases} \tag{21}$$

and satisfies lemma 3 and 4.

We define the Lyapunov functional

$$\mathscr{L}(w) = \int_{-\infty}^{\infty} e^{-cz} \left\{ \frac{1}{2} w_z^2 - F(w) \right\} dz \tag{22}$$

where $F(v) = \int_0^v f(s)$ and the functional

$$\mathscr{Q}(w) = \int_{-\infty}^{\infty} e^{-cz} \{ w_{zz} - cw_z + f(w) \}^2 \, dz. \tag{23}$$

LEMMA 5. *$\mathscr{L}(w)$, $d/dt\, \mathscr{L}(w)$, $d^2/dt^2\, \mathscr{L}(w)$ and $\mathscr{Q}(w)$ are well-defined and uniformly bounded for all $t \geqq \delta > 0$ and*

$$\left| \frac{d}{dt} \mathscr{L}(w) + \mathscr{Q}(w) \right| \leqq K_5 e^{-\nu t} \tag{24}$$

for some $\nu > 0$.

PROOF. Lemma 3 implies

$$|\mathscr{L}(w)| \leqq C_1 \int_{-mt-1}^{\infty} e^{-cz} \min\{1, e^{2\lambda_1 z} + e^{-2\mu t + 2\lambda_1 z}\} \, dz \leqq C_2$$

for all $t \geqq 0$. Here we have used the fact that $2\lambda_+ - c = (c^2 - 4f'(0))^{1/2} > 0$ and $2\mu + m(2\lambda_1 - c) > 0$. The same estimate holds for $\mathscr{Q}(w)$, $d/dt\, \mathscr{L}(w)$, $(d/dt)^2\, \mathscr{L}(w)$. A straightforward calculation shows that

$$\frac{d}{dt} \mathscr{L}(w) + \mathscr{Q}(w) = -\int_{-\infty}^{\infty} e^{-cz} [w_{zz} - cw_z + f(w)] N(w) dz.$$

Since $N(w) \neq 0$ only if $-mt - 1 \leqq z \leqq -mt$,

$$\begin{aligned} \left| \frac{d}{dt} \mathscr{L}(w) + \mathscr{Q}(w) \right| &\leqq C_3 \int_{-\infty}^{-mt} e^{-cz} \{ e^{2\lambda_+ z} + e^{-2\mu t + 2\lambda_1 z} \} \, dz \\ &\leqq C_4 (\exp\{-(c^2 - 4f'(0))^{1/2} mt\} \\ &\quad + \exp\{-[2\mu + m(2\lambda_1 - c)] t\}) \\ &\leqq K_5 e^{-\nu t}. \end{aligned}$$

LEMMA 6. $\lim_{t \to \infty} \mathscr{Q}(w) = 0$.

PROOF. Assume the assertion is false. Then there exists $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that $\mathscr{Q}(w(\cdot, t_n)) \geqq \varepsilon$. Hence $d/dt\ \mathscr{L}(w(\cdot, t_n)) \leqq -\varepsilon/2$ for $n \geqq N_0$. Since $|(d/dt)^2 \mathscr{L}| \leqq K_6$, this implies

$$\frac{d}{dt} \mathscr{L}(w(., t)) \leqq -\varepsilon/2 + K_6|t - t_n|$$

for $n \geqq N_0$. Using $d/dt\ \mathscr{L} \leqq d/dt\ \mathscr{L} + \mathscr{Q} \leqq K_5 e^{-\nu t}$ we get $\lim_{t\to\infty} \mathscr{L}(t) = -\infty$ which contradicts lemma 5.

For the further considerations it is useful to define the limit set ω of a trajectory $\{w(\cdot, t) | t > 0\}$: $\omega = \{W \in C^2(-\infty, \infty) |$ there exists a sequence $t_n \to \infty$ such that $w(\cdot, t_n) \to W$ in $C^2[-L, L]$ for each $L > 0\}$.

LEMMA 7. *The limit-set ω is of the form $\omega = \{w \mid w(z) = U(z - y)$ $y \in \omega_R\}$, U being as usual, the travelling front we are concerned with. Here ω_R is a nonvoid, bounded set in* $\mathbf{R}$.

PROOF. Lemma 4 implies that ω is nonvoid. Let $W \in \omega$. By Lemma 1 $U(z - z_1) \leqq W(z) \leqq U(z - z_2)$ for all $z \in \mathbf{R}$ and by lemma 6 $\mathscr{Q}(W) = 0$. Hence W satisfies the differential equation $W_{zz} - cW_z + f(W) = 0$ and $W(z) = U(z - y)$ with $y \in [z_1, z_2]$.

The following lemma only uses upper stability. This will allow us to use the same lemma without change in the proof of theorem 2.

LEMMA 8. *The set ω_R contains exactly one point z_0 and*

$$\lim_{t\to\infty} \|v(\cdot, t) - U(\cdot, -z_0)\| = 0,$$

$\|\cdot\|$ being the weighted norm.

PROOF. Take $z_0 \in \omega_R$. By lemma 3 there exists K_7 such that

$$|v(z, t) - U(z - z_0)| \leqq K_7 e^{\lambda_1 z}(e^{-\mu t} + e^{(\lambda_+ - \lambda_1)z}), \tag{25}$$

for $z \leqq 0$ and $t \geqq \delta$. Let $\varepsilon > 0$ be given. There exist L and T_1 such that

$$U(z - z_0) + \varepsilon \min \exp \lambda_1 z \geqq 1 \text{ for } z \geqq L, \tag{26}$$

$$K_7 e^{(\lambda_+ - \lambda_1)z} \leqq \varepsilon/2 \quad \text{for} \quad z \leqq -L, \tag{27}$$

and

$$K_7 e^{-\mu t} \leqq \varepsilon/2 \text{ for } t \geqq T_1. \tag{28}$$

Let (t_n) be a sequence such that $\lim_{n\to\infty} w(z, t_n) = U(z - z_0)$. Then $v(z, t_n) = w(z, t_n)$ and

$$|v(z, t_n) - U(z - z_0)| \leqq \varepsilon \min \exp \lambda_1 z \tag{29}$$

for $|z| \leqq L$ and $t_n \geqq T_2$. Putting together (25) through (29) we get for $t_n \geqq \max(T_1, T_2)$,

$$v(z, t) \leqq U(z - z_0) + \varepsilon \min \exp \lambda_1 z \tag{30}$$

for all $z \in \mathbf{R}$ and hence from the stability lemma 2

$$v(z, t) \leqq U(z - z_0) + K_3 \varepsilon \min \exp z$$

for all $z \in \mathbf{R}$ and all $t \geqq T_3$. Because $\varepsilon > 0$ is arbitrary, this implies $W(z) \leqq U(z - z_0)$ for all $z \in \mathbf{R}$ and $W \in \omega$. Hence $z_0 = \max \omega_R$ and ω_R contains only one point.

It remains to get a lower bound for $v(z, t)$. By lemma 1 there exist z_7, T_4 such that

$$v(z, t) \geqq U(z - z_7) - \varepsilon e^{-\mu t} \min \exp \lambda_1 z \tag{31}$$

for $t \geqq T_4$ and all $z \in \mathbf{R}$. There exist L and T_5 such that

$$U(z - z_0) - \varepsilon \min \exp \lambda_1 z \leqq 0 \tag{32}$$

for $z \leqq -L$, and

$$U(z - z_0) - \varepsilon \min \exp \lambda_1 z \leqq U(z - z_7) - \varepsilon e^{-\mu t} \min \exp \lambda_1 z \tag{33}$$

for $z \geqq L$ and $t \geqq T_4$. By definition of the limit set $v(z, t) = w(z, t)$ and

$$|v(z, t) - U(z - z_0)| \leqq \varepsilon \min \exp \lambda_1 z \tag{34}$$

for $|z| \leqq L$ and $t \geqq T_5$. Together we get from (31) through (34) that

$$v(z, t) \geqq U(z - z_0) - \varepsilon \min \exp \lambda_1 z \tag{35}$$

for all $t \geqq \max(T_4, T_5)$ and all $z \in \mathbf{R}$. Now (30) and (35) prove the lemma.

LEMMA 9. *There exist constants* $\nu > 0$ and $K_8 > 0$ *such that*

$$\|v(\cdot, t) - U(\cdot - z_0)\| \leqq K_8 e^{-\nu t}$$

where $\| \quad \|$ *is the weighted norm.*

PROOF. By lemma 3

$$e^{-\lambda_1 z}|v(z, t) - w(z, t)| \leqq K_9(e^{-(\lambda_+ - \lambda_1)mt} + e^{-\mu t}).$$

Hence $\|v(\cdot, t) - w(\cdot, t)\| = 0\,(e^{-\nu t})$ and one can consider the truncation w instead of v. Define

$$h(z, t) = w(z, t) - U(z - z_0 - \alpha(t))$$

and

$$y(z, t) = e^{-cz/2}\, h(z, t).$$

As in [4] one chooses $\alpha(t) \in C^1$, $\lim_{t \to \infty} \alpha(t) = 0$ such that

$$\int_{\infty}^{\infty} e^{-cz/2}\, y(z, t) U'(z - z_0 - \alpha(t))\, dz = 0. \tag{36}$$

The function $y(z, t)$ satisfies a diffusion equation

$$y_t = -Ay + \alpha' e^{-cz/2}U' + R \tag{37}$$

where A denotes the operator

$$Ay = -y_{zz} + \left[\frac{1}{4}c^2 - f'(U)\right]y$$

and the remainder term satisfies the estimates $R \leqq K_{10}(hy + r)$ with

$$r(z, r) = \exp\{(c/2 - \lambda_+)mt\} + \exp\{-(\mu + (\lambda_1 - c/2)m)t\}$$

for $-mt - 1 \leqq z \leqq -mt$ and $r(z, t) = 0$ otherwise. Hence r decays exponentially.

The operator A is selfadjoint in $L^2(-\infty, \infty)$. Its continuum spectrum lies to the right of $c^2/4 - f'(0) > 0$ and zero is the smallest eigenvalue with the positive eigenfunction $e^{-cz/2}U' \in L^2(-\infty, \infty)$. Let a denote the smallest positive eigenvalue of A. Multiplying (37) with y and using the orthogonality (36) one gets ($\| \quad \|_2$ denotes the L^2-norm)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y\|_2^2 &= -(Ay, y) + (R, y)\\ &\leqq -a\|y\|_2^2 + K_{11}(\sup_{z\in R}|h(z, t)|\|y\|_2^2 + e^{-\nu t}\|y\|_2)\end{aligned}$$

since $\lim_{t\to\infty}\sup_{z\in R}|h(z, t)| = 0$, this implies $\|y\|_2 = O(e^{-\nu t})$. Since y_z is uniformly bounded by lemma 3, a standard interpolation lemma implies

$$\sup_{z\in\mathbf{R}}|y(z, t)| = O(e^{-\nu t})$$

Distinguish the cases $|z| \leqq nt$ and $|z| \geqq nt$. If $|z| \geqq nt$, the inequalities from lemma 1 and 3

$$|h(z, t)| \leqq K_{12}(e^{-\mu t} + e^{-\lambda_2 z})$$

for $z \geqq 0$ and

$$e^{-\lambda_1 z}\,|h(z, t)| \leqq K_{12}(e^{-\mu t} + e^{(\lambda_+ - \lambda_1)z})$$

for $z \leqq 0$ with $\lambda_2 > 0$ and $\lambda_+ - \lambda_1 < 0$ prove exponential decay. If $|z| \leqq nt$, we get

$$\begin{aligned}|h(z, t)| &\leqq \exp\{cz/2\}|y(z, t)|\\ &= O(\exp\{(cn/2 - \nu)t\})\end{aligned}$$

for $z \geqq 0$ and

$$\begin{aligned}\exp(-\lambda_1 z)\,|h(z, t)| &\leqq \exp\{-|c/2 - \lambda_1|z\}|y(z, t)|\\ &= O(\exp\{(|c/2 - \lambda_1|n - \nu)t\})\end{aligned}$$

for $z \leqq 0$, which proves exponential decay for $n > 0$ small enough. Thus

$$\|h(\cdot, t)\| = O(e^{-\nu t}).$$

Exactly as in [4], one shows that $\alpha(t) = O(e^{-\nu t})$. Since $\| U(\cdot - z_0) - U(\cdot - z_0 - \alpha(t)\| \leqq K_2\alpha(t)$, the result follows.

REMARK. The crucial argument concerning the spectrum of the operator A is taken from Sattinger [11], but applied to a somewhat different situation. In [11] it is assumed that $f(u) \leqq f'(0)u$ and hence $c(f)^2 \leqq 4f'(0)$ and U is a pulled front. But Sattinger then takes $c > c(f)$ to get $c^2/4 - f'(0) > 0$ and hence considers not the minimal but higher velocity fronts. In that case $e^{-cz/2}\, U' \notin L^2(-\infty, \infty)$ which differs from our situation.

The proof of theorem 2 is obtained along the same lines as above. To prove estimates from below (Lemma 11) we shall, however, need a preliminary result (Lemma 10) about the behavior of $u(0, t)$. The rest of the proof then proceeds exactly as for Theorem 1. Let us now go into the details. It is clearly sufficient to consider the case $x \leqq 0$. We start with a preliminary lemma.

LEMMA 10. *Under the assumptions of theorem* 2, (*i.e., if L is chosen large enough*) *and for* q_0 *such that* $1 - q_0 < \alpha_1 + \eta$, *there exist constants* ξ *and* $\nu > 0$ *such that*

$$U(ct - \xi) - q_0 e^{-\nu t} \leqq u(0, t)$$

for all $t \geqq 0$.

PROOF. Assume $1 - q_0 < 1 - q_0^* < \alpha_1 + \eta$. Consider the comparison problem

$$u_t^* = u_{xx}^* + f^*(u^*)$$
$$u^*(x, 0) = \varphi(x) \text{ for all } x \in \mathbf{R}$$

where f^* is chosen such that $f^*(u) \leqq f(u)$ for all $u \in [0, 1]$ and f^* satisfies

(g0) $\qquad f^* \in C^2[0, 1],\quad f^*(0) = f^*(1) = 0,$

(g1) $\qquad f^{*\prime}(0) < 0,\quad f^{*\prime}(1) < 0,$

(g2) $\qquad \int_0^1 f^*(u)\, du > 0$, and

(g3) $\qquad f^*(u) > 0$ for $\alpha_1 < u < 1$.

To the above modified problem, theorem 3.2 in [4] can be applied (which is the analogue of our theorem 2). Hence if L is chosen large enough according to lemma 6.1 from [4], there exist constants z_8 and $\nu > 0$ such that

(13) $$2U^*(c^*t - z_8) - 1 - q_0^* e^{-\nu t} \leqq u^*(0, t)$$

for all $t \geqq 0$ where U^* denotes the front for the modified problem. Here

$$U_{zz}^* - c^*U_z^* + f^*(U^*) = 0$$

and c^* is the front velocity. Condition ($g2$) implies $c^* > 0$. (Note that c^* has the sign reversed as compared with the notation in [4].) After possibly diminishing the value of ν, there exists ξ such that

$$U(ct - \xi) - q_0 e^{-\nu t} = 2U^*(c^* t - z_8) - 1 - q_0^* e^{-\nu t}. \tag{39}$$

by a standard comparison theorem

$$u^*(x, t) \leqq u(x, t) \text{ for all } x \in \mathbf{R}, t \geqq 0. \tag{40}$$

Now (38), (39) and (40) imply the lemma.

LEMMA 11. *Under the hypotheses of theorem* 2 *there exist constants* $z_1, z_2, z_3, q_0, \mu > 0$ *and* $T > 0$ *such that*

$$u(x, t) \leqq U(x + ct - z_2) + q_0 e^{-\mu t} \min \exp \lambda_1(x + ct - z_3) \tag{41}$$

for all $x \leqq 0, t \geqq 0$ *and*

$$U(x + ct' - z_1) - q_0 e^{-\mu t'} \min \exp \lambda_1(x + ct' - z_3) \leqq u(x, t) \tag{42}$$

for all $x \leqq 0$ *and* $t \geqq T$. *Here* λ_1 *is chosen as in the proof of theorem* 1 *and* $t' = t - T$.

PROOF. The upper bound follows directly from lemma 1. To get the lower bound, we use again a moving coordinate frame corresponding to the variables $(z, t') = (x + c(t - T), t - T)$; $v(z, t') = u(x, t)$. The quarterplane $\{(x, t) \mid x \leqq 0 \text{ and } t \geqq T\}$ corresponds to the region $\{(z, t') \mid z \leqq ct', t' \geqq 0\}$ with boundary $B_1 \cup B_2$ where $B_1 = \{(z, t') \mid z \leqq 0, t' = 0\}$ and $B_2 = \{(z, t') \mid z = ct', t' \geqq 0\}$.

For the subfunction again use the Ansatz (13), (14) with t replaced by t'. The relevant quantities q_0, μ, z_4 and $\xi(t')$ are determined by the following modification of the procedure given in lemma 1.

1) Choose q_0 such that $\alpha_1 < 1 - q_0 < \alpha_1 + \eta$.
2) through 6) as in lemma 1, in 2) choose $\mu \leqq \nu$ from lemma 10.
7) Choose T such that $u(0, t) \geqq U(z_4)$ for all $t \geqq T$.
8) Choose $\xi(0)$ such that $\underline{v}(z, t') \leqq v(z, t')$ for $(z, t') \in B_1 \cup B_2$.
7) is possible by lemma 10. We show that 8) is also possible. Choose $\xi(0) + z_5 \geqq 0$ and $\xi(0) \geqq \xi$. As shown in lemma 1 $\underline{v}(z, 0) \leqq 0 \leqq v(z, 0)$ for $z \leqq \xi(0) + z_5$. Thus the choice $\xi(0) + z_5 \geqq 0$ guarantees 7) for $(z, t') \in B_1$. If $(z, t') = (ct', t') \in B_2$, we have to distinguish two cases.

a) $ct' \leqq \xi(0) \leqq z_4$.

$$\begin{aligned} \underline{v}(ct', t) &\leqq U(ct' - \xi(t')) \leqq U(ct' - \xi(0)) \leqq U(z_4) \\ &\leqq u(0, t) = v(ct', t') \end{aligned}$$

b) $ct' - \xi(0) \geqq z_4$.

$$\underline{v}(ct', t') \leqq U(ct' - \xi(t')) - q_0 e^{-\mu t'} \leqq U(ct - \xi(0)) - q_0 e^{-\nu t}$$
$$\leqq u(0, t) = v(ct', t')$$

because $t' \leqq t$, $\mu \leqq \nu$ and $\xi(0) \leqq \xi$ as well as lemma 10. Thus 8) can be fullfilled. The rest is clear from lemma 1.

For the rest of the proof we use again the moving coordinate frame $(z, t) = (x + ct, t)$; $v(z, t) = u(x, t)$. Lemmas 2 through 7 can be applied without change. Define the "left truncations" $v_\ell(z, t)$ and $w_\ell(z, t)$ by

$$v_\ell(z, t) = 1 - \zeta(ct - z)(1 - v(z, t))$$
$$w_\ell(z, t) = 1 - \zeta(cy - z)(1 - w(z, t)).$$

Then

$$v_\ell(z, t) = \begin{cases} v(z, t) & \text{for } z \leqq ct \\ 1 & \text{for } z \geqq ct + 1 \end{cases}$$

and similarly for w.

Lemma 8 shows in our case

$$\lim_{t\to\infty} \max_{z \leqq ct} \frac{|v(z, t) - U(z - z_0)|}{\min \exp \lambda_1 z} = 0.$$

Hence $\lim_{t\to\infty} \|v_\ell(\cdot, t) - U(\cdot - z_0)\| = 0$. Now lemma 9 can be proved for the left truncation, hence

$$\|v_\ell(\cdot, t) - U(\cdot - z_0)\| \leqq Ke^{-\nu t}$$

which proves theorem 2.

The corollary follows from Weinberger's result [1], stating that $\lim_{t\to\infty} u(x, t) = 1$ uniformly on compact sets.

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